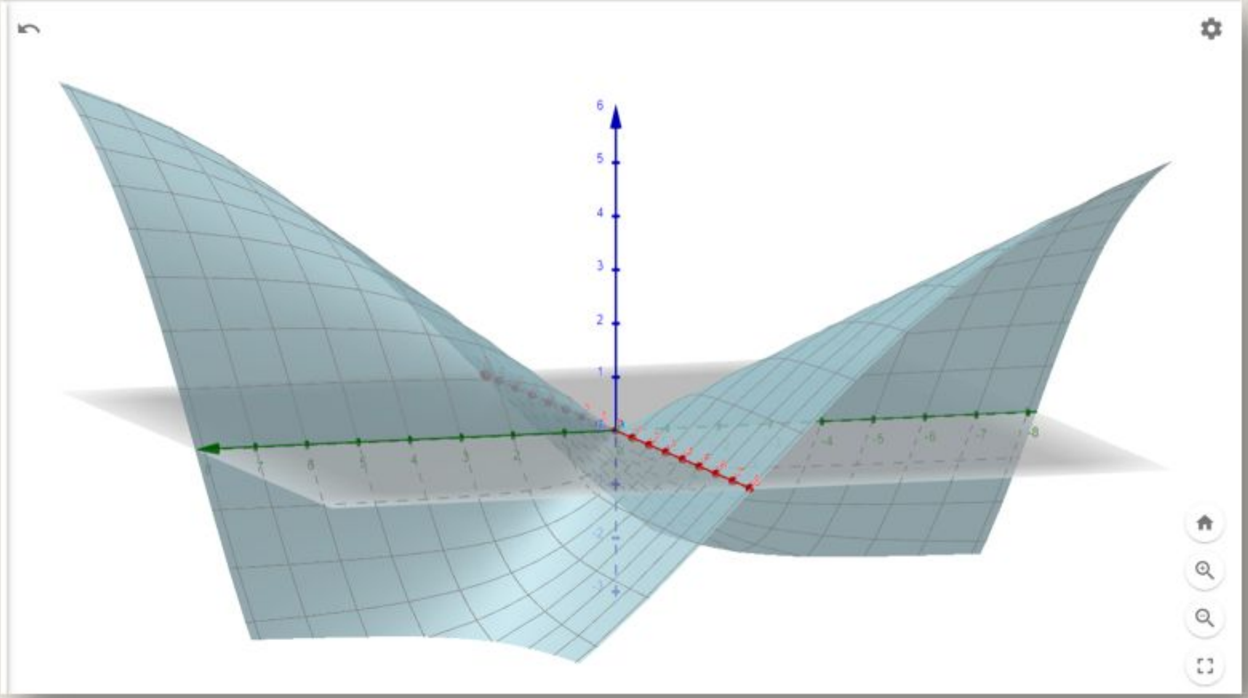
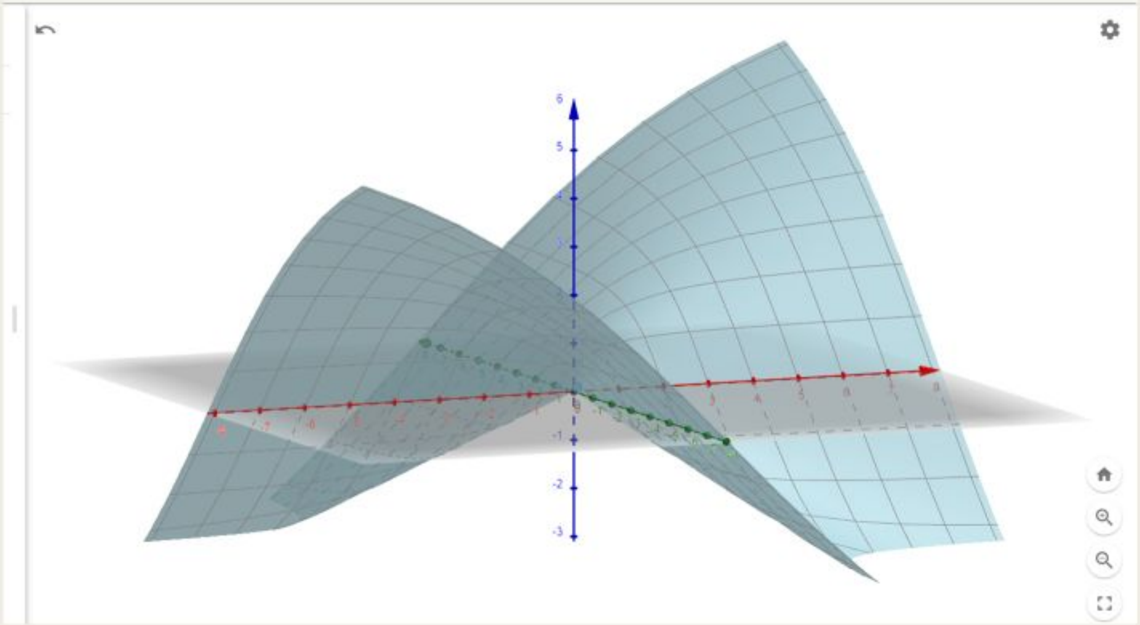
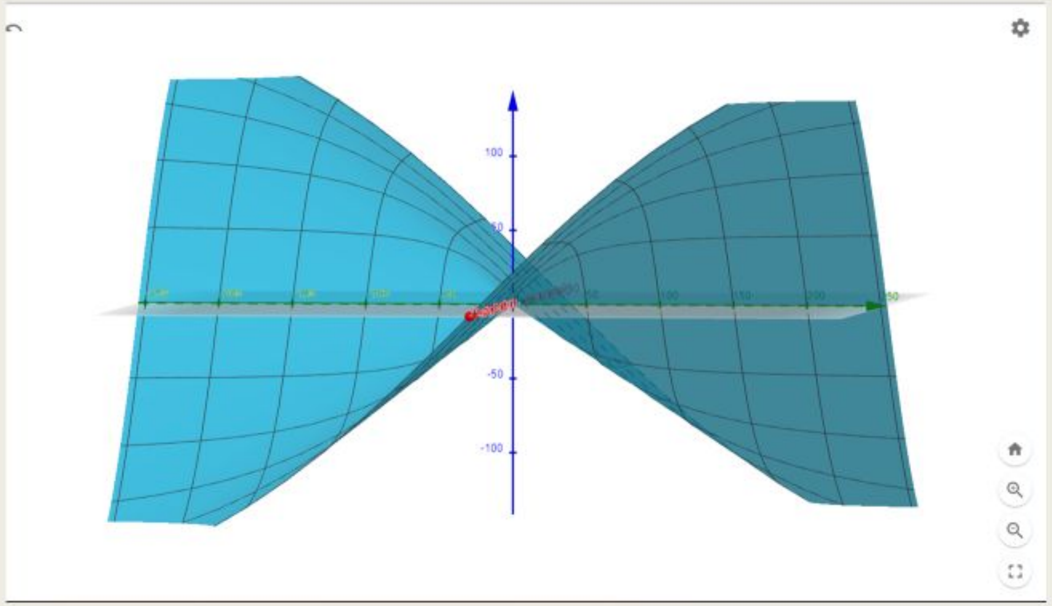
