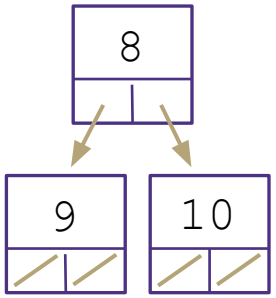
Lecture 14: Heap Percolations

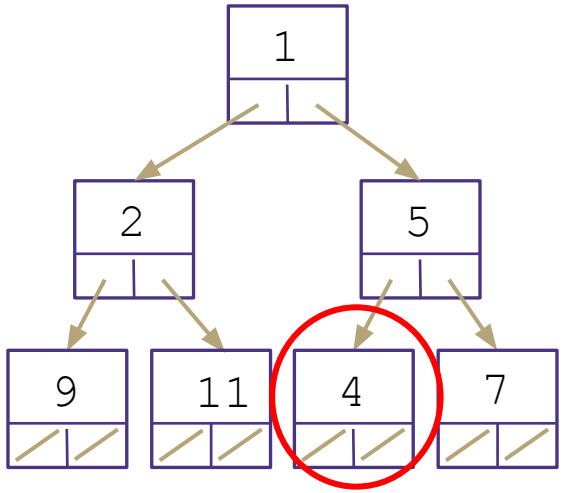
CSE 373 Data Structures and Algorithms

Warm Up

Are the following trees valid min heaps?



Valid



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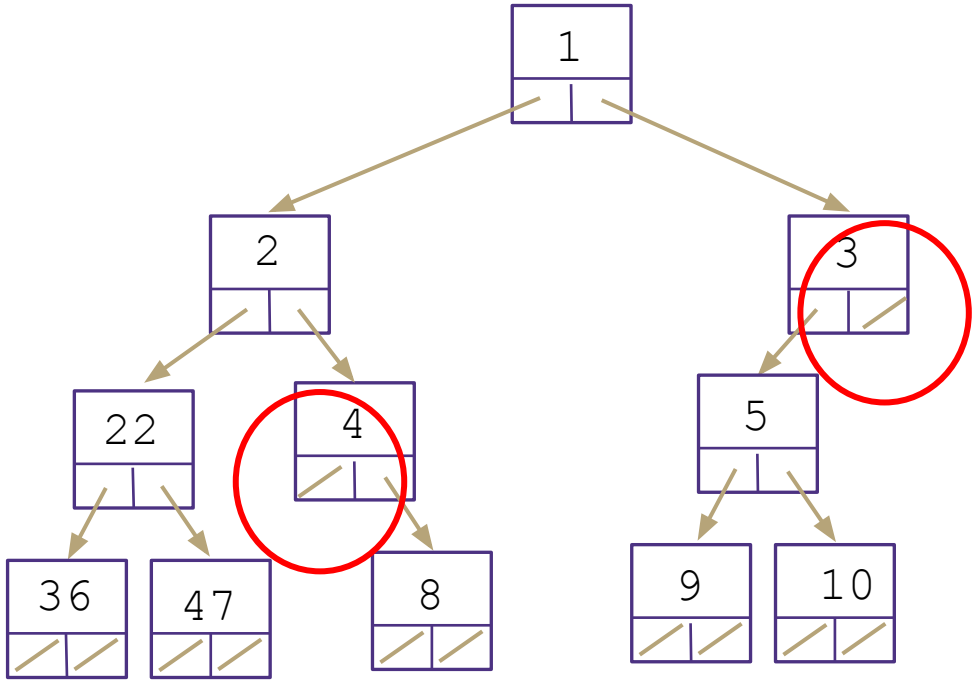


Invalid

Announcements

P2 due today!

EX3 due Friday

Simulated Midterm out this Friday

- Designs due Monday at 12pm (NOT AM)
- Design Reviews due Wednesday at 11:59pm

NO LATE ASSIGNMENTS ACCEPTED

Your toolbox so far…

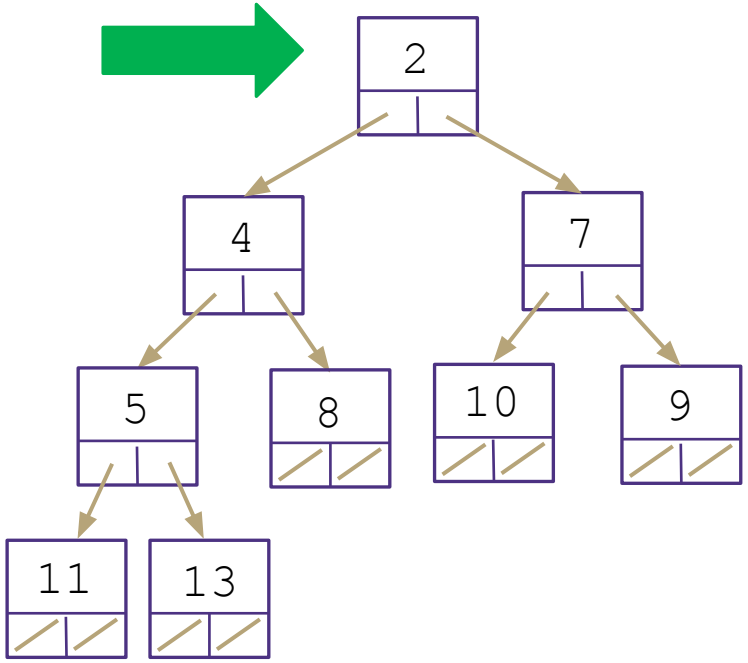
- ADT
  - List – flexibility, easy movement of elements within structure
  - Stack – optimized for first in last out ordering
  - Queue – optimized for first in first out ordering
  - Dictionary (Map) – stores two pieces of data at each entry
  - Priority Queue - optimized for highest priority out first
- Data Structure Implementation
  - Array – easy look up, hard to rearrange
  - Linked Nodes – hard to look up, easy to rearrange
  - Hash Table – constant time look up, no ordering of data
  - BST – efficient look up, possibility of bad worst case
  - AVL Tree – efficient look up, protects against bad worst case, hard to implement
  - Heap - efficient for Min or Max values

Priority Queue / heaps roadmap

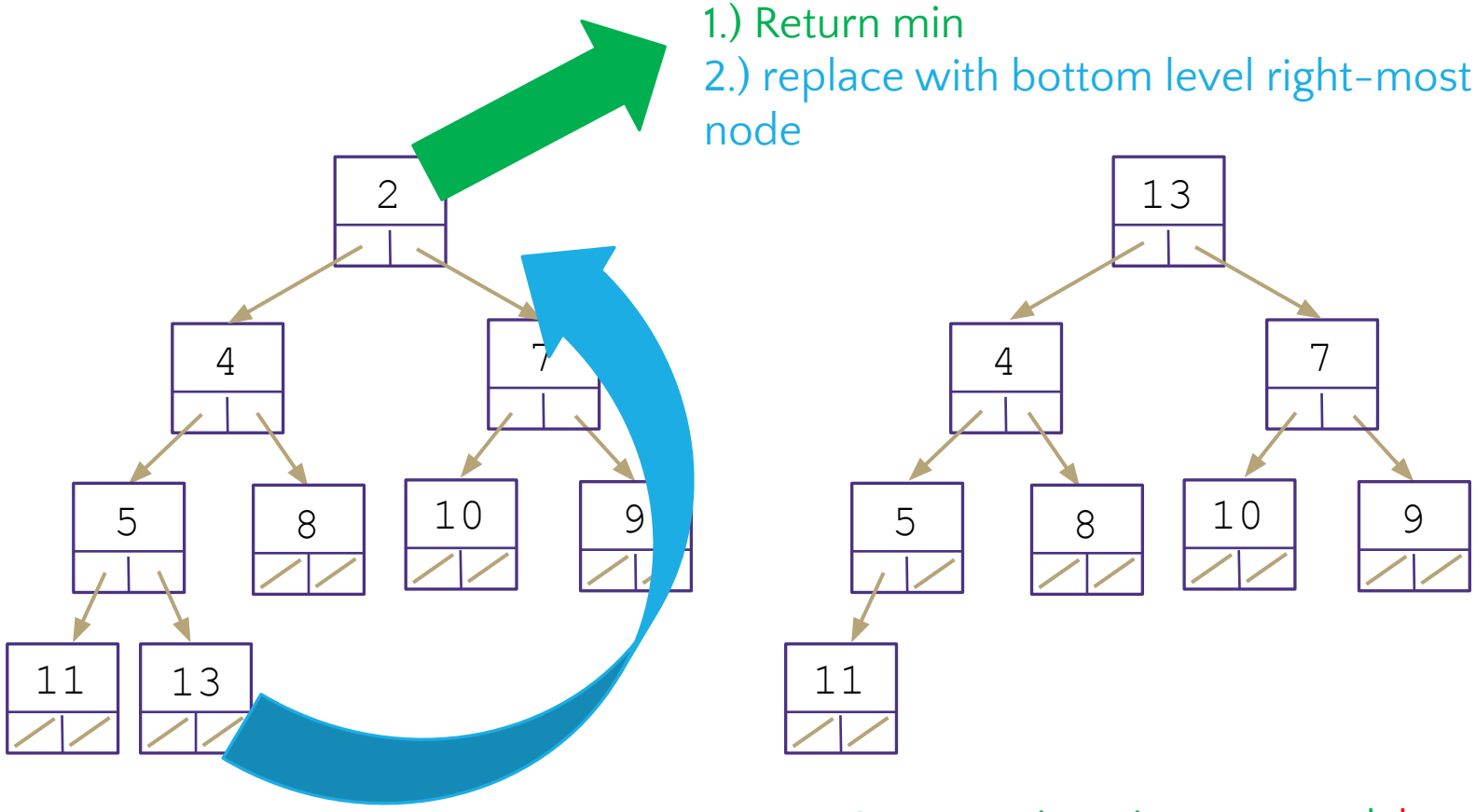
- PriorityQueue ADT
- PriorityQueue implementations with current toolkit
- Binary Heap idea + invariants
- **Binary Heap methods**
- Binary Heap implementation details

