Note

The live version of this lecture used the term isometry many times. A better term is “one to one correspondence” or “bijection”. I’ve updated the slides accordingly.

- Unimportant technical note: The “isometry” is a consequence of the bijection. Essentially, because of the bijection, we guarantee that distances from roots to leaves are the same in both the 2-3 case and the LLRB case, hence the term “isometric”, meaning approximately “of equal measure”. “Isometry” was an unnecessarily technical term that obfuscated what was really important. Don’t worry about this small paragraph if you don’t know what I mean.

I was just asked to balance a Binary Search Tree by JFK's airport immigration. Welcome to America.

RETWEETS 8,772 LIKES 7,520

8:26 AM - 26 Feb 2017 from Manhattan, NY

CS61B

Lecture 18: Balanced Search Trees

- Tree Rotation
- Left Leaning Red-Black Trees
- Maintaining Correspondence Through Rotation

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The Bad News

2-3 trees (and 2-3-4 trees) are a real pain to implement, and suffer from performance problems. Issues include:

- Maintaining different node types.
- Interconversion of nodes between 2-nodes and 3-nodes.
- Walking up the tree to split nodes.

fantasy 2-3 code via Kevin Wayne

```
public void put(Key key, Value val) {
   Node x = root;
   while (x.getTheCorrectChildKey(key) != null) {
      x = x.getTheCorrectChildKey();
      if (x.is4Node()) { x.split(); }
   }
   if (x.is2Node()) { x.make3Node(key, val); }
   if (x.is3Node()) { x.make4Node(key, val); }
}
```

“Beautiful algorithms are, unfortunately, not always the most useful.” - Knuth

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BST Structure and Tree Rotation

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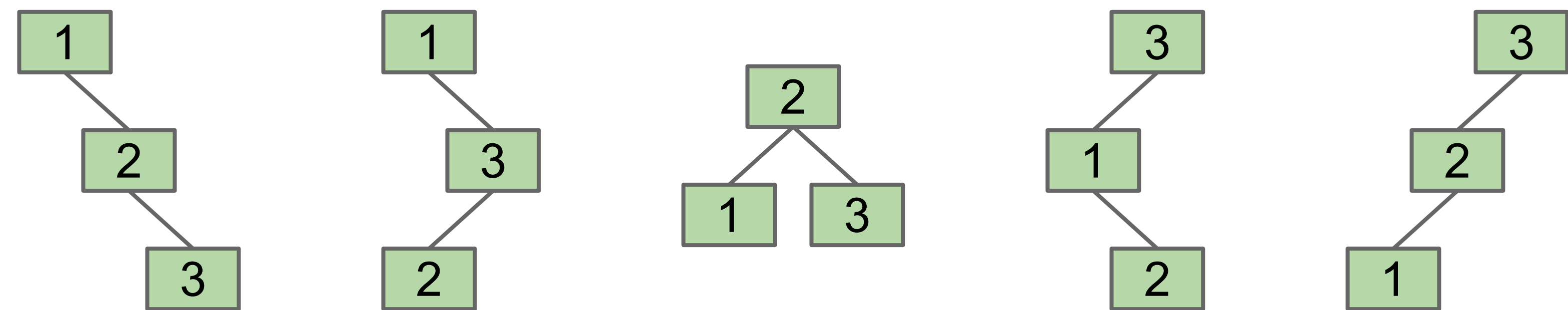
BSTs

Suppose we have a BST with the numbers 1, 2, 3. Five possible BSTs.

- The specific BST you get is based on the insertion order.
- More generally, for N items, there are Catalan(N) different BSTs.



Given any BST, it is possible to move to a different configuration using “rotation”.

- In general, can move from any configuration to any other in 2n - 6 rotations (see Rotation Distance, Triangulations, and Hyperbolic Geometry or Amy Liu).

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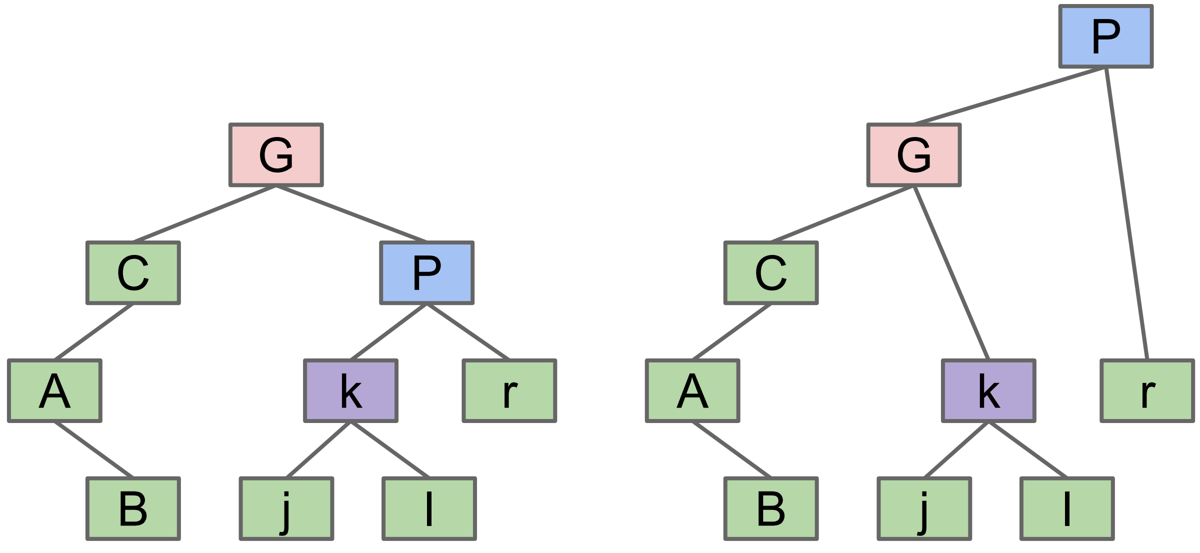
Tree Rotation Definition

rotateLeft(G): Let x be the right child of G. Make G the **new left child** of x.