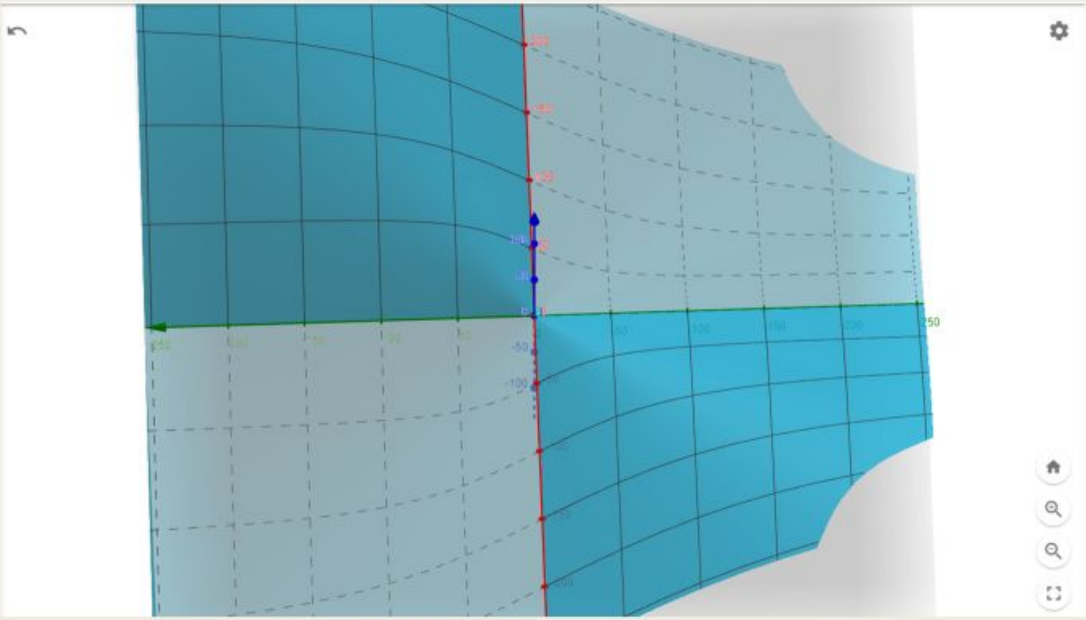
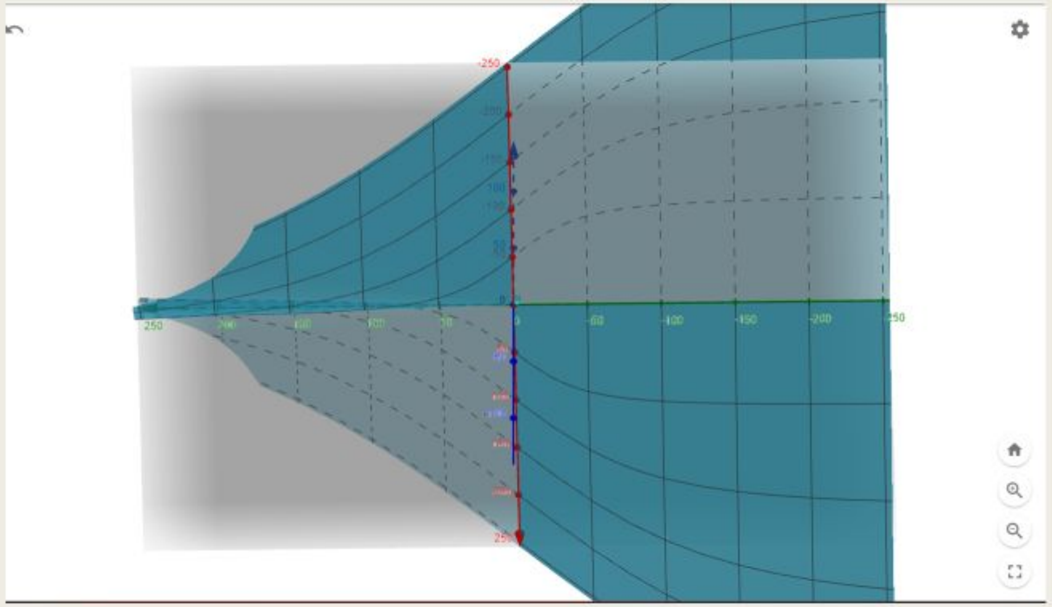
Limits, Continuity and differentiability
SEMESTER- I (BMG1CC1A)