Implementing peekMin()

Runtime: $\Theta(1)$



Implementing removeMin()



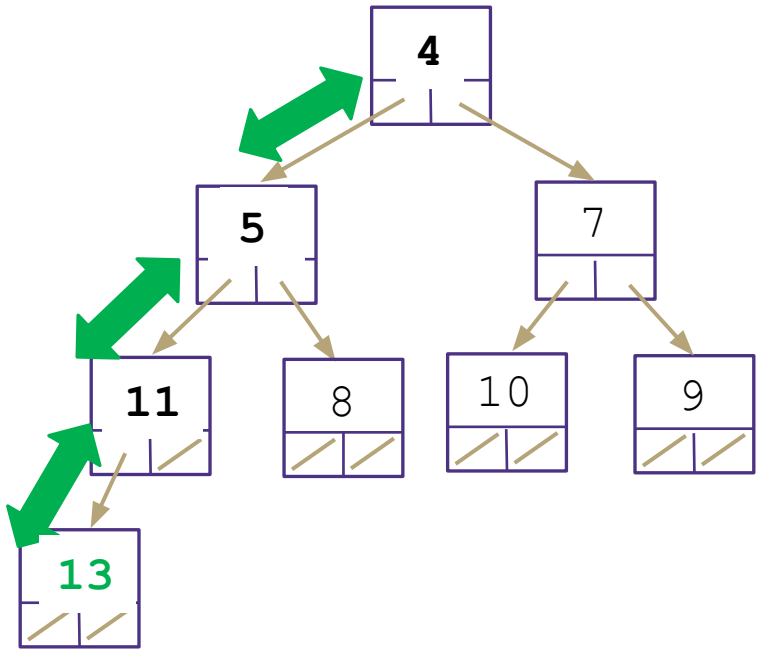
1.) Return min
2.) replace with bottom level right-most node

Structure invariant restored, heap invariant broken

Implementing removeMin() – percolateDown

3.) percolateDown()

Recursively swap parent with **smallest** child until parent is smaller than both children (or we're at a leaf).



What's the worst-case running time?
Have to:

1. Find last element
2. Move it to top spot
3. Swap until invariant restored

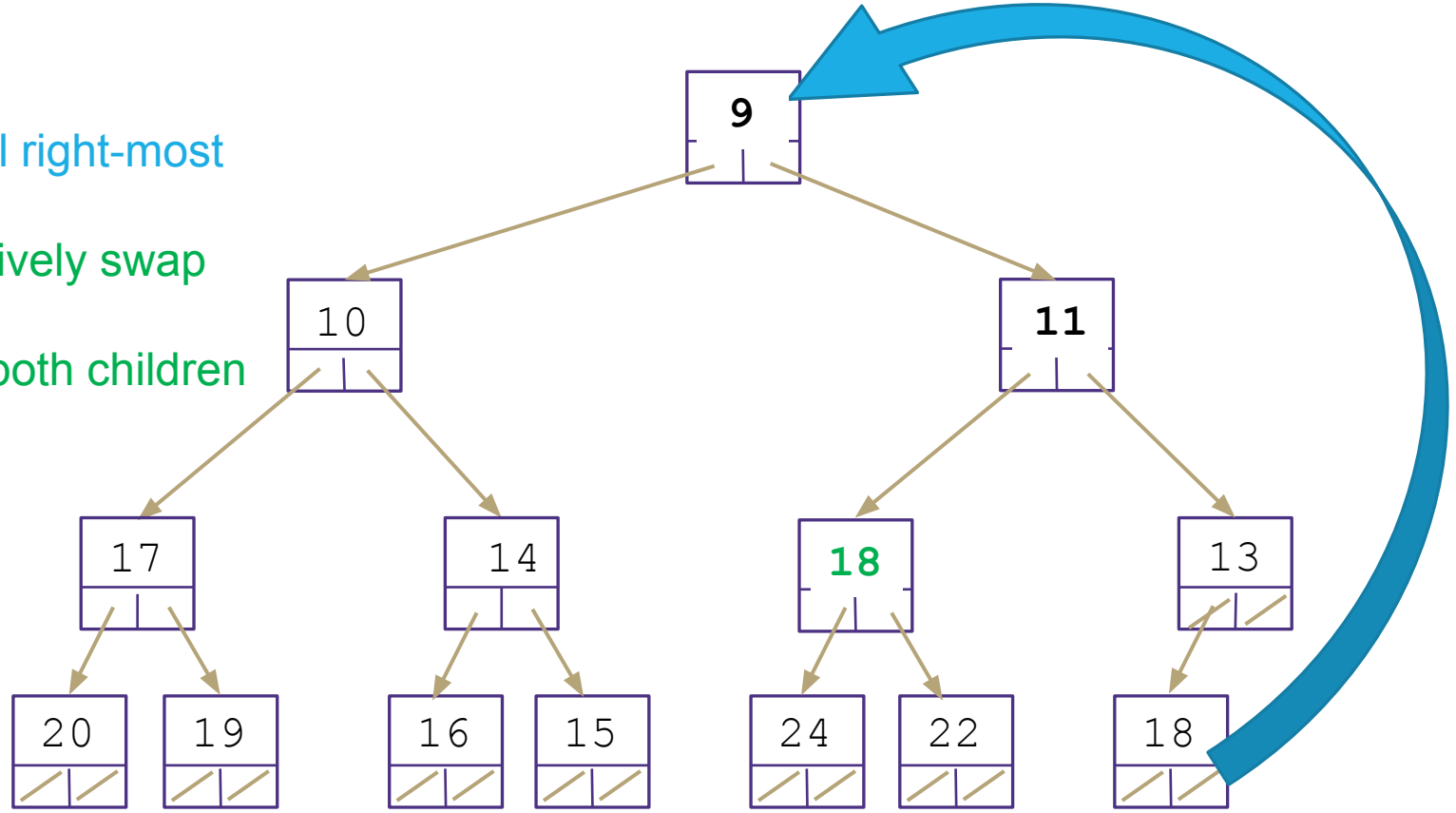
(how many times do we have to swap?)

this is why we want to keep the height of the tree small! The height of these tree structures (BST, AVL, heaps) directly correlates with the worst case runtimes

Structure invariant restored, heap invariant restored

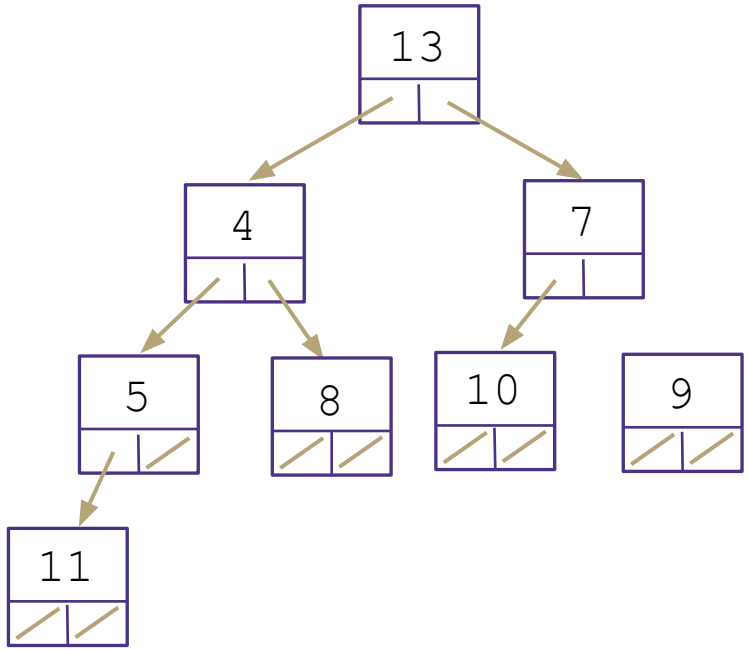
Practice: removeMin()

1.) Remove min node
2.) replace with bottom level right-most node
3.) percolateDown - Recursively swap parent with **smallest** child
until parent is smaller than both children (or we're at a leaf).



