Escape Velocity, Pistons, and Integrals

By Eli

How this is going to work

Basically, you’re going to read the slides on the powerpoint, similar to a newspaper. You should be in present mode so that when you finish a slide, you can just click your mouse to move on to the next one. What follows is meant to be a presentation that threads together black holes, pistons, and of course, some math. At the end we see how this is relevant to real life by showing how these concepts are employed with pneumatic motors, a type of motor that runs on renewable energy. Hopefully this presentation will spark an urge to explore these concepts further, and maybe that’ll lead to a contribution to society (automobiles powered by pneumatic motors?). All you need to understand the following concepts is rudimentary calculus and high school physics.

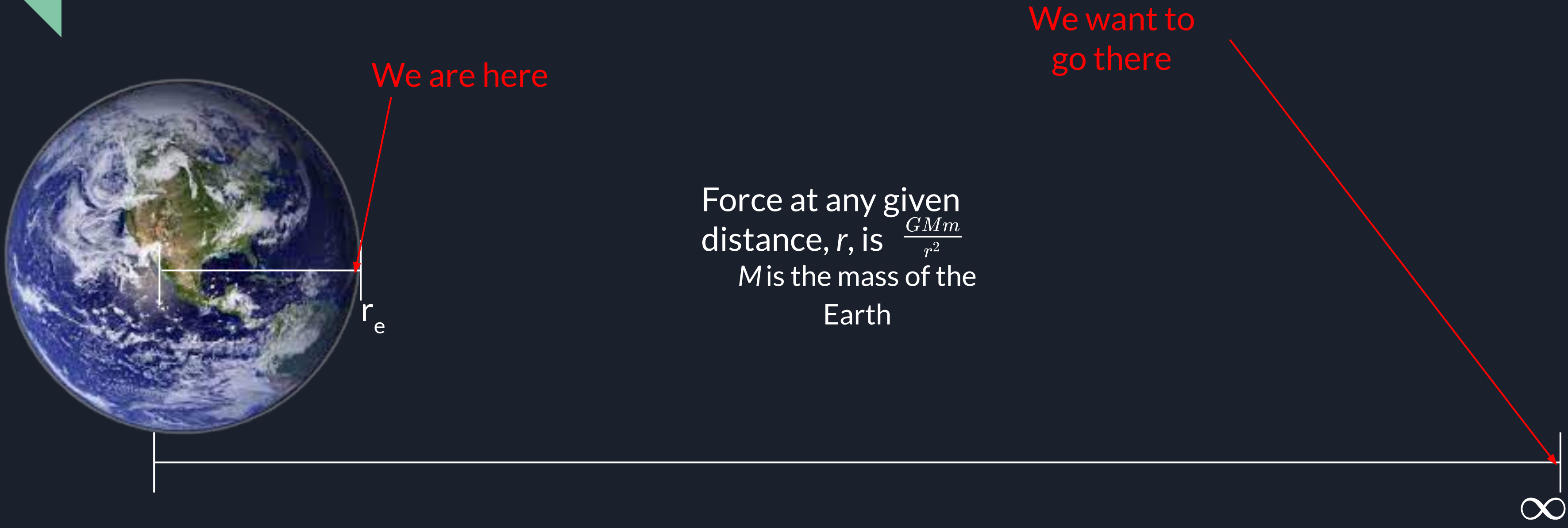
DISCLAIMER

I am still a student when it comes to physics and math in general. Shockingly, I’m not an expert in the field who decided to take up a pseudonym in order to present this topic (though I wouldn’t be offended if you thought I was :). Anyways, as such nothing in here is guaranteed to be 100% accurate, so proceed with caution.

A (seemingly) Innocuous Problem

So you want to go to Mars, eh? Well . . . probably not. I don’t want to go either. But let’s say you did! In order to design the most efficient rocket, you would need to know the escape velocity of an object on Earth. What minimum speed does a person require to exit the Earth’s gravitational field? The astute among you probably know that you can’t exit the Earth’s gravitational field. It will pull on you even in the far reaches of the Universe, albeit with a miniscule force. However, if we travel an infinitely far distance from the Earth, we will experience no gravitational force from the Earth, and thus we will have “exited” its influence. Well now we have a classic integration problem that shouldn’t be too difficult to solve. The obvious way to do this is to find the energy it takes to travel infinitely far away and then equate that to $\frac{mv^2}{2}$. In order to find the energy, we know that energy is defined as the ability to exert a force over a distance; after all, a joule is literally recorded in the units of $\text{Newtons} \cdot \text{meters}$. Therefore, we can just integrate the gravitational force for each miniscule step in distance, all the way to infinity.