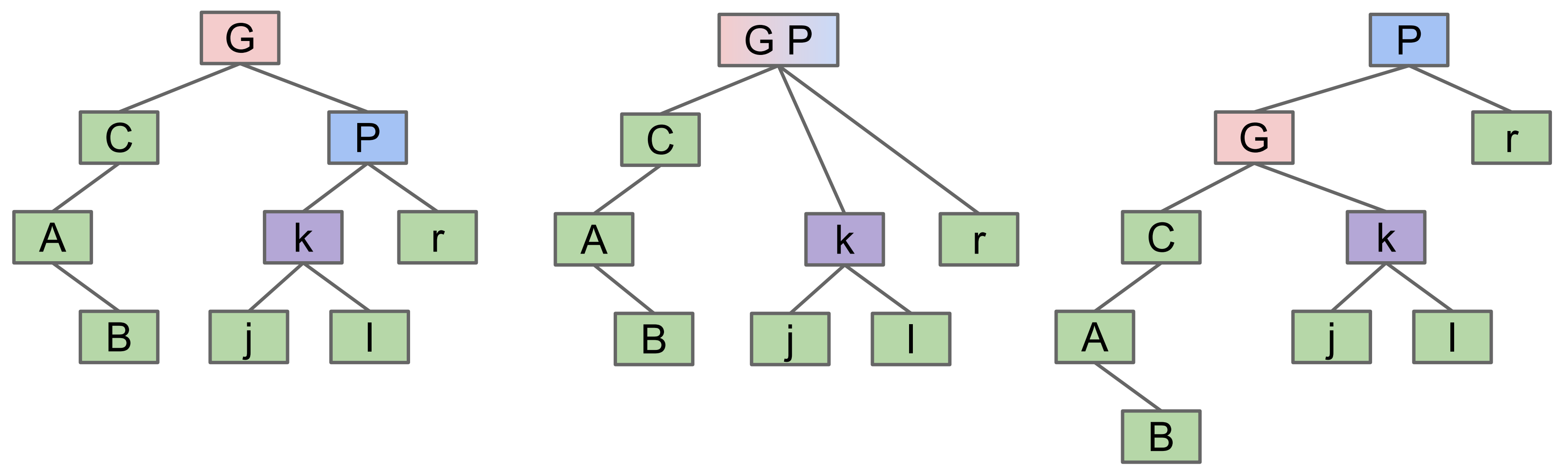
- Preserves search tree property. No change to semantics of tree.



Tree Rotation Definition

rotateLeft(G): Let x be the right child of G. Make G the **new left child** of x.

- Can think of as temporarily merging G and P, then sending G down and **left**.
- Preserves search tree property. No change to semantics of tree.

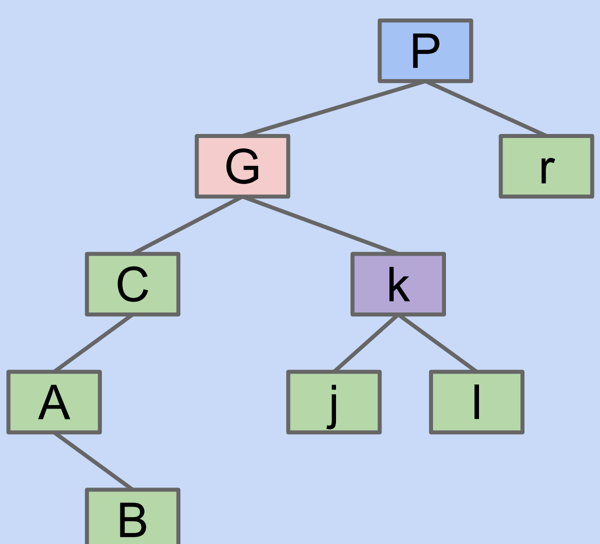


For this example rotateLeft(G) increased height of tree!

Your Turn

rotateRight(P): Let x be the left child of P. Make P the **new right child** of x.

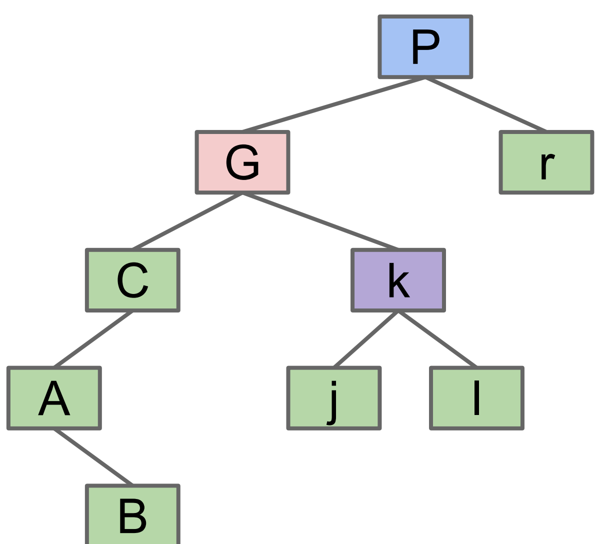
- Can think of as temporarily merging G and P, then sending P down and **right**.



Your Turn

rotateRight(P): Let x be the left child of P. Make P the **new right child** of x.

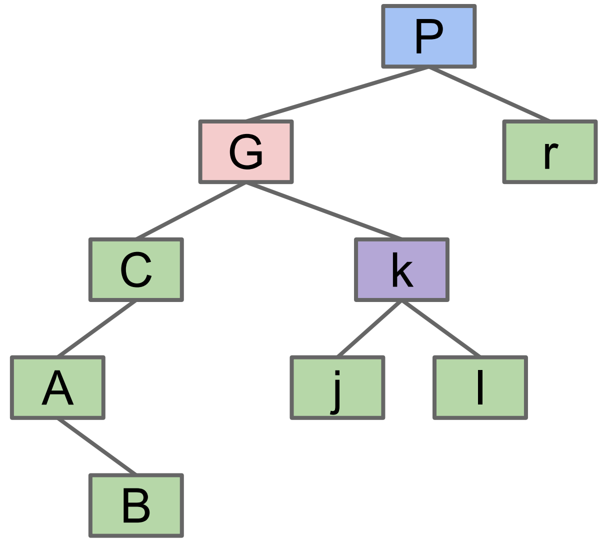
- Can think of as temporarily merging G and P, then sending P down and **right**.