Important note about heap invariant

Why does `percolateDown` swap with the smallest child instead of just any child?



If we swap 13 and 7, the heap invariant isn't restored!

7 is greater than 4 (it's not the smallest child!) so it will violate the invariant.

Implementing add()

add() Algorithm:

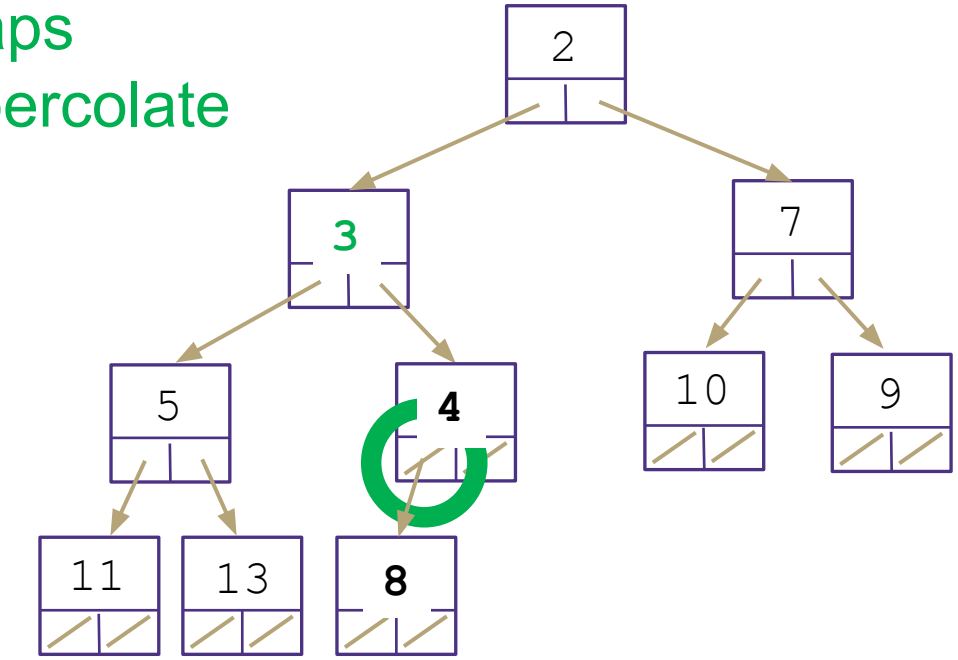
- Insert a node on the bottom level that ensure no gaps
- Fix heap invariant by percolate **UP**

i.e. swap with parent,

until your parent is

smaller than you

(or you're the root).



Worst case runtime is similar to removeMin and percolateDown – might have to do log(n) swaps, so the worst-case runtime is Theta(log(n))

Practice: Building a minHeap

Construct a Min Binary Heap by adding the following values in this order:

5, 10, 15, 20, 7, 2

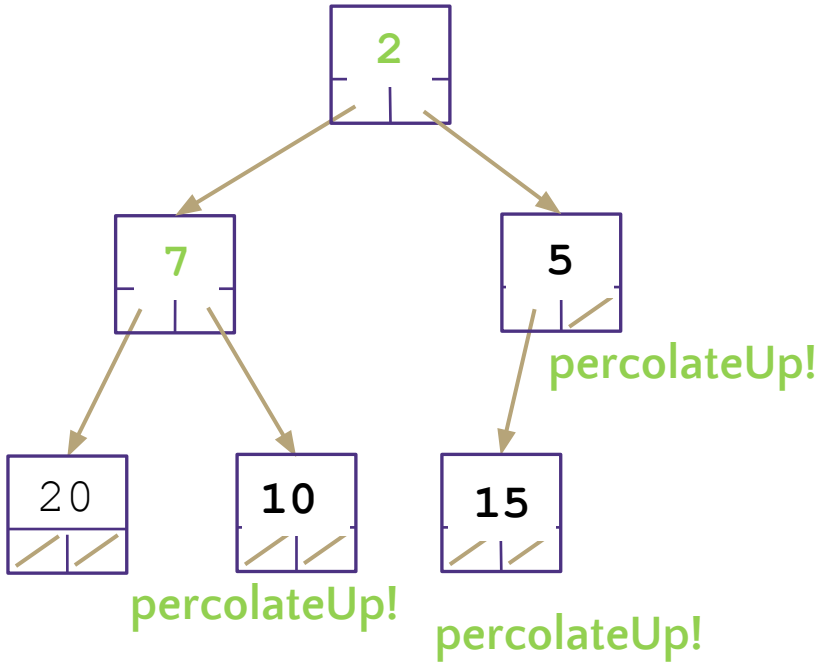
Add() Algorithm:

- 1.) Insert a node on the bottom level that ensures no gaps
- 2.)Fix heap invariant by percolate **UP**

i.e. swap with parent, until your parent is smaller than you (or you're the root).

Min Binary Heap Invariants

1. **Binary Tree** – each node has at most 2 children
2. **Min Heap** – each node's children are larger than itself
3. **Level Complete** - new nodes are added from left to right completely filling each level before creating a new one



minHeap runtimes

removeMin():

- remove root node
- Find last node in tree and swap to top level
- Percolate down to fix heap invariant

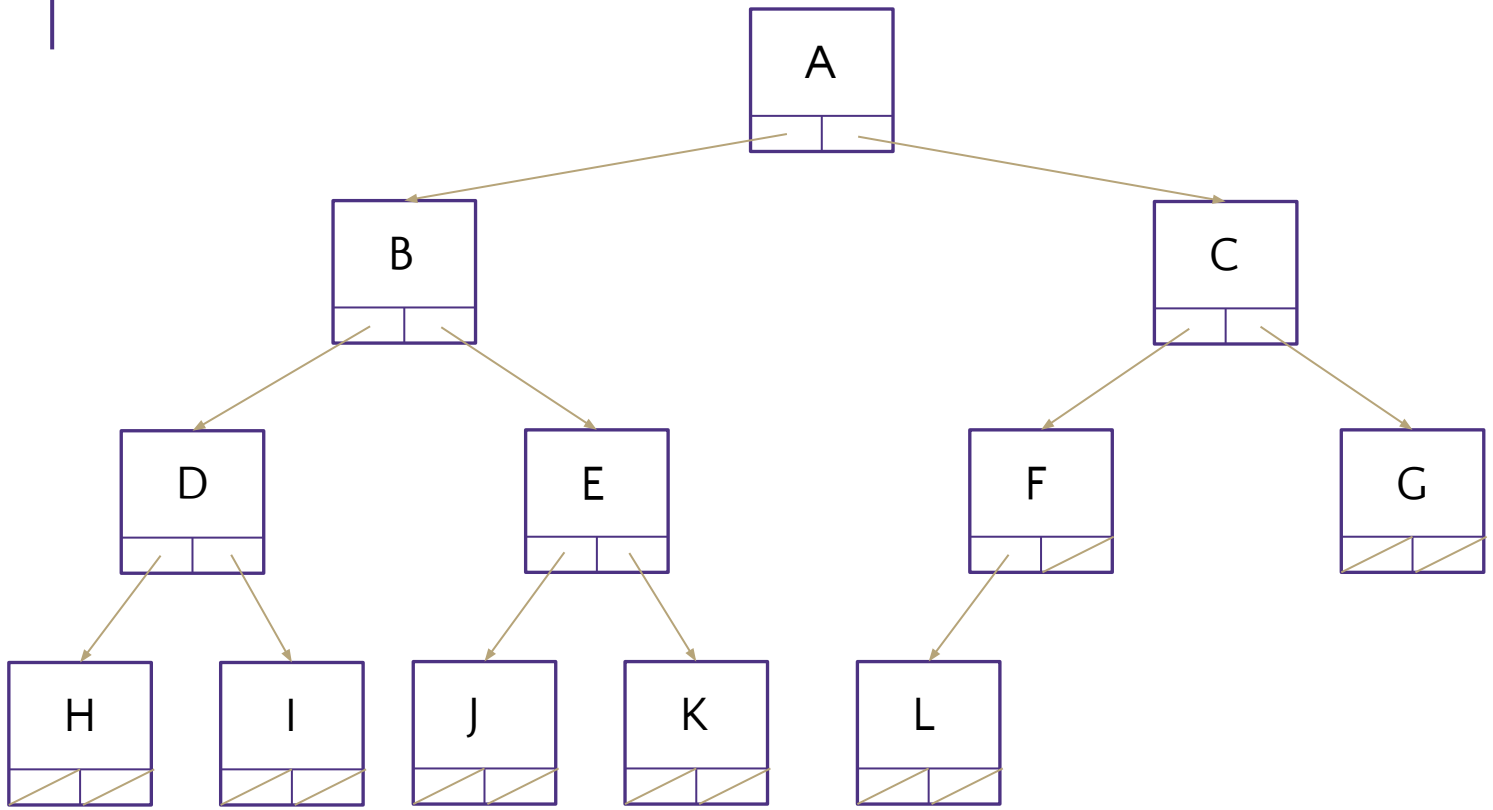
add():

- Insert new node into next available spot
- Percolate up to fix heap invariant

Finding the last node/next available spot is the hard part.
You can do it in $\Theta(\log n)$ time on complete trees, with some extra class variables…
But it's NOT fun

And there's a much better way!