The Diagram



Force at any given distance, *r*, is $\frac{GMm}{r^2}$

M is the mass of the Earth

The Math

Alright, seems simple enough. We want to compute $\int_{r_e}^{\infty} \frac{GMm}{r^2}\,dr$. Since the masses and gravitational constant, are . . . well . . . constant, we can factor them out, and then we have the familiar integral:

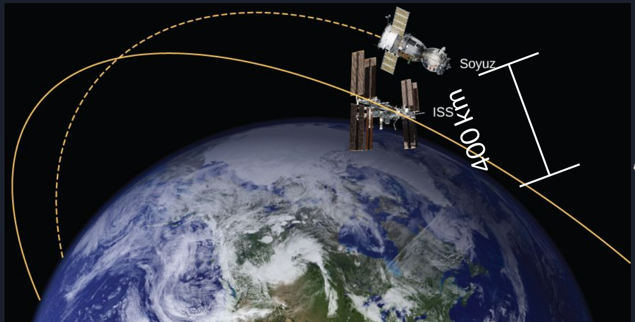
$$\begin{aligned}\int_{r_e}^{\infty} \frac{GMm}{r^2}\,dr &= GMm \int_{r_e}^{\infty} \frac{1}{r^2}\,dr \\ &= GMm\left(-\frac{1}{\infty} + C\right) \\ &= GMm\left(-\frac{1}{\infty} + \frac{1}{r_e}\right) \\ &= \frac{GMm}{r_e}\end{aligned}$$

Setting this equal to $\frac{mv^2}{2}$ yields:

$$\begin{aligned}\frac{mv^2}{2} &= \frac{GMm}{r_e} \\ v^2 &= \frac{2GM}{r_e} \\ v &= \sqrt{\frac{2GM}{r_e}}\end{aligned}$$

Results

We have now concluded that if you were at the Earth's surface, your escape velocity would need to be $\sqrt{\frac{2GM}{r_e}}$, where r_e is the radius of the Earth. However, the logic we used was applicable if you were *any* distance from the Earth. Look back on the math and convince yourself of this point. Choosing r_e as the starting point is arbitrary. Instead, we could have said we were at the International Space Station, hovering 400 km above the Earth. Your radius would be $r_e + 400 \cdot 10^3$, and thus your escape velocity would need to be less, namely $\sqrt{\frac{2GM}{r_e+400\cdot 10^3}}$ (for those that are curious, the escape velocity at the Earth's surface is $11,200\,\frac{\text{m}}{\text{s}}$ while at the ISS it is $10,800\,\frac{\text{m}}{\text{s}}$). Now, what happens if you were closer to the Earth's center? Then your radius would be less and your escape velocity would need to be higher. In the limit, as you become closer and closer to the Earth's center, your escape velocity would spiral out of control, eventually reaching infinity. This would mean you need infinite energy to escape the Earth! This can't be right; we're clearly missing something. . .

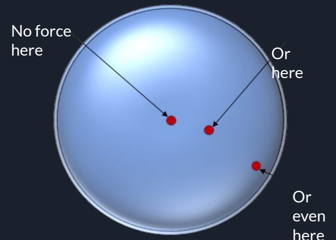


$$v_{esc} = \sqrt{\frac{2GM}{r_e}}$$

Resolution

The key to solving this problem is the fact that if you were digging through the Earth, the gravitational force would not be $\frac{GMm}{r^2}$, rather it weakens as you come closer to the center of the Earth. After all, if you are at the center of the Earth, by symmetry you can experience no gravitational force. In fact, if you are inside a uniform, spherical shell, you experience no gravitational force at all.