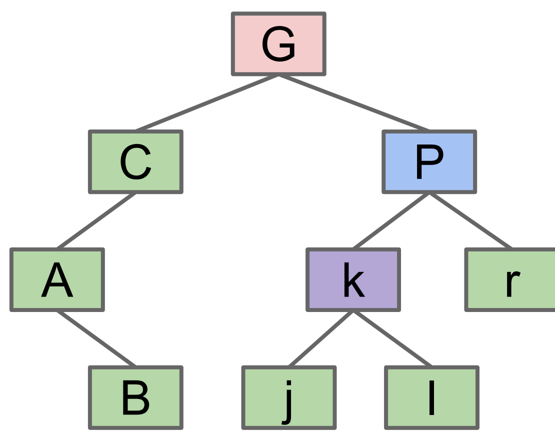


Your Turn

rotateRight(P): Let x be the left child of P. Make P the **new right child** of x.

- Can think of as temporarily merging G and P, then sending P down and **right**.
- Note: k was G's right child. Now it is P's left child.

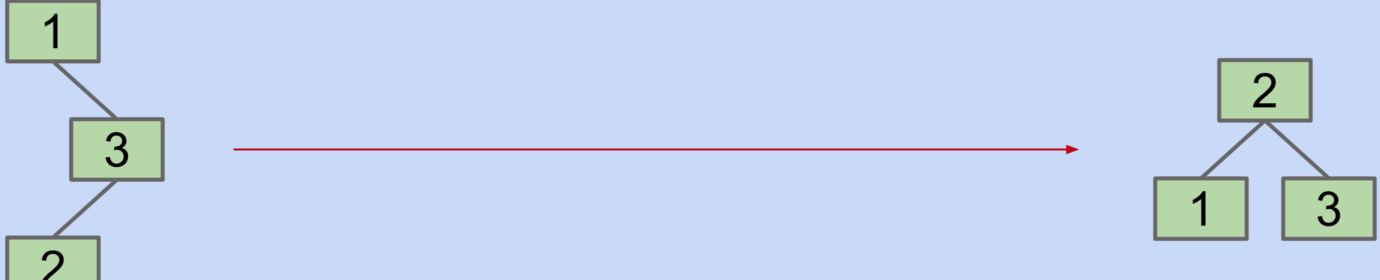




For this example rotateRight(P) decreased height of tree!

BSTs

Give a sequence of rotation operations that balances the tree on the left.



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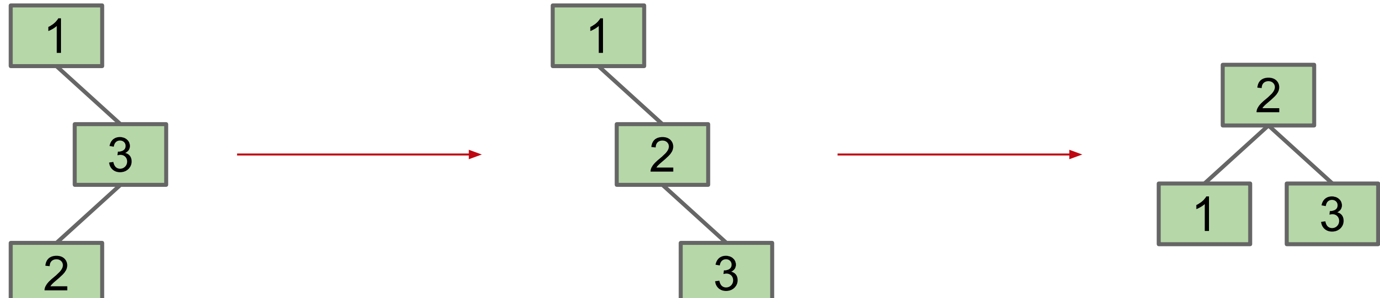
BSTs

Give a sequence of rotation operations that balances the tree on the left.

- rotateRight(3)
- rotateLeft(1)



There are other correct answers as well!

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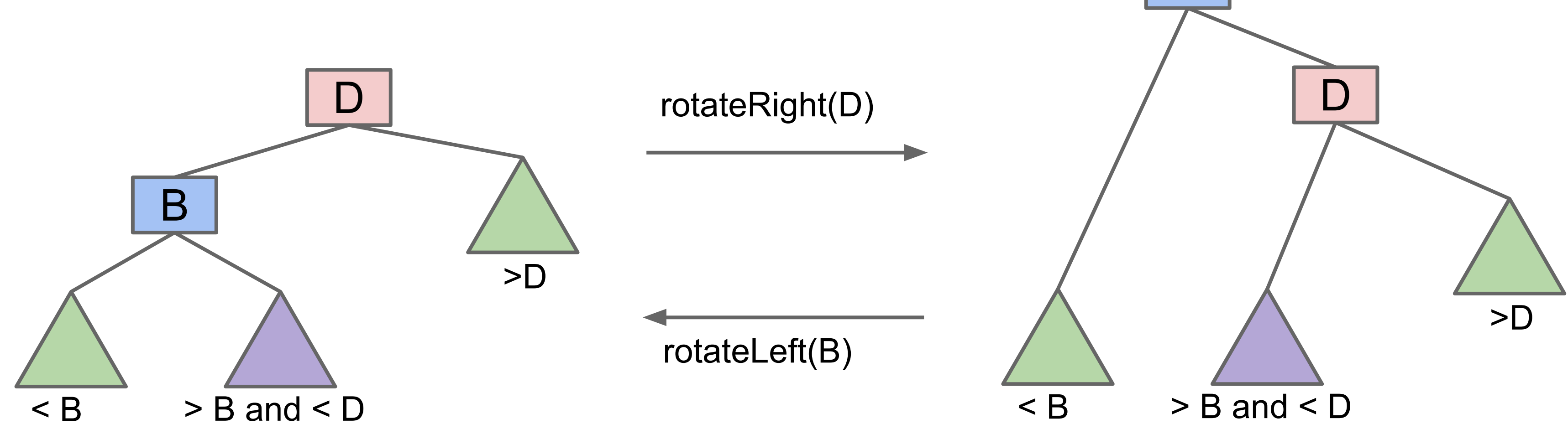
Rotation for Balance

Rotation:

- Can shorten (or lengthen) a tree.
- Preserves search tree property.