Implement Heaps with an array



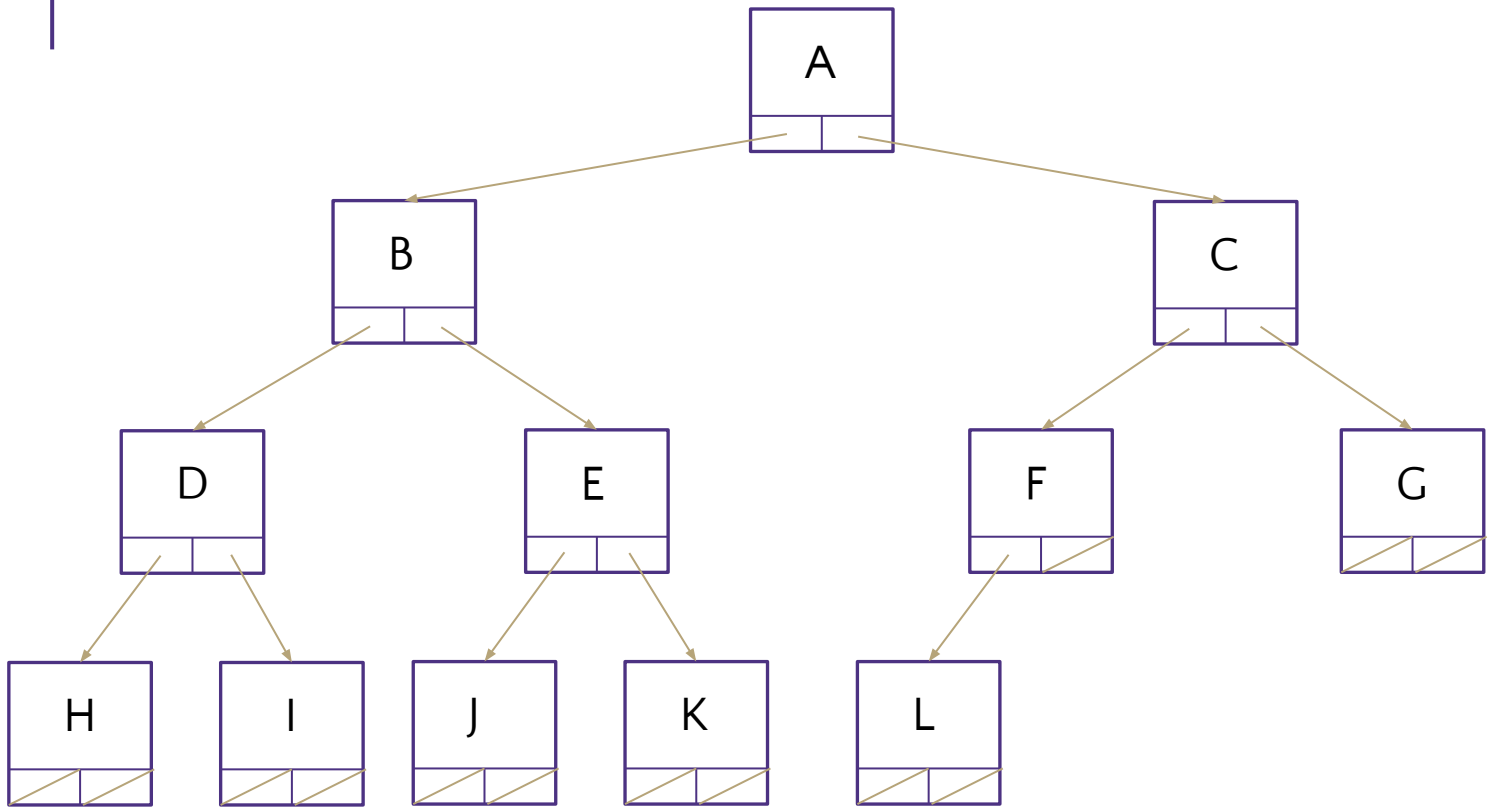
Fill array in **level-order** from left to right

0	1	2	3	4	5	6	7	8	9	10	11	12	13
A	B	C	D	E	F	G	H	I	J	K	L		

We map our binary-tree representation of a heap into an array implementation where you fill in the array in level-order from left to right.

The array implementation of a heap is what people actually implement, but the tree drawing is how to think of it conceptually. Everything we've discussed about the tree representation still is true!

Implement Heaps with an array



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0	1	2	3	4	5	6	7	8	9	10	11	12	13
A	B	C	D	E	F	G	H	I	J	K	L		

How do we find the minimum node?

$$peekMin() = arr[0]$$

How do we find the last node?

$$lastNode() = arr[size - 1]$$

How do we find the next open space?

$$openSpace() = arr[size]$$

How do we find a node's left child?

$$leftChild(i) = 2i + 1$$

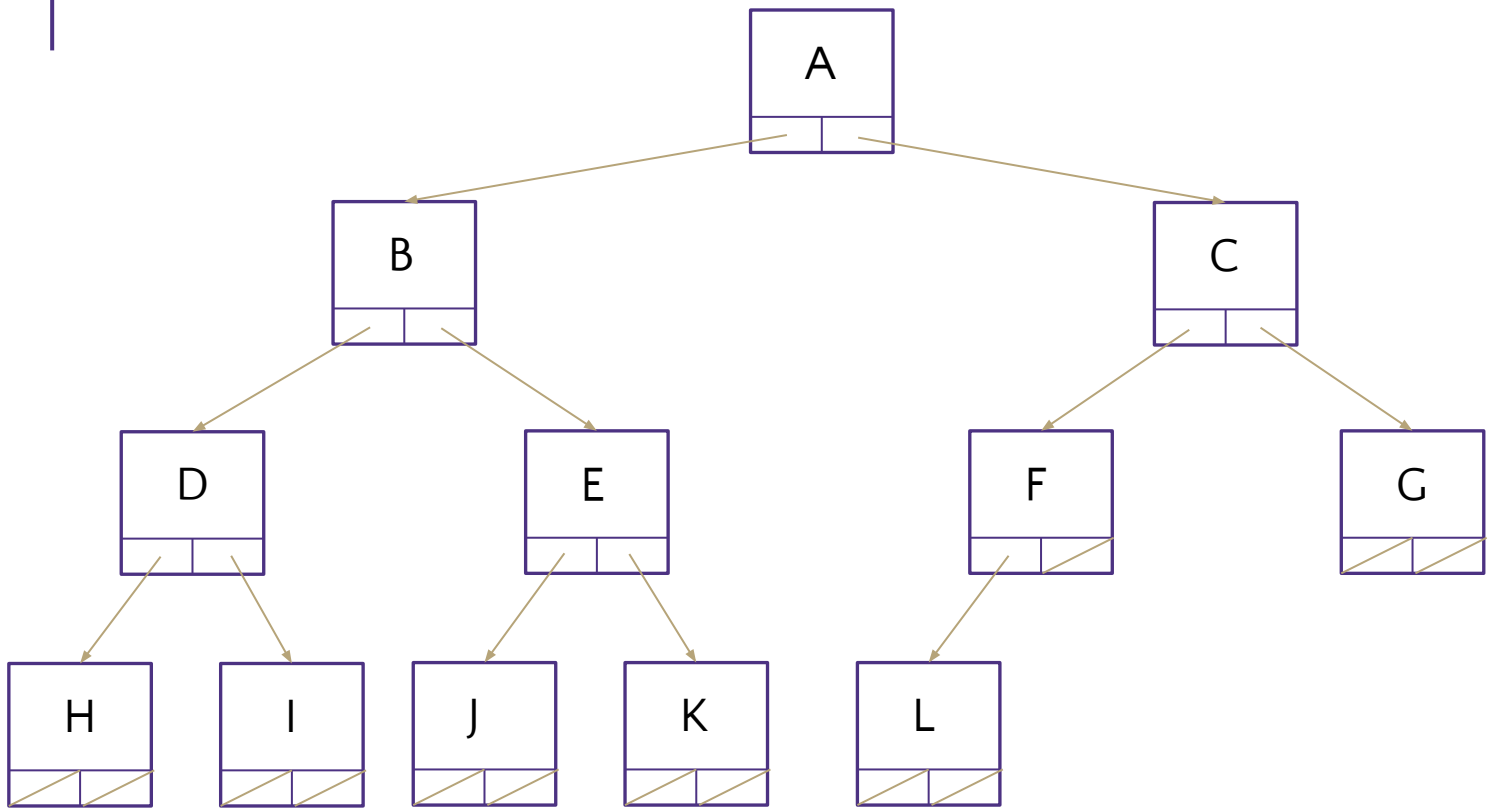
How do we find a node's right child?

$$rightChild(i) = 2i + 2$$

How do we find a node's parent?

$$parent(i) = \frac{(i-1)}{2}$$

Implement Heaps with an array



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How do we find the minimum node?