We take the function is $f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

Now we check it's

- Limit
- Continuity
- differentiability



Limit:- Let $D \subseteq \mathbb{R}^2, f: D \to \mathbb{R}$ be a function and (a, b) be a limit point of D. Let $(x, y) \in$ D and $(x, y) \to (a, b)$ in any manner. f is said to tend to the limit A if to each $\varepsilon(> 0)$ there is a $\delta(> 0)$ such that

$|f(x, y) - \mathrm{A}| < \varepsilon$ whenever $(x, y) \in \mathrm{N}_\delta(a, b), a\ \delta -\ nbd$ of (a, b).

We may choose $\mathrm{N}_\delta(a, b) = \{(x, y)/0 < |x - a| < \delta, 0 < |y - b| < \delta\}$
The $\lim_{(x,y)\to(a,b)} f(x, y) = \mathrm{A}$ is called **double limit or simultaneous limit**.

Find the limit of $f(x,y) = \frac{xy}{\sqrt{x^2+y^2}}$

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = \left|\frac{xy}{\sqrt{x^2+y^2}} - 0\right|$$

$$= \frac{|x||y|}{\sqrt{x^2+y^2}}$$

$$\leq \frac{\sqrt{x^2+y^2}.\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}$$

$$= \sqrt{x^2 + y^2} < \varepsilon \text{ ,if } x^2 + y^2 < \delta^2 (= \varepsilon^2)$$

Iterated or Repeated limit:-

Let N be a certain nbd of (a, b) and $f(x, y)$ be defined on N. For a fixed value of y, $\lim_{x\to a} f(x, y)$, if exists, will involve y.

$\therefore$ $\lim_{x\to a} f(x, y)$ will be different for different values of y and thus $\lim_{x\to a} f(x, y)$ will be a function of y.