Rotation for Balance

Rotation:

- Can shorten (or lengthen) a tree.
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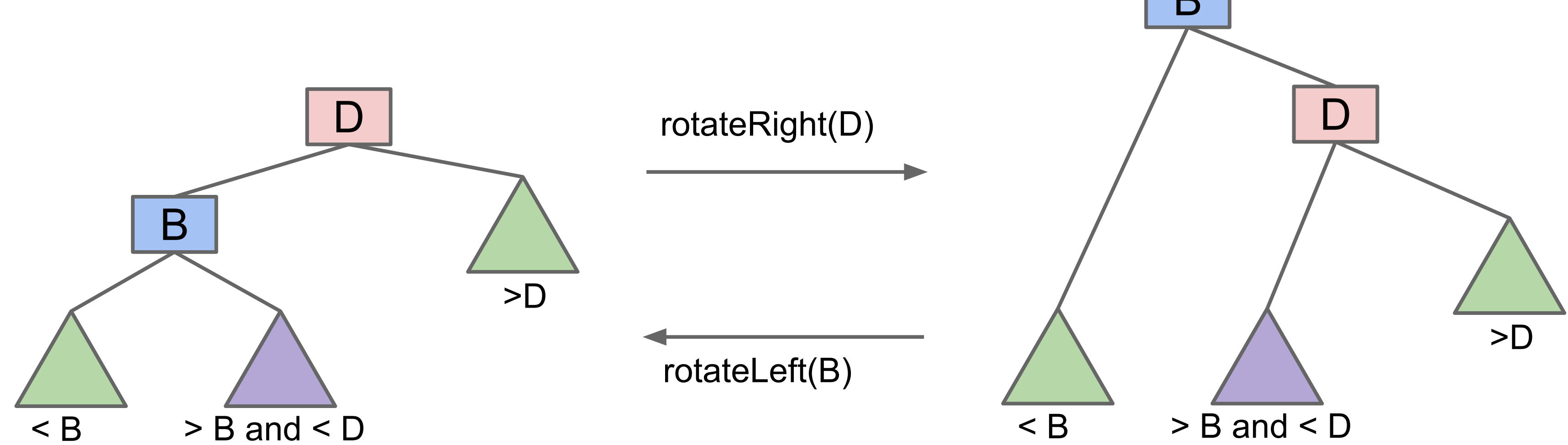


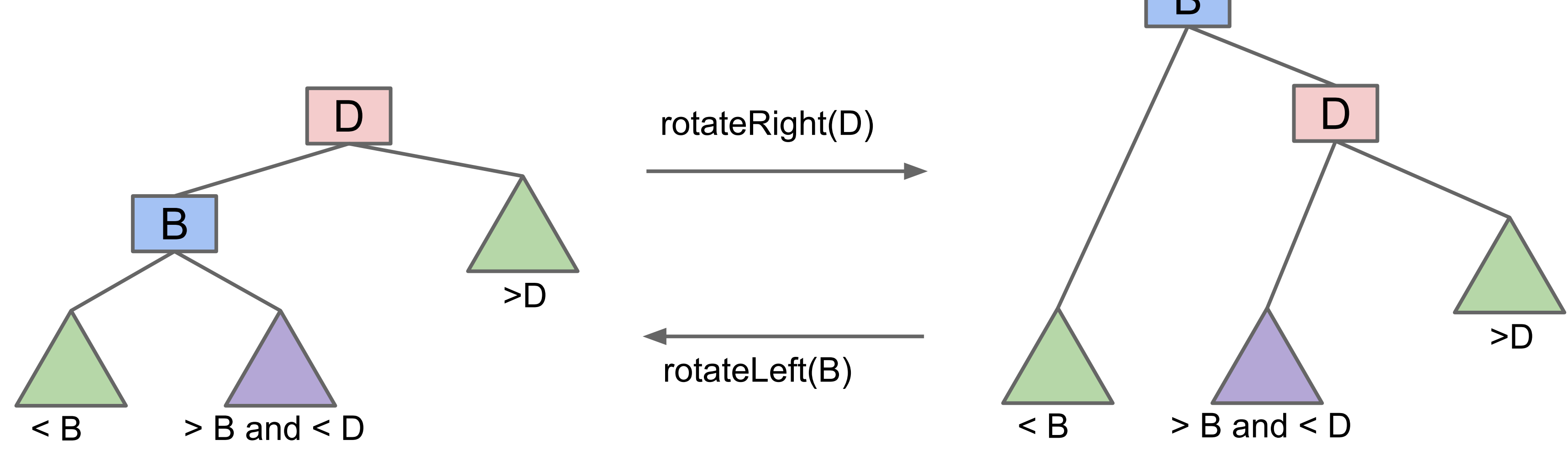
Can use rotation to balance a BST: Demo

- Rotation allows balancing of a BST in O(N) moves.

Rotation: An Alternate Approach to Balance

Rotation:

- Can shorten (or lengthen) a tree.
- Preserves search tree property.



Paying O(n) to occasionally balance a tree is not ideal. In this lecture, we'll see a better way to achieve balance through rotation. But first...

Red-Black Trees

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Search Trees

There are many types of search trees:

- **Binary search trees**: Can balance using rotation, but we have no algorithm for doing so (yet).
- **2-3 trees**: Balanced by construction, i.e. no rotations required.

Let's try something clever, but strange.

Our goal: Build a BST that is structurally identical to a 2-3 tree.

- Since 2-3 trees are balanced, so will our special BSTs.