How do we find the last node?

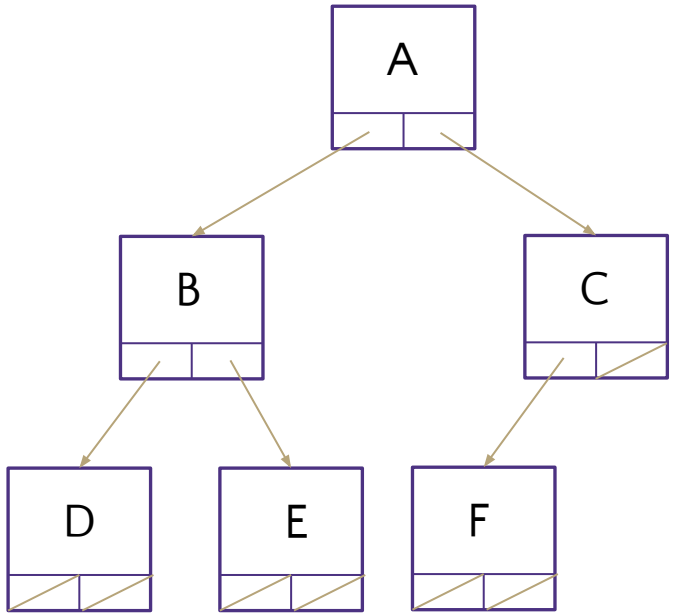
How do we find the next open space?

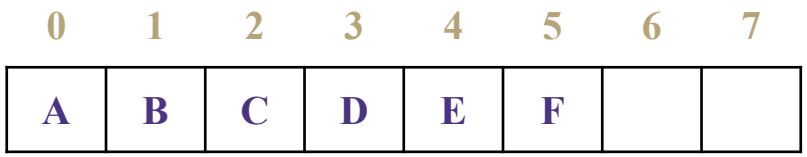
How do we find a node's left child?

How do we find a node's right child?

How do we find a node's parent?

Heap Implementation Runtimes





Implementation	add	removeMin	Peek
Array-based heap	worst: $\Theta(\log n)$ in-practice: $\Theta(1)$	worst: $\Theta(\log n)$ in-practice: $\Theta(\log n)$	$\Theta(1)$

We've matched the **asymptotic worst-case** behavior of AVL trees.

But we're actually doing better!

- The constant factors for array accesses are better.
- The tree can be a constant factor shorter because of stricter height invariants.
- In-practice case for add is really good.
- A heap is MUCH simpler to implement.

Are heaps always better? AVL vs Heaps

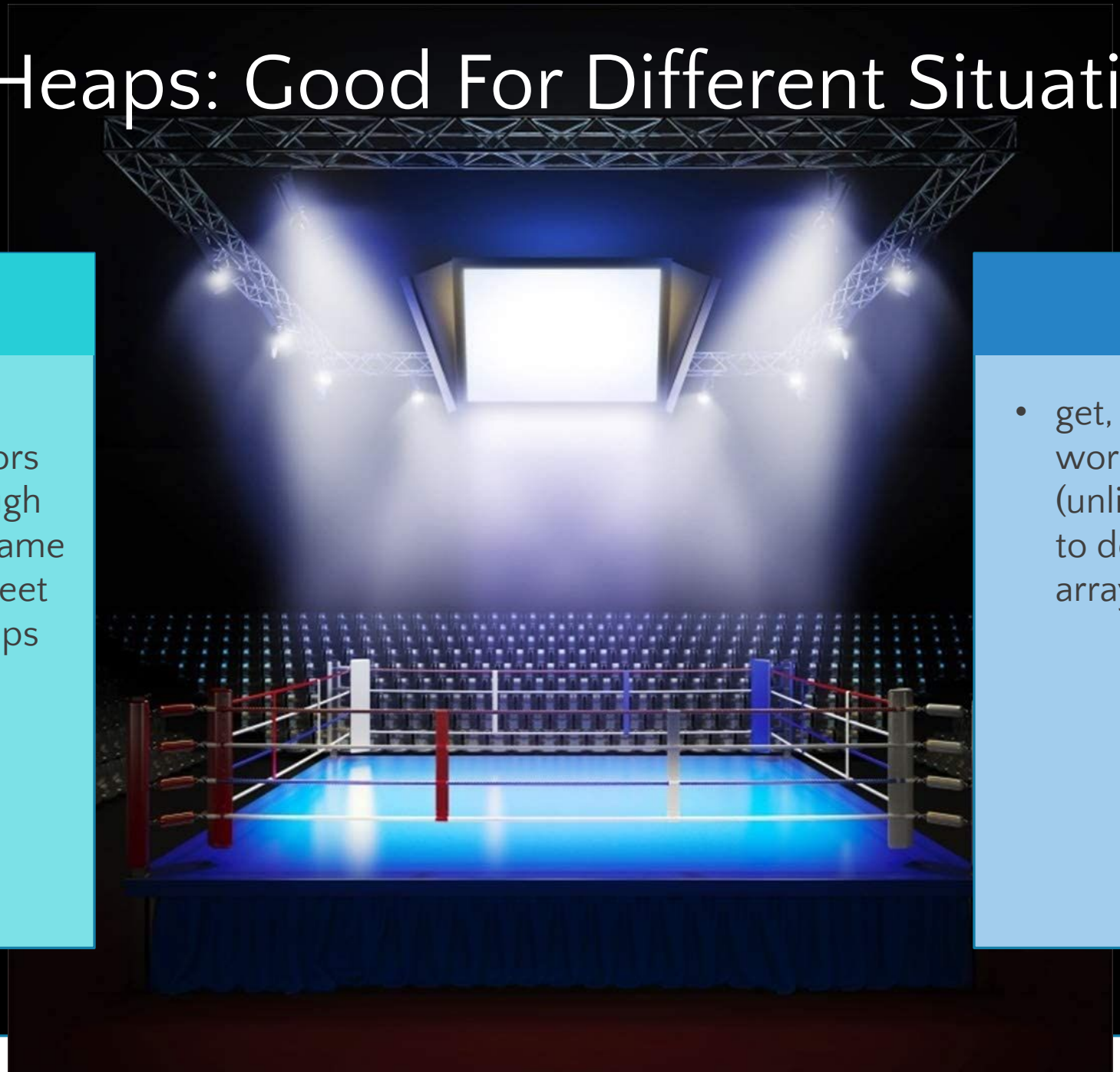
- The really amazing things about heaps over AVL implementations are the constant factors (e.g. 1.2n instead of 2n) and the sweet sweet Theta(1) in-practice `add` time.

- The really amazing things about AVL implementations over heaps is that AVL trees are absolutely sorted, and they guarantee worst-case be able to find (contains/get) in Theta(log(n)) time.

If heaps have to implement methods like contains/get/ (more generally: finding a particular value inside the data structure) – it pretty much just has to loop through the array and incur a worst case Theta(n) runtime.

Heaps are stuck at Theta(n) runtime and we can’t do anything more clever…. aha, just kidding.. unless…?

AVL vs Heaps: Good For Different Situations



HEAPS

- removeMin: much better constant factors than AVL Trees, though asymptotically the same
- add: in-practice, sweet sweet Θ(1) (few swaps usually required)

PriorityQueue

AVL TREES

- get, containsKey: worst-case (log n) time (unlike Heap, which has to do a linear scan of the array)

Map/Set

Project 3

Build a heap! Alongside hash maps, heaps are one of the most useful data structures to know – and pop up many more times this quarter!

- You'll also get practice using multiple data structures together to implement an ADT!
- Directly apply the invariants we've talked so much about in lecture! Even has an invariant checker to verify this (a *great* defensive programming technique!)

MIN PRIORITY QUEUE ADT

State

Set of comparable values (*ordered based on "priority"*)

Behavior

add(value) – add a new element to the collection

removeMin() – returns the element with the smallest priority, removes it from the collection

peekMin() – find, but do not remove the element with the smallest priority

changePriority(item, priority) *– update the priority of an element*

contains(item) *– check if an element exists in the priority queue*

Project 3 Tips

Project 3 adds changePriority and contains to the PriorityQueue ADT, which aren't efficient on a heap alone

You should utilize an extra data structure for changePriority!

- Doesn't affect *correctness* of PQ, just runtime. Please use a built-in Java collection instead of implementing your own (although you could in theory).

changePriority Implementation Strategy:

- implement without regards to efficiency (without the extra data structure) at first
- analyze your code's runtime and figure out which parts are inefficient
- reflect on the data structures we've learned and see how any of them could be useful in improving the slow parts in your code

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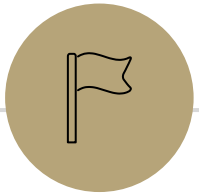


More Priority Queue Operations

More Operations

Min Priority Queue ADT

state

Set of comparable values
- Ordered based on "priority"

behavior

add(value) – add a new element to the collection

removeMin() – returns the element with the smallest priority, removes it from the collection

peekMin() – find, but do not remove the element with the smallest priority

We'll use priority queues for lots of things later in the quarter.