Let $\lim_{x\to a} f(x, y) = \phi(y)$

If then, $\lim_{y\to b} \phi(y)$ exists and is equal to A, we write

$$\lim_{y\to b} \left\{\lim_{x\to a} f(x, y)\right\} = A \qquad \text{---------- (1)}$$

We change the order of obtaining limit. Keeping x fixed, if $\lim_{y\to b} f(x, y)$ exists it will be a function of x say $\Psi(x)$. If $\lim_{x\to a} \Psi(x)$ exists and is equal to B we write

$$\lim_{x\to a} \left\{\lim_{y\to b} f(x, y)\right\} = \mathrm{B} \qquad \text{---------- (2)}$$

This limits given in (1) and (2) are called **iterated or repeated limits.**

Here for $f(x,y) = \frac{xy}{\sqrt{x^2+y^2}}$

$$\lim_{y\to 0} \left\{\lim_{x\to 0} f(x, y)\right\} = \lim_{y\to 0} \left\{\lim_{x\to 0} \frac{xy}{\sqrt{x^2+y^2}}\right\} = 0 \,,$$

$$similarly\,, \qquad \lim_{x\to 0} \left\{\lim_{y\to 0} f(x, y)\right\} = \lim_{x\to 0} \left\{\lim_{y\to 0} \frac{xy}{\sqrt{x^2+y^2}}\right\} = 0$$

f IS CONTINUOUS AT (0,0)

Continuity :-Let D be a non-empty subset of $\mathbb{R}^2$ and $f: \mathrm{D} \to \mathbb{R}$ be a function. Let $(a, b) \in \mathrm{D}$. We say that f is continuous at (a, b) if given $\varepsilon > 0, \exists \delta > 0$ such that $|f(x, y) - f(a, b)| < \varepsilon \; \forall x \; \mathrm{N}_{\delta}(a, b) \cap D$ In other words, f is continuous at $(a, b) \in D$ if $\lim_{(h,k)\to(0,0)} f(a + h, b + k) = f(a, b)$ where $(a + h, b + k) \in \mathrm{D}$. f is called continuous on D if f is cont. at every point of D.

Note : If $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist, the question of continuity of f at (a, b) does not arise.

Theorem :- If $f(x, y)$ is continuous at (a, b) and $f(a, b) \neq 0$ then there exists a neighbourhood of (a, b) where $f(x, y)$ and $f(a, b)$ maintain the same sign.

Theorem :- If $f(x, y)$ be continuous at (a, b), then the functions $f(x, b)$ and $f(a, y)$ are continuous at $x = a$ and $y = b$ respectively.

Prove that the function

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$ Is continuous at (0,0)

Let $\varepsilon > 0$.

Now $|f(x,y) - f(0,0)| = \left|\frac{xy}{\sqrt{x^2+y^2}} - 0\right|$

$$= \left|\frac{xy}{\sqrt{x^2+y^2}}\right| = \frac{|x||y|}{\sqrt{x^2+y^2}} \leq \frac{\sqrt{x^2+y^2}\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}$$

$$= \sqrt{x^2+y^2} < \varepsilon$$

if $x^2 + y^2 < \delta^2$ where $\delta = \varepsilon$

Thus for chosen $\varepsilon > 0, \exists$ a positive δ such that

$|f(x,y) - f(0,0)| < \varepsilon$ whenever $x^2 + y^2 < \delta^2$

This proves that f is continuous at (0,0).

PARTIAL DIFFERENTIABILITY OF f AT (0,0)

Let f be a function of two independent variables x and y defined over a domain $\mathrm{D} \subset \mathbb{R}^2$ and $\mathrm{P}(a, b) \in \mathrm{D}$. Let N be a *nbd* of (a, b) such that $\mathrm{N} \subset \mathrm{D}$ and $\mathrm{Q}(a + h, b + k) \in \mathrm{N}$. If α be the angle that the line joining $\overrightarrow{PQ}$ makes with the positive direction of x-axis, then the direction cosines of $\overrightarrow{PQ}$ are
$l = \cos\alpha$ and $m = \sin\alpha$.

If $\rho = \sqrt{h^2 + k^2}$

Then $h = \rho\cos\alpha, k = \rho\sin\alpha$, and $\rho \to 0$ as $\mathrm{Q} \to \mathrm{P}$.