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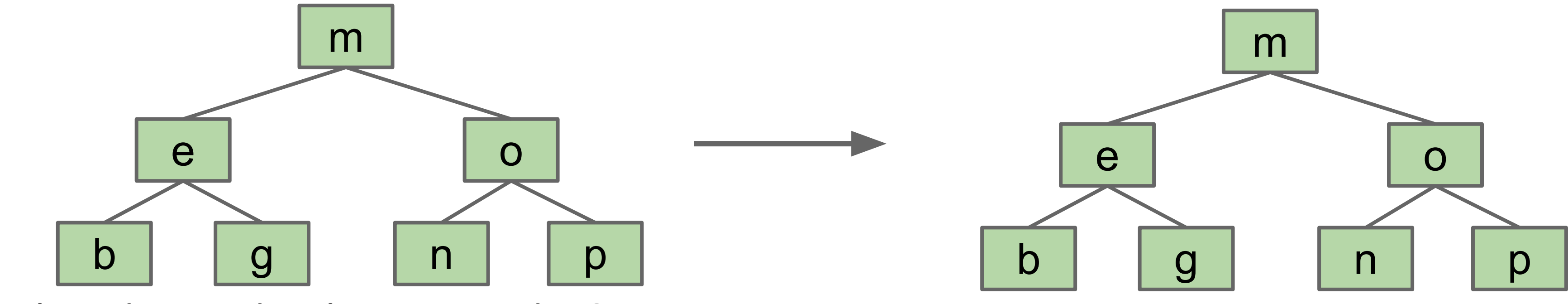
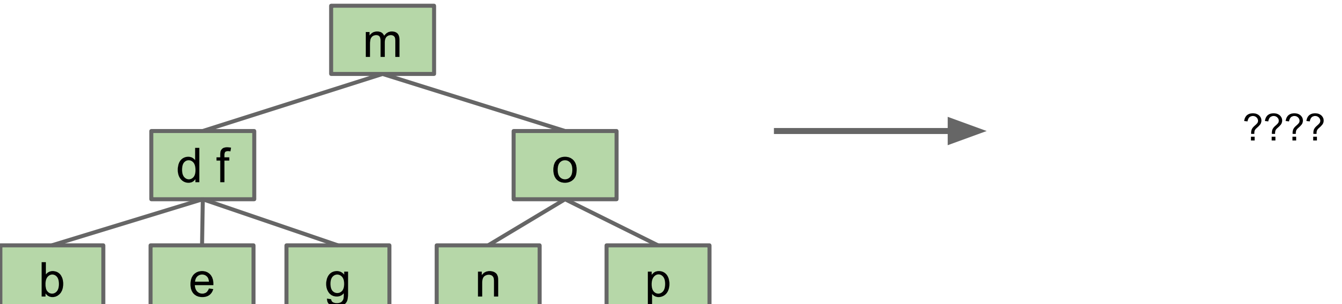
Representing a 2-3 Tree as a BST

A 2-3 tree with only 2-nodes is trivial.

- BST is exactly the same!



What do we do about 3-nodes?



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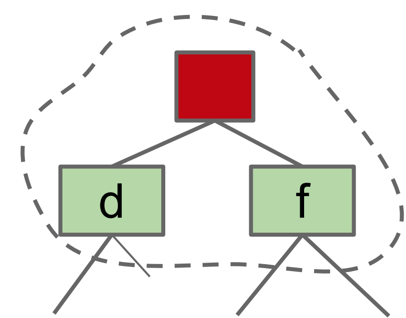
Representing a 2-3 Tree as a BST: Dealing with 3-Nodes

Possibility 1: Create dummy “glue” nodes.

m
d f
o
b
e
g
n
p

m
o
d
f
n
p
b
e
g

Result is inelegant. Wasted link. Code will be ugly.



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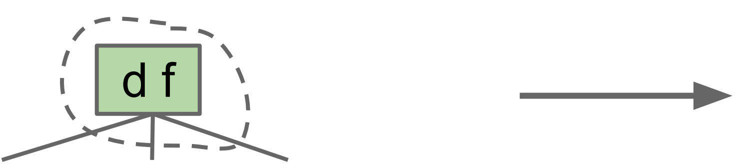
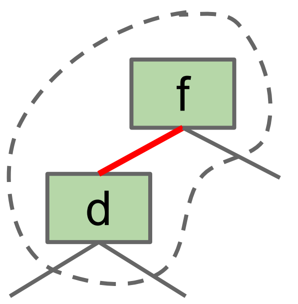
Representing a 2-3 Tree as a BST: Dealing with 3-Nodes

Possibility 2: Create "glue" links with the smaller item **off to the left**.

Idea is commonly used in practice (e.g. java.util.TreeSet).





For convenience, we'll mark glue links as "**red**".

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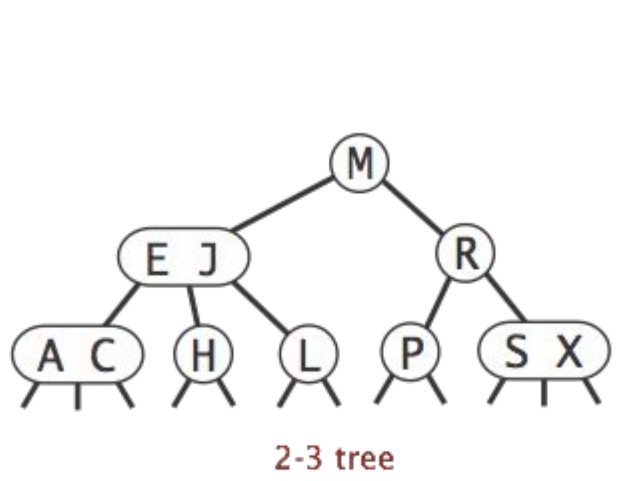
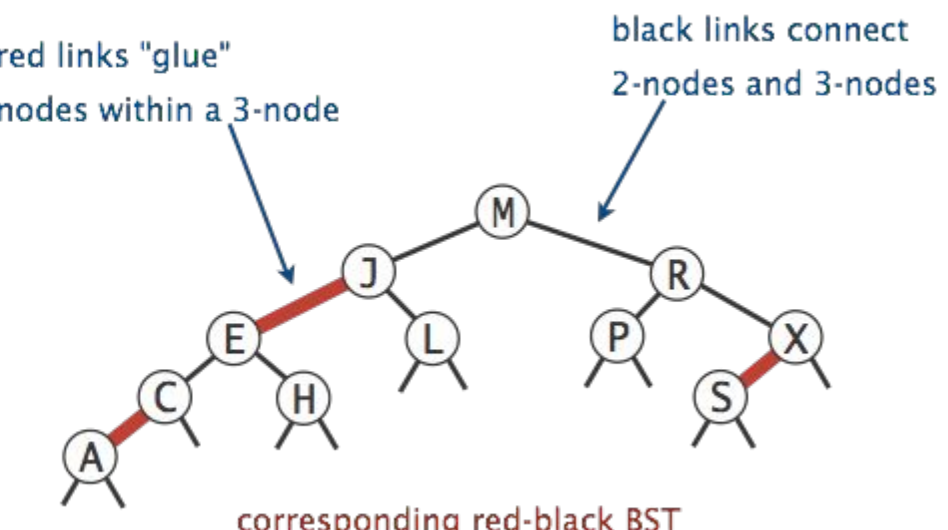
Left-Leaning Red Black Binary Search Tree (LLRB)

A BST with left glue links that represents a 2-3 tree is often called a “Left Leaning Red Black Binary Search Tree” or LLRB.

- LLRBs are normal BSTs!
- There is a 1-1 correspondence between an LLRB and an equivalent 2-3 tree.
- The red is just a convenient fiction. Red links don’t “do” anything special.