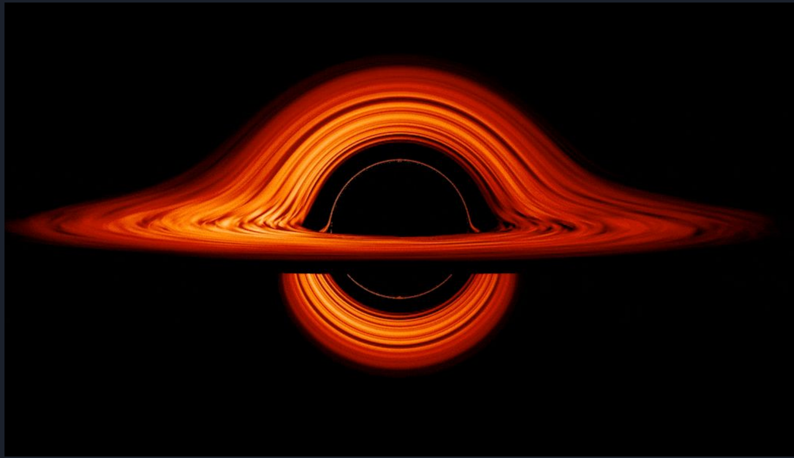
This result is known as the shell theorem, but its discussion is saved for another time. Anyways, if you were to dig through the crust of the Earth, you would be inside a shell (the crust of the Earth), and would thus experience no force from the crust. You could pretend as if the mass of the continents and oceans didn't exist. Therefore, our logic from the previous argument doesn't hold because we assume the gravitational force at every point in space is $\frac{GMm}{r^2}$.



Hollow Shell

Going Down the Rabbit Hole (literally and figuratively)

An interesting question at this point would be: what if the Earth's radius really was smaller? What if we really could get very close to the Earth's center without having to dig through the Earth and forsake the gravitational force of the outer shells? The force throughout the whole process would genuinely be $\frac{GMm}{r^2}$. Well, we still find that the escape velocity required is $\sqrt{\frac{2GM}{r}}$. However, as we come closer and closer, this velocity balloons to infinity. If the Earth's radius was 2 meters, but still had all its mass, we'd find that at its surface the escape velocity would be approximately **2 million meters a second**. Alright, now what if the radius was 8 mm? Then at its surface, the escape velocity would be equivalent to the speed of light, approximately 3 million meters a second! The reason this is significant is because nothing (with mass) can go faster than the speed of light. Therefore, nothing can escape this super dense mass once it touches the surface. Another word for this super dense mass is a black hole.



Relevance

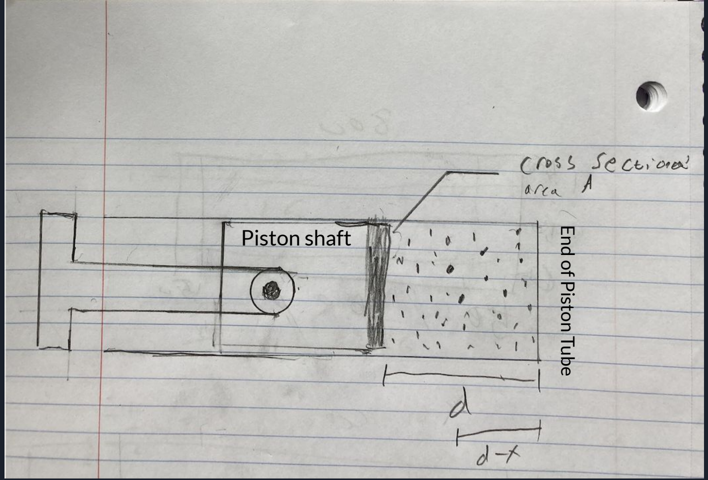
Alright, that’s an interesting way to introduce black holes and it’s a nice math problem, but is there a practical application of this? Well, I’m glad you asked. After all, that’s how I got inspiration to do this presentation. Remember how we discussed earlier that the escape velocity of a 2m radius Earth would be 2 million meters a second? That’s a lot of gravity, and a lot of energy stored in it. In fact, if an object were to fall from the edge of the universe into that “mini-Earth”, it would accrue a speed of 2 million meters a second due to energy conservation. A way to see this is that when an object leaves “mini-Earth”, it converts 2 million meters a second into potential energy, so when it enters, it’s potential energy is converted into 2 million meters a second. In case you were unaware, that is quite a bit of energy. Unfortunately, we humans do not have ready access to super dense masses where we can cash in terajoules of potential energy, but we do have analogues of that. These analogues may someday be used to solve the climate crisis. And no, these analogues do not cost billions of dollars to make in a lab.