Let's add them to our ADT now.

Some of these will be **asymptotically** faster for a heap than an AVL tree!

BuildHeap(elements $e_1, \ldots, e_n$)
Given n elements, create a heap containing exactly those n elements.

Even More Operations

BuildHeap(elements $e_1, \ldots, e_n$) – Given n elements, create a heap containing exactly those n elements.

Try 1: Just call insert n times.

Worst case running time?

n calls, each worst case $\Theta(\log n)$. So it's $\Theta(n \log n)$ right?

That proof isn't valid. There's no guarantee that we're getting the worst case every time!

Proof is right if we just want an $O()$ bound

- But it's not clear if it's tight.

BuildHeap Running Time

Let's try again for a Theta bound.

The problem last time was making sure we always hit the worst case.

If we insert the elements in decreasing order **we will!**

- Every node will have to percolate all the way up to the root.

So we really have n $\Theta(\log n)$ operations. QED.

There's still a bug with this proof!

BuildHeap Running Time (again)

Let's try once more.

Saying the worst case was decreasing order was a good start.

What are the actual running times?

It's $\Theta(h)$, where h is the **current** height.

- The tree isn't height $\log n$ at the beginning.

But most nodes are inserted in the last two levels of the tree.

- For most nodes, h is $\Theta(\log n)$.

The number of operations is at least

$\frac{n}{2} \cdot \Omega(\log n) = \Omega(n \log n)$.

Can We Do Better?

- What's causing the n add strategy to take so long?
  - Most nodes are near the bottom, and might need to percolate all the way up.

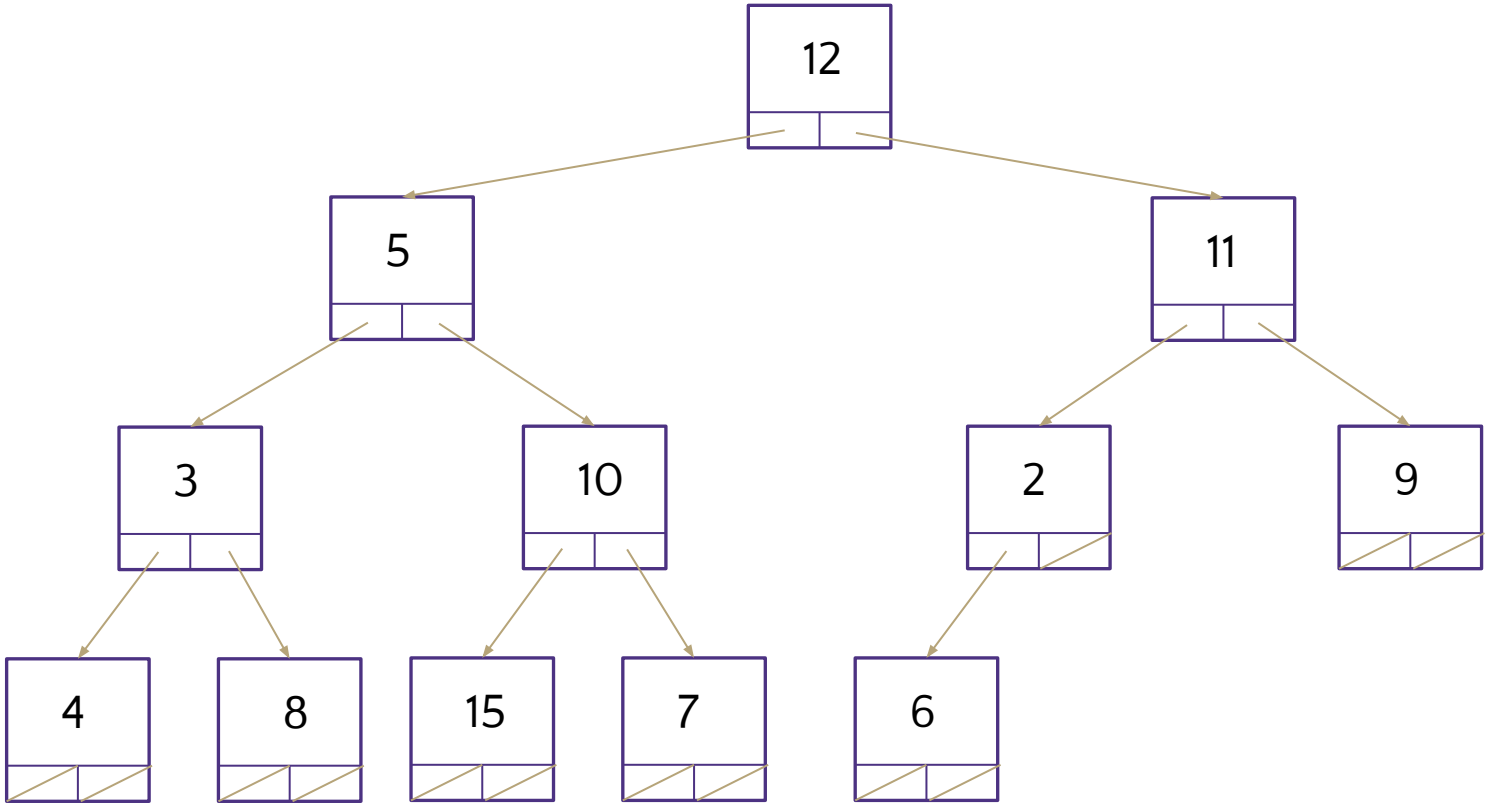
- Idea 2: Dump everything in the array, and percolate things down until the heap invariant is satisfied
  - Intuition: this could be faster!
  - The bottom two levels of the tree have $\Omega(n)$ nodes, the top two have 3 nodes
  - Maybe we can make "most of the nodes" go only a constant distance

Floyd's buildHeap algorithm

Build a tree with the values:
12, 5, 11, 3, 10, 2, 9, 4, 8, 15, 7, 6

1. Add all values to back of array

2. percolateDown(parent) starting at last index



0	1	2	3	4	5	6	7	8	9	10	11	12	13
12	5	11	3	10	2	9	4	8	15	7	6		

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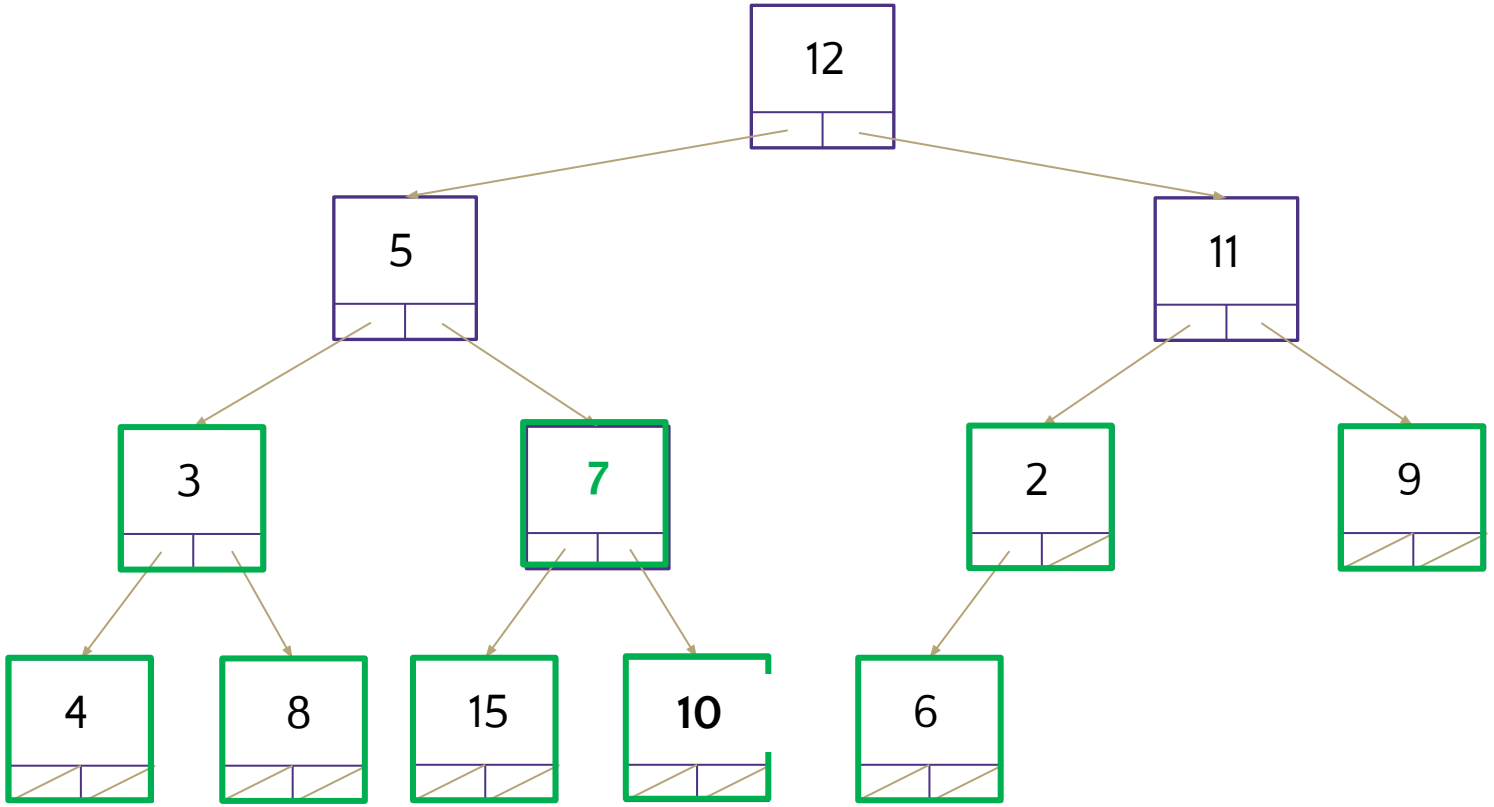
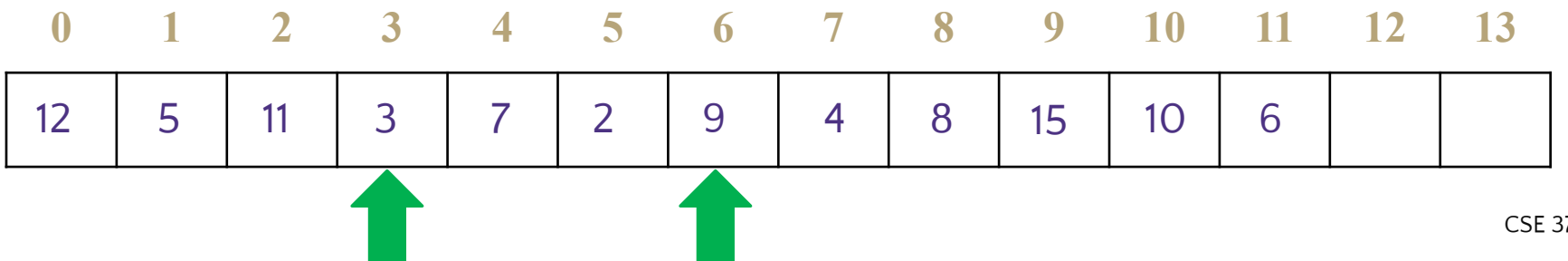
Build a tree with the values:
12, 5, 11, 3, 10, 2, 9, 4, 8, 15, 7, 6