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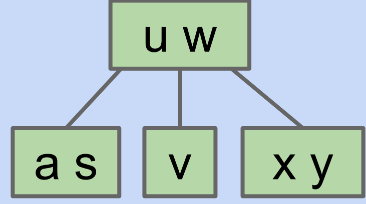
Left-Leaning Red Black Binary Search Tree (LLRB)

Draw the LLRB corresponding to the 2-3 tree shown below.



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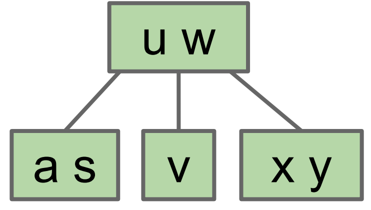

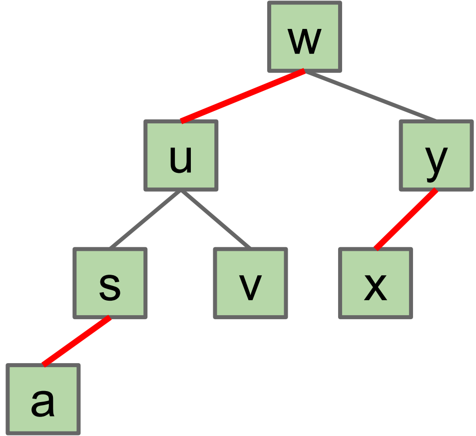

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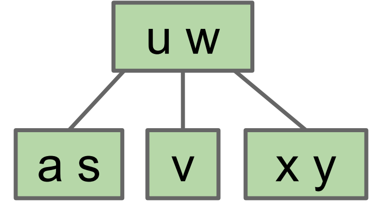
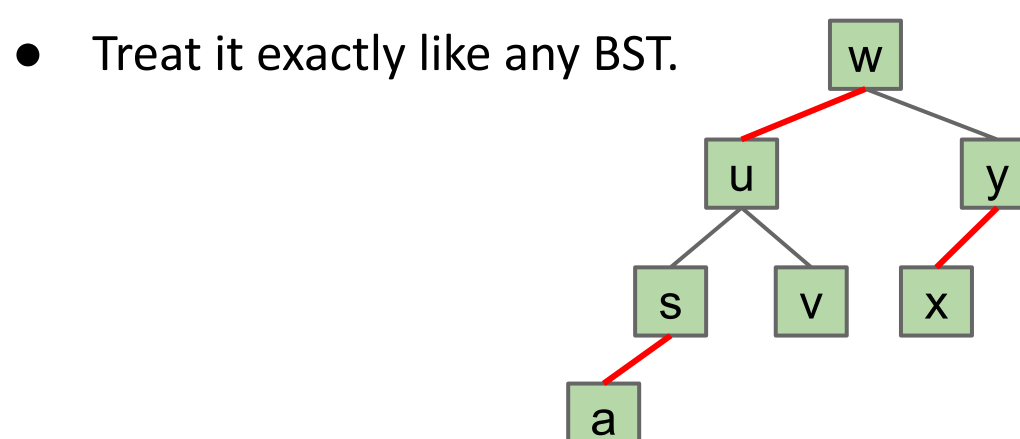
Left-Leaning Red Black Binary Search Tree (LLRB)

Draw the LLRB corresponding to the 2-3 tree shown below.



Searching an LLRB tree for a key is easy.

- Treat it exactly like any BST.



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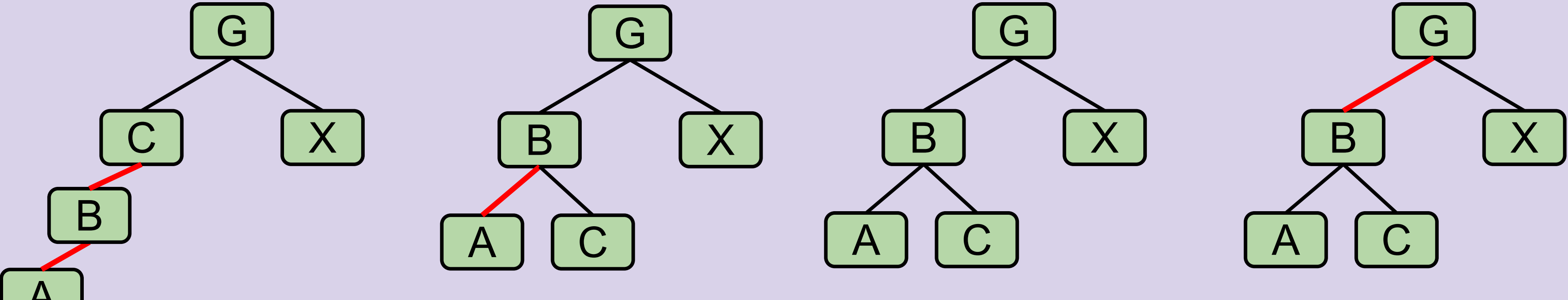
LLRB Problem #1: yellkey.com/person

How many of these are valid LLRBs, i.e. have a 1-1 correspondence with a valid 2-3 tree?



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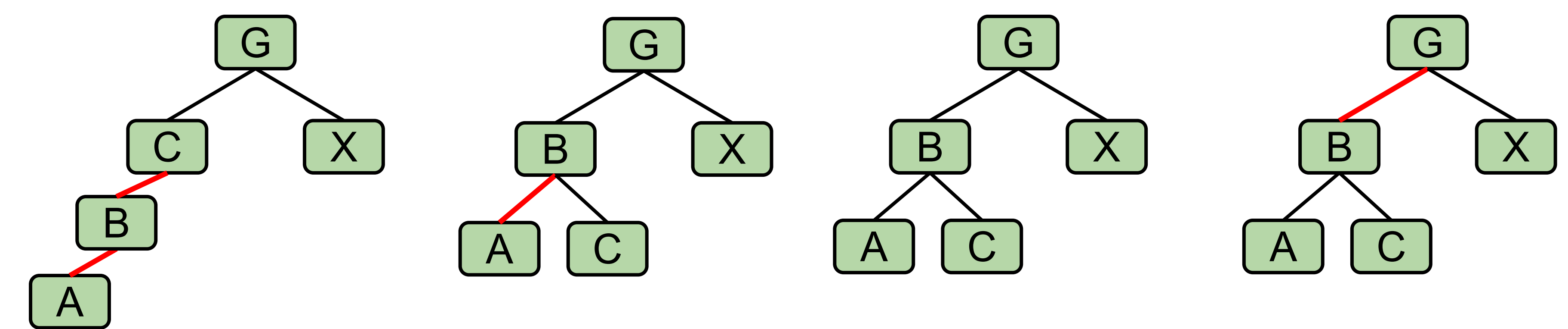
BY NC SA

LLRB Problem #1: yellkey.com/person

How many of these are valid LLRBs, i.e. have a 1-1 correspondence with a valid 2-3 tree?