THE PISTON

Pistons! These remarkably simple feats of engineering are capable of possessing an unlimited amount of potential energy, just like black holes. If you concentrated an amount of mass to occupy zero volume, as you get closer and closer, your gravitational force balloons to infinity, resulting in the integral having infinite energy. The principle is the same with pistons. If you compressed air to occupy minimal volume, the pressurized force balloons to infinity, causing the integral to have infinite energy. We'll get to the math on the next slide, but there's a key difference I want to point out here. With black holes, the mass is already compressed to zero volume, so we can just cash in on the potential energy. With pistons, we don't have air compressed to zero volume. There are no natural pistons in space where you can find pressurized air in soda bottles. We have to make those pistons ourselves. "Then what's the point? We're just making the potential energy ourselves by compressing the pistons." That is true, and their use does not reside in the fact that they are energy sources; rather, they are energy storers. We can take a piston, compress it using all our available energy, secure it so that it can't move, and then move it somewhere else. To harness the energy again, we can hook it up to a turbine, release the air, and it will spin the turbine, retrieving the energy.

Also, we will act as if the temperature remains constant throughout the compression of the gas, even though the momentum from the piston accelerates the gas particles and consequently heats it up. This is because the temperature only adds to our energy, and we can still show that you can store infinite energy without having to factor in the increase in temperature. With that out of the way, let's do some math!

The Math



We have the diagrammed piston above, and we want to figure out how much energy it has if we were to move the shaft forward an extra *x* meters. Assume that at this point, the pressure in the piston is the same as the atmospheric pressure, so it is in equilibrium. Then, the force on the piston shaft created by moving the shaft *x* meters forwards is given by:

$$PV = nRT$$
$$P \cdot A(d-x) = nRT$$
$$P = \frac{nRT}{A(d-x)}$$
$$A \cdot P = \frac{nRT}{(d-x)}$$
$$F = \frac{nRT}{(d-x)}$$

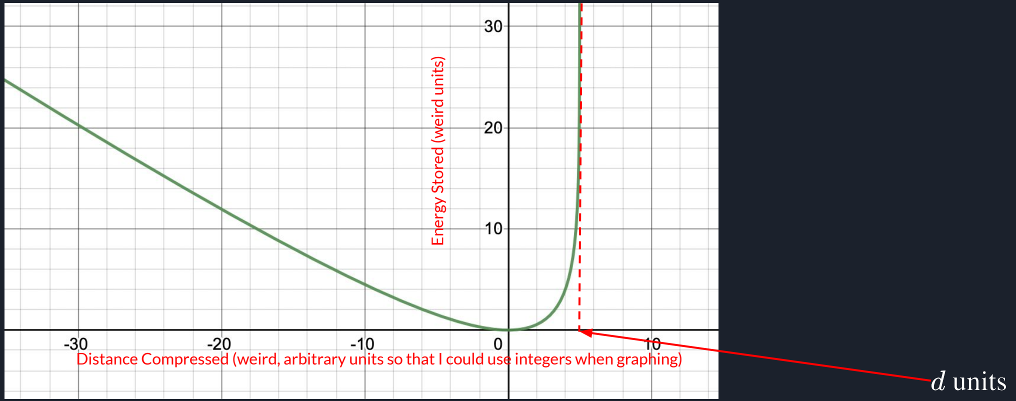
To find the energy it has, we find the net work performed on the piston if it were to be displaced *x* meters. Naturally, this quantity will be negative because we are trying to store energy. To do this, we note that the net force on the piston at any given point is the force exerted by the air pressure on the left minus the force exerted by the compressed gas on the right. If we let F_0 be the force exerted by the atmosphere, then $F_{net} = F_0 - \frac{nRT}{d-x}$. This makes sense; at $x = d$ we get infinite leftwards force. At $x = 0$, we find that $F_{net} = F_0 - \frac{nRT}{d} = 0$. Remember, we defined *d* such that the forces were in equilibrium.