$\lim_{\rho\to 0} \frac{f(a+\rho\cos\alpha, b+\rho\sin\alpha) - f(a,b)}{\rho}$, if exists, is called the derivative of $f(x, y)$ at (a, b) in the direction α and is denoted by $\mathrm{D}_\alpha f(a, b)$. If $\alpha = 0$ the derivative is denoted by $\frac{\partial f(a,b)}{\partial x}$ and if $\alpha = \frac{\pi}{2}$, the derivative is $\frac{\partial f(a,b)}{\partial y}$, and these derivatives are called the **Partial derivatives** of $f(x, y)$ at (a, b) with respect to x and y respectively.

Thus, $\lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$, if exists, is $\frac{\partial f}{\partial x}$ or, f_x and $\lim_{k\to 0} \frac{f(x,y+k) - f(x,y)}{k}$, if exists is $\frac{\partial f}{\partial y}$ or, f_{y^*}

- [ ] Find the partial derivative of $f(x,y) = \frac{xy}{\sqrt{x^2+y^2}}$ at $(0,0)$

Here f(x,y) is continuous at (0,0)

Now we know that the partial derivative of f(x,y) with respect to x is

$$f_x \text{ or } \frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

At (0,0), $f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h}$

$$= \lim_{h\to 0} \frac{0-0}{h} = 0$$

we know that the partial derivative of f(x , y) with respect to y is

$$f_y \text{ or } \frac{\partial f}{\partial y} = \lim_{k\to 0} \frac{f(x,y+k) - f(x,y)}{k}$$

$$\text{At } (0,0),\ f_y(0,0) = \lim_{k \to 0} \frac{f(0,0+k) - f(0,0)}{k}$$

$$= \lim_{k \to 0} \frac{0-0}{k} = 0$$

Since $f_x(0,0) = f_y(0,0) = 0$ therefore partial derivative exist at (0,0)

DIFFERENTIABILITY OF f AT (0,0)

Let D $\subseteq \mathbb{R} \times \mathbb{R}$ be the domian of definition of $f(x, y)$, a function of two independent variables x and y

Let $(x, y) \in$ D be an interior point of D and N be a nbd of (x, y) lying in D.Let again,$(x + h, y + k) \in$ N.

Then $\Delta f = f(x + h, y + k) - f(x, y)$ is called the increment of $f(x, y)$ at (x, y) Let $\rho = \sqrt{h^2 + k^2}$, f is

said to be differentiable at (a, b) if we can express Δf at (a, b) in the form $\Delta f = \mathrm{Ah} + \mathrm{Bk} + \varepsilon\rho$ where

A and B are independent of h, k and $\varepsilon \to 0$ as $\rho \to 0$ $\mathrm{A}h + \mathrm{B}k$ is called the Total Differential of f at

(a, b) denoted by df, and $\varepsilon\rho$ is called the error in taking df in place of Δf.

Thus, if a function f is differentiable at any point $(x, y) \in$ D

$\varepsilon = \frac{\Delta f - df}{\rho}$ should tend to zero as $(h, k) \to (0,0)$.

❑ Check the differentiability of $f(x,y) = \frac{xy}{\sqrt{x^2+y^2}}$ at (0,0)

partial derivatives at (0,0)

$$\lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{0}{h} - 0}{h} = \lim_{h\to 0} \frac{0}{h} = 0.$$

∴ The partial derivative $f_x(0,0)$ exists and is equal to 0 .

Similarly, $\lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = 0.$ ∴ $f_y(0,0) = 0.$

Supposing that the function is differentiable at (0,0) we can write

$$f(0+h, 0+k) - f(0,0) = hf_x(0,0) + kf_y(0,0) + \varepsilon\sqrt{h^2+k^2}$$

where ε should tend to zero as $(h,k) \to (0,0)$.

Now, since $f(0,0) = 0 f(h,k) = h.0 + k.0 + \varepsilon\sqrt{h^2 + k^2} \Rightarrow \varepsilon = \frac{\text{[illegible]}}{h^2+k^2}$ which does not tend to a unique limit as $(h,k) \to (0,0)$.

In fact, if $(h,k) \to (0,0)$ along $k = mh$ we see that $\lim_{(h,k)\to(0,0)} \varepsilon = \lim_{(h,k)\to(0,0)} \frac{hk}{h^2+k^2} = \lim_{h\to 0} \frac{mh^2}{h^2(1+m^2)} =$

$\frac{m}{1+m^2}$, a function of m.

$\therefore \lim_{(h,k)\to(0,0)} \varepsilon$ dose not exist . Hence f(x,y) is not differentiable at (0,0)