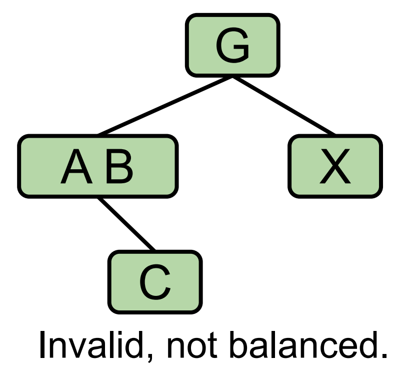
Equivalent 2-3

G

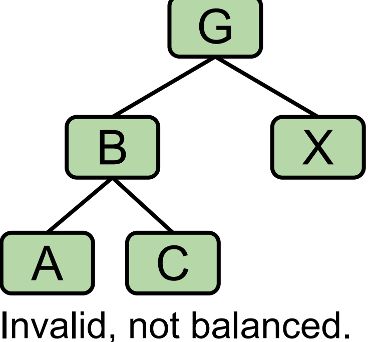
A B C

X

Invalid, has 4 node.



Invalid, not balanced.



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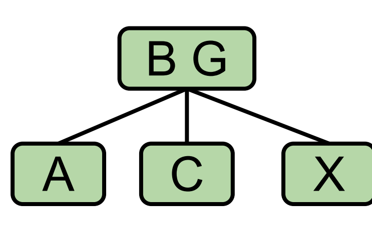

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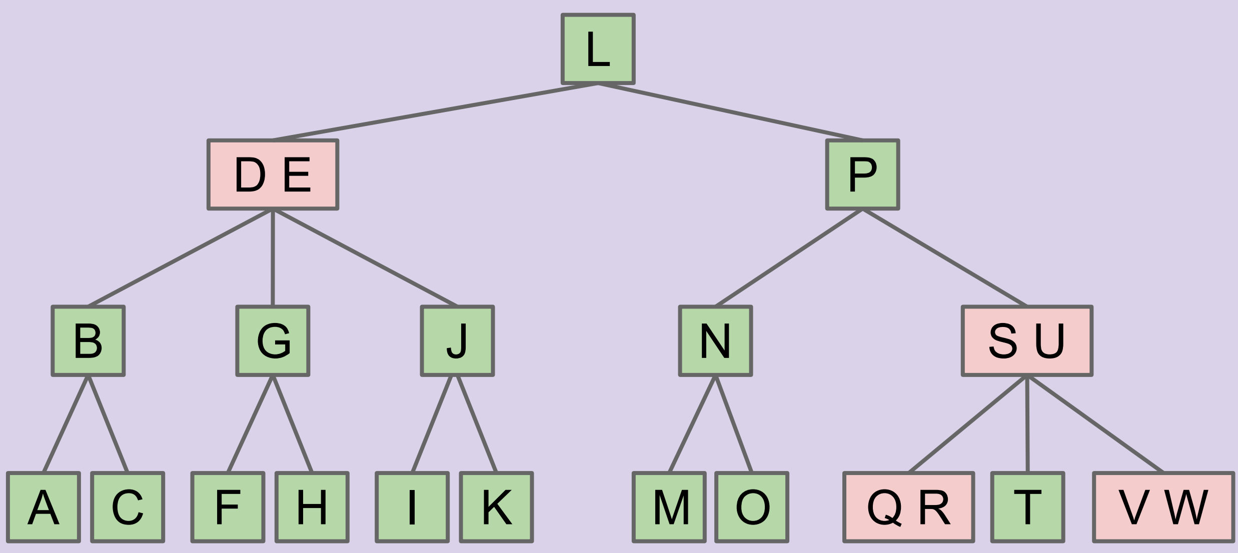
CC BY NC SA

LLRB Problem #2: yellkey.com/leave

How tall is the corresponding LLRB for the 2-3 tree below? (3 - nodes in pink)

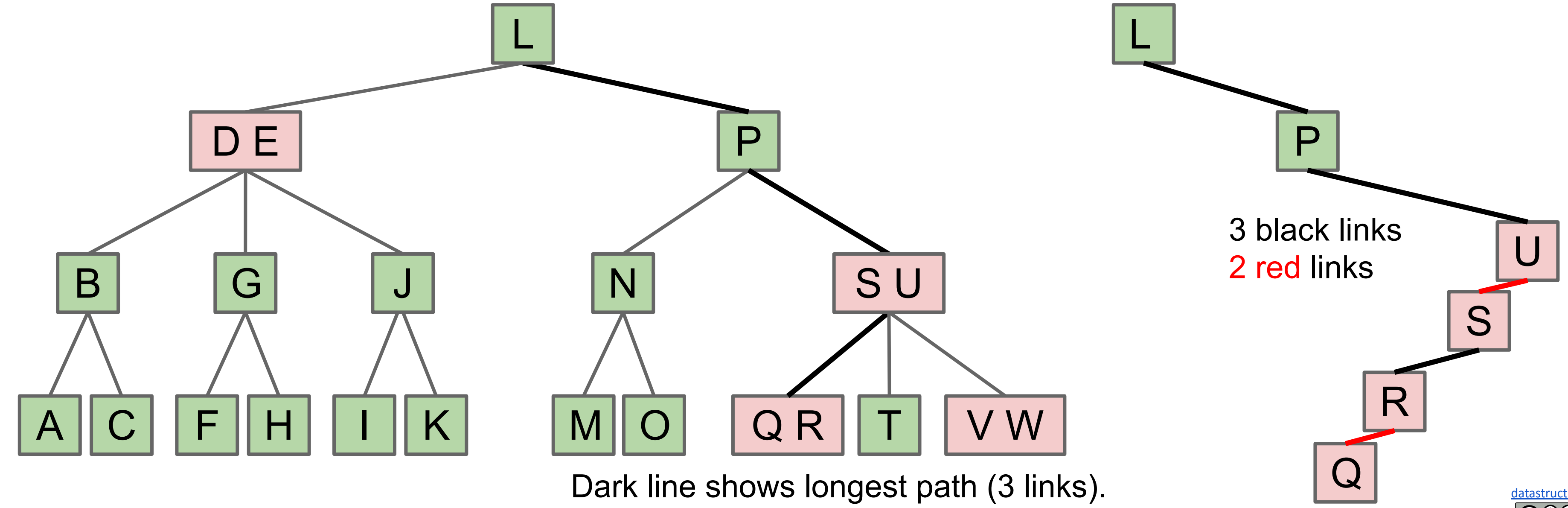


Note: The version of this slide used in live lecture was slightly different.