Now all that is left to do is to integrate to find the total work done.

$$W_{tot}(x) = \int_0^x \left(F_0 - \frac{nRT}{d-s}\right) ds$$
$$= F_0 \cdot x - nRT \int_0^x \frac{1}{d-s} ds$$
$$= F_0 \cdot x - nRT(-\ln(d-x) + C)$$
$$= F_0 \cdot x - nRT(-\ln(d-x) + \ln(d))$$

Results

Alright, we have now discovered that if you displace a piston shaft *x* meters from an equilibrium position (distance of *d*), the energy stored is equivalent to $|F_0 \cdot x - nRT(-\ln(d-x)+\ln(d))|$. The graph of that function looks something like this:



As you can see, you can get any amount of energy stored in the piston by pushing it from in between $[0, d)$.

Real World Applications

Naturally, people have built motors that run off pressurized air, and these are known as pneumatic motors. A way to visualize a simple pneumatic motor is to take an internal combustion engine, and instead of igniting gasoline in the chamber, compressed air is let into the chamber which pushes out the piston. There have been many attempts at getting them to power automobiles, with no success so far. That's because it's a lot harder than it sounds. Theoretically showing that pneumatic motors would be able to contain unlimited amounts of energy* doesn't actually translate to real world success. There's still a problem figuring out how to physically put that energy in a piston. Additionally, you need to build strong vessels and containers to store the compressed air, and you also must deal with many other complications, such as the changing temperatures of compressing and expanding air. You also will not get constant power when extracting the energy from the piston, which poses a real problem. Overall, it's a tough engineering challenge, but one with enormous benefits. It's not everyday that you come up with a potential solution for the climate crisis.

*though not really because the ideal gas law doesn't work at such extremes, but you can still store a tremendous amount of energy in a piston that you might as well call it infinite

Conclusion

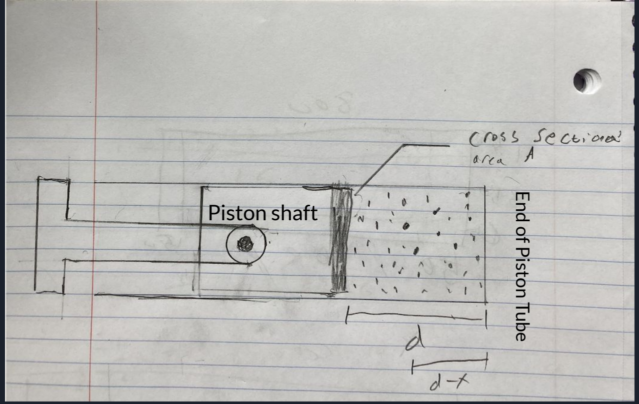
We went over the formula for escape velocity, and we saw how it related to black holes. We then connected it to pneumatic motors and pistons, and showed how this theoretically allows for infinite energy storage. There are many more things I could dive into, but I believe that's enough for the day. This is the official end of my SoME1 project, but there are still some following slides for those of you who are still interested (if you are, congrats! This isn't the most enthralling slideshow). The following slides are simply other topics I wanted to include, but it dragged on the project and deviated from the main point. Therefore, they're kind of in the "back" for those who want more, and I personally think there are some very cool ideas there.

Bonus Content 1: More physics

A few of you may be wondering why I didn't include electromagnetic force as an analogue to gravity, and the reason is that opposites attract in electromagnetism. With gravity, matter attracts matter, antimatter attracts antimatter (theorized), and matter repels antimatter (theorized). For those of you who are unaware, you can just think of antimatter as stuff that looks exactly the same as regular matter, except the charge is multiplied by negative one, and instead of having gravitational attraction, it has gravitational repulsion (this gravitational stuff is theorized, not proven). In electromagnetism, positive repels positive, negative repels negative, and positive attracts negative. Therefore, it's exceedingly difficult to amass a big positive charge in one place, because the positive charges repel each other. Theoretically though, you could just have a piston in which the end of the tube is positively charged, and the end of the shaft is as well. This piston could be evacuated of all air, and you could also store infinite energy by pushing the shaft towards the end of the tube. However, the charge would leak out into the atmosphere, making it ineffective. Also, it's pretty much just a capacitor, but it's the same charge on each side.