1. Add all values to back of array
2. percolateDown(parent) starting at last index
   1. percolateDown level 4
   2. percolateDown level 3



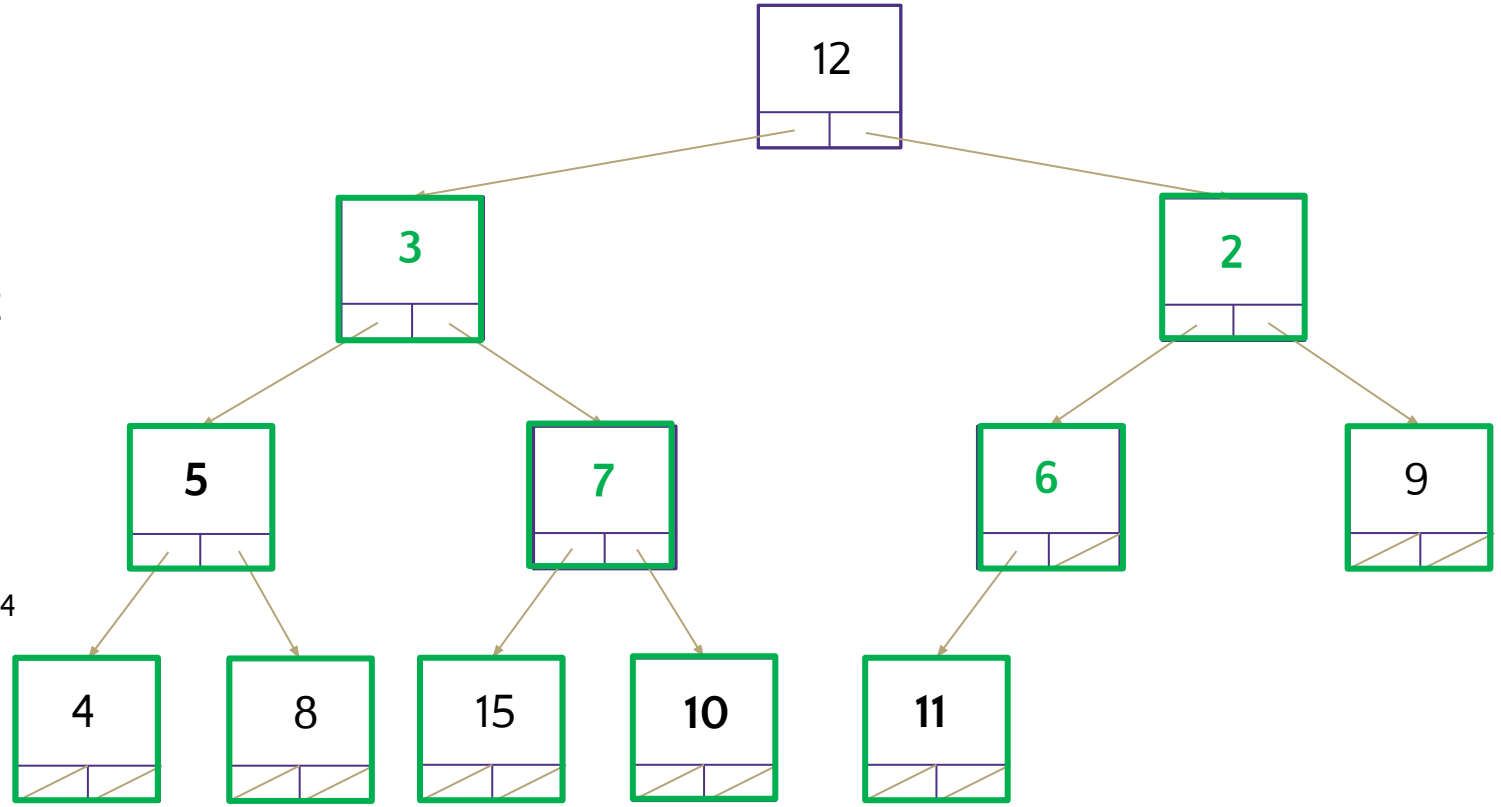


Floyd's buildHeap algorithm

Build a tree with the values:
12, 5, 11, 3, 10, 2, 9, 4, 8, 15,
7, 6

1. Add all values to back of array
2. percolateDown(parent) starting at last index
   1. percolateDown level 4
   2. percolateDown level 3
   3. percolateDown level 2