Another interesting concept that's worth thinking about is if there is an analogue of black holes with charge. Basically, if you condense matter closely enough, the escape velocity exceeds the speed of light because of gravitation. If you condense charge close enough (assuming you can get all that charge to be smushed up against each other; possibly by using neutrons and the strong electromagnetic force), could you make a charge-black hole? Something in which the force is so great that you could never escape it once you come too close? The answer is no, because the force depends on the charge of the object. With gravity, the force varies with your inertia, while with electromagnetism, it varies with charge. Therefore, you could have *0* charge and walk past the charge-black hole unaffected.

Bonus Content 2: More Math



Let's look back on our original piston diagram and our derived equations. One interesting side-excursion to journey through would be to consider what happens if $F_0 = 0$. In fact, this is the case in space; there is no outside air-pressure. You might protest because we defined *d* in the beginning to be in equilibrium with F_0, but that decision was arbitrary. The net force at a distance *d* need not be *0*. It could be any amount, and all the math remains the same. It was just conceptually easier to think of the starting point as having no net force. Anyways, this way we can set $F_0 = 0$, have *d* be some arbitrary distance, say 1 meter, and then we can look at our equation for net work done. If we make *x* be a positive displacement, it's still pretty much the same thing; the energy balloons to infinity. However, if we make *x* negative; we see that the work done becomes positive. This means that the piston shaft will be pushed leftwards without us having do anything. And the work keeps on being positive. The limit as x approaches $-\infty$ is ∞, meaning that infinite work can be done, and as such infinite energy accrued.

$$
\begin{aligned}
W_{\text{tot}}(x) &= \int_0^x \left(F_0 - \frac{nRT}{d-s} \right) ds \\
&= F_0 \cdot x - nRT \int_0^x \frac{1}{d-s}\, ds \\
&= F_0 \cdot x - nRT(-\ln(d-x) + C) \\
&= F_0 \cdot x - nRT(-\ln(d-x) + \ln(d))
\end{aligned}
$$

Assume this is *0*.

As we displace x to $-\infty$, the work done goes to positive infinity, yielding infinite energy.

Bonus Content 2: More Math (cont’d)

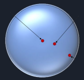
$$
\begin{aligned}
W_{\text{tot}}(x) &= \int_0^x \left(F_0 - \frac{nRT}{d-s} \right) ds \\
&= F_0 \cdot x - nRT \int_0^x \frac{1}{d-s}\, ds \\
&= F_0 \cdot x - nRT(-\ln(d-x) + C) \\
&= F_0 \cdot x - nRT(-\ln(d-x) + \ln(d))
\end{aligned}
$$

Phrased more simply, our problem statement is that if we bring compressed air to outer space, and then we let the piston be pushed infinitely outwards by the gas, then we should get infinite energy. This is due to the fact that the force varies inversely to displacement, and the integral of $\int_1^\infty \frac{1}{x}\,dx$ is undefined. Of course, we’re missing something. That thing is that as the gas expands, the temperature falls to *0*. Thus, the term nRT goes to *0* in $W_{\text{tot}}(x) = F_0 \cdot x - nRT(-\ln(d-x) + \ln(d))$. Therefore, we can’t just blow up *x* to negative infinity and get infinite energy.

Interestingly, there is no analogue of this when it comes to gravity. This is because the force is proportional to $\frac{1}{r^2}$, and the integral of $\frac{1}{r^2}$ to ∞ is definite. This is good, because otherwise there’d be no such thing as escape velocity. No matter how far you’d go, you would still need more and more energy to exit the Earth’s gravitational field. In fact, if we assume that matter gravitationally *repels* antimatter, we can prove that gravity cannot vary with $\frac{1}{r}$. Imagine we had an antimatter particle, and we put it down near Earth. If gravity was proportional to $\frac{1}{r}$, then after moving so that it is a distance r_2 from the center of the Earth, the work done on it would be $GMm \int_{r_e}^{r_2} \frac{1}{r}\, dr = GMm(\ln r_2 - \ln r_e)$. Therefore, as r_2 goes to infinity, the particle will gain infinite energy, breaking energy conservation.

Images used

https://en.m.wikipedia.org/wiki/File:Earth_Western_Hemisphere_transparent_background.png



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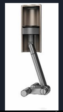

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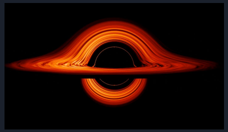

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