keep percolating down
like normal here and swap 5 and 4

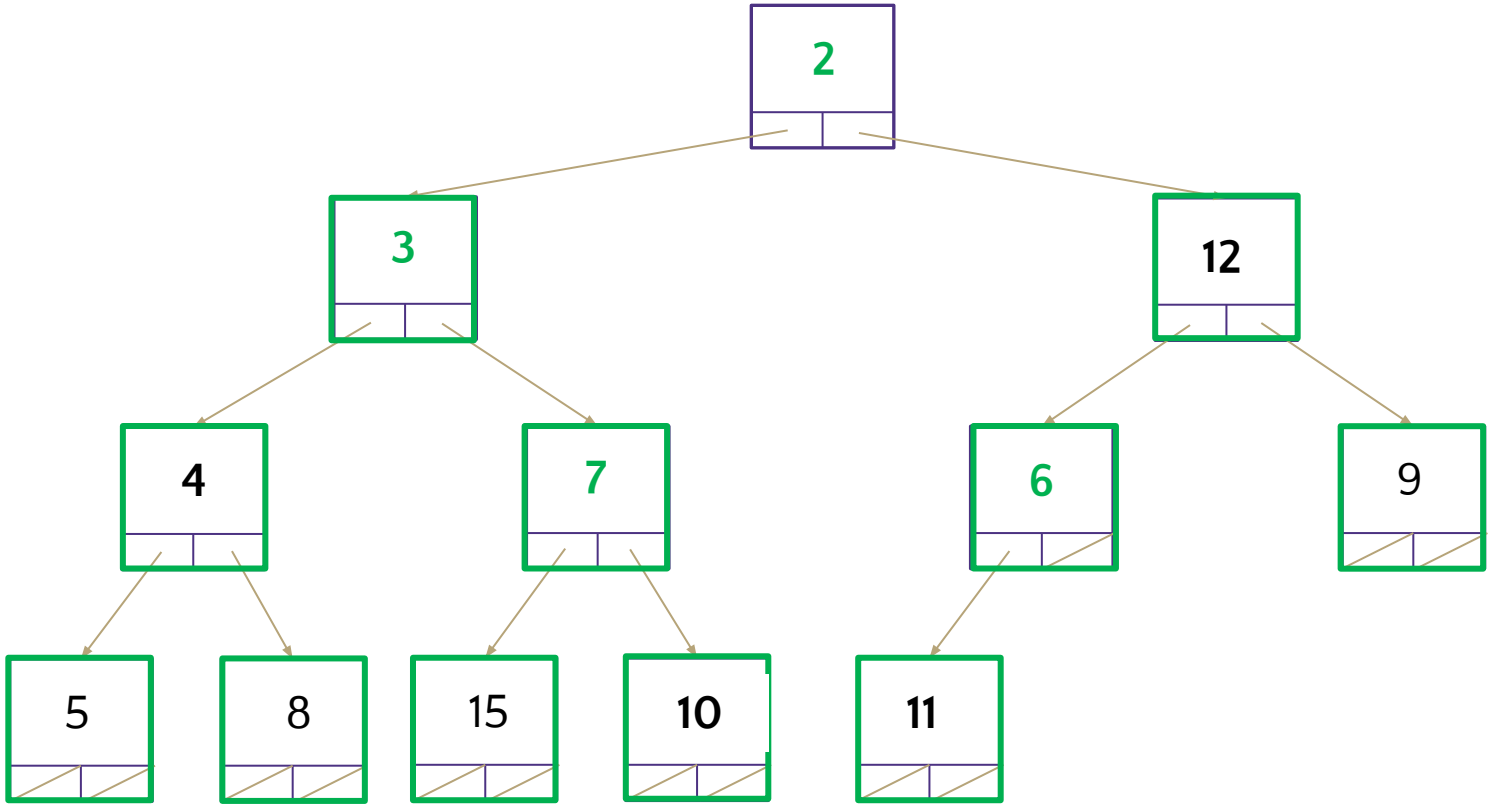


0	1	2	3	4	5	6	7	8	9	10	11	12	13
12	3	2	5	7	6	9	4	8	15	10	11		

Floyd's buildHeap algorithm

Build a tree with the values:
12, 5, 11, 3, 10, 2, 9, 4, 8, 15, 7, 6

1. Add all values to back of array
2. percolateDown(parent) starting at last index
    1. percolateDown level 4
    2. percolateDown level 3
    3. percolateDown level 2
    4. percolateDown level 1



0	1	2	3	4	5	6	7	8	9	10	11	12	13
12	3	2	4	7	6	9	5	8	15	10	11		

Floyd's buildHeap algorithm

Build a tree with the values:
12, 5, 11, 3, 10, 2, 9, 4, 8, 15, 7, 6

1. Add all values to back of array
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    1. percolateDown level 4
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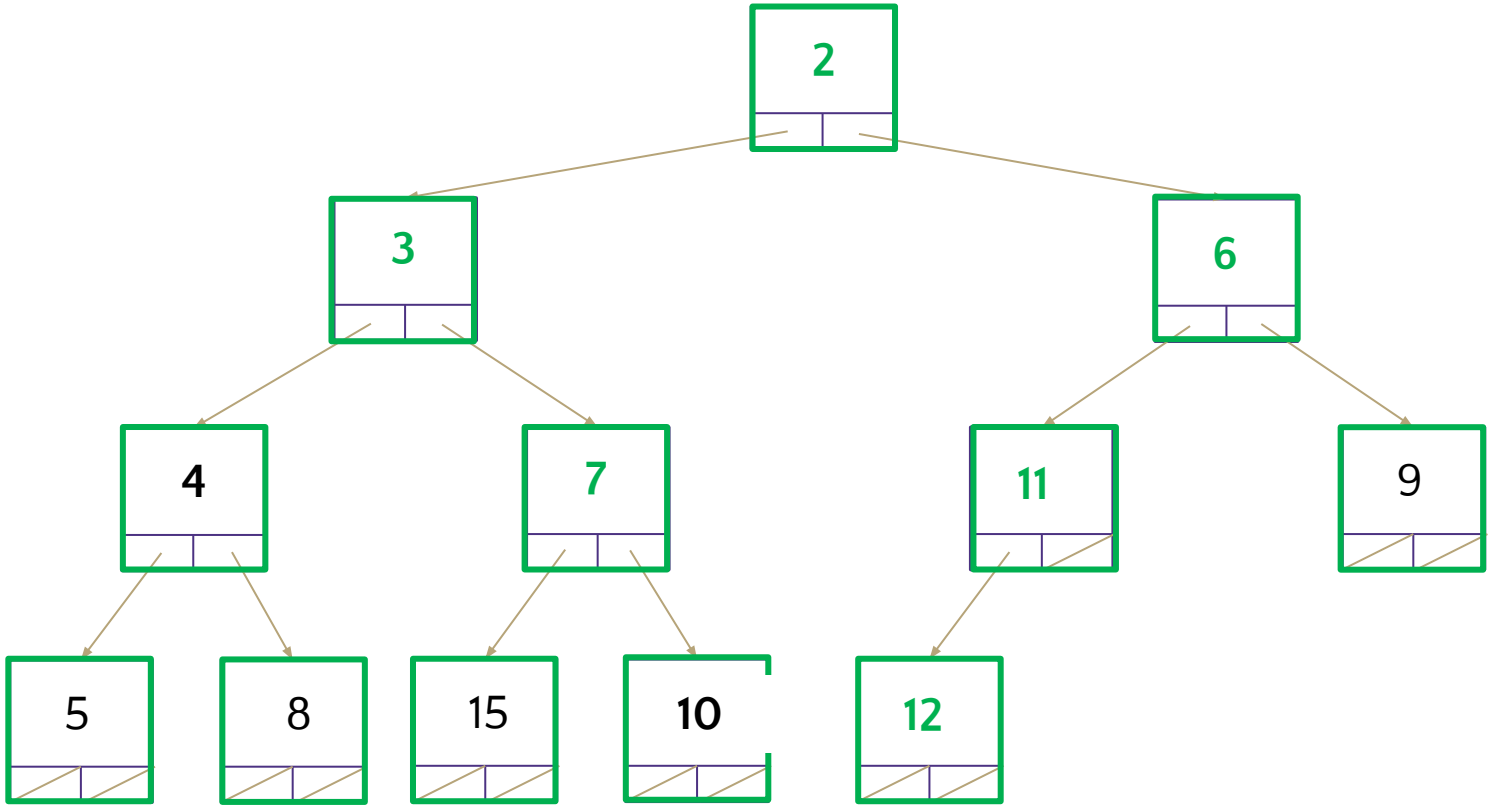

0	1	2	3	4	5	6	7	8	9	10	11	12	13
2	3	6	4	7	11	9	5	8	15	10	12		

Is It Really Faster? **Floyd's buildHeap runs in O(n) time!**

percolateDown() has worst case log n in general, but for most of these nodes, it has a much smaller worst case!

- n/2 nodes in the tree are leaves, have 0 levels to travel
- n/4 nodes have at most 1 level to travel
- n/8 nodes have at most 2 levels to travel
- etc…

$$\text{worst-case-work(n)} \approx \underbrace{\frac{n}{2} \cdot 1}_{\text{much of the work}} + \underbrace{\frac{n}{4} \cdot 2}_{\text{a little less}} + \underbrace{\frac{n}{8} \cdot 3}_{\text{a little less}} + \cdots + \underbrace{1 \cdot (\log n)}_{\text{barely anything}}$$

Intuition: Even though there are log n levels, each level does a smaller and smaller amount of work. Even with infinite levels, as we sum smaller and smaller values (think $\frac{1}{2^i}$) we converge to a constant factor of n.

Optional Slide Floyd's buildHeap Summation

- $n/2 \cdot 1 + n/4 \cdot 2 + n/8 \cdot 3 + \cdots + 1 \cdot (\log n)$

factor out n

work(n) $\approx n\left(\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\cdots+\frac{\log n}{n}\right)$ find a pattern -> powers of 2 work(n) $\approx n\left(\frac{1}{2^1}+\frac{2}{2^2}+\frac{3}{2^3}+\cdots+\frac{\log n}{2^{\log n}}\right)$ Summation!

$$work(n) \approx n\sum_{i=1}^{?}\frac{i}{2^i}$$? = upper limit should give last term

We don't have a summation for this! Let's make it look more like a summation we do know.

Infinite geometric series

$$work(n) \leq n\sum_{i=1}^{\log n}\frac{\left(\frac{3}{2}\right)^i}{2^i} \qquad if\ -1 < x < 1\ then \sum_{i=0}^{\infty} x^i = \frac{1}{1-x} = x \qquad work(n) \approx n\sum_{i=1}^{\log n}\frac{i}{2^i} \leq n\sum_{i=0}^{\infty}\left(\frac{3}{4}\right)^i = n * 4$$

Floyd's buildHeap runs in O(n) time!

Even More Operations

These operations will be useful in a few weeks…

IncreaseKey(element,priority) Given an element of the heap and a new, larger priority, update that object's priority.

DecreaseKey(element,priority) Given an element of the heap and a new, smaller priority, update that object's priority.

Delete(element) Given an element of the heap, remove that element.

Should just be going to the right spot and percolating…

Going to the right spot is the tricky part.

In the programming projects, you'll use a dictionary to find an element quickly.