LLRB Problem #2: yellkey.com/leave

How tall is the corresponding LLRB for the 2-3 tree below? (3 - nodes in pink)

- Each 3-node becomes two nodes in the LLRB.
- Total height is 3 (black) + 2 (red) = 5.
- More generally, an LLRB has no more than ~2x the height of its 2-3 tree.



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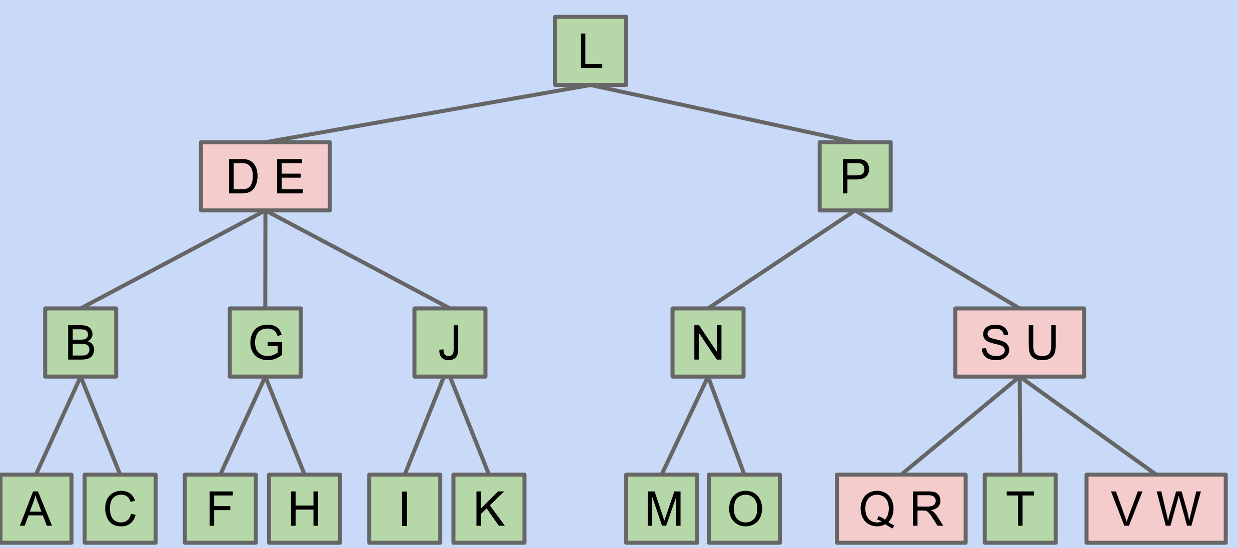
LLRB Height

Suppose we have a 2-3 tree of height H.

- What is the maximum height of the corresponding LLRB?



LLRB Height

Suppose we have a 2-3 tree of height H.

- What is the maximum height of the corresponding LLRB?
  - Total height is H (black) + H + 1 (red) = 2H + 1.

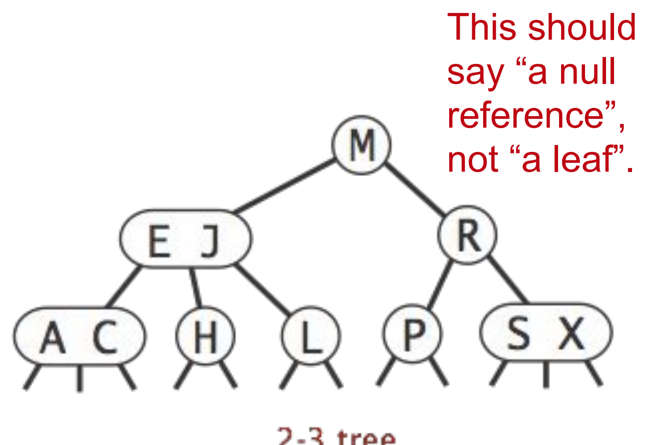
Worst case would be if these were both 3 nodes.

Left-Leaning Red Black Binary Search Tree (LLRB) Properties

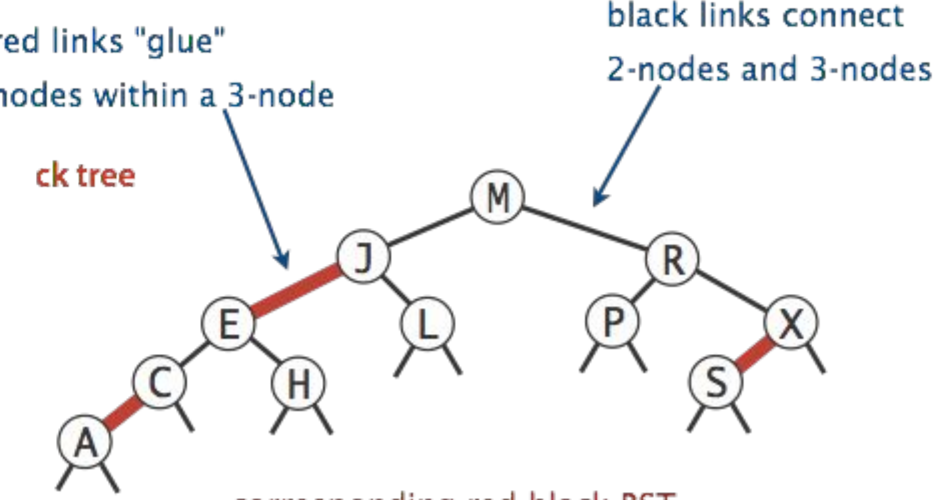
Some handy LLRB properties:

- No node has two red links [otherwise it'd be analogous to a 4 node, which are disallowed in 2-3 trees].
- Every path from root to ~~a leaf~~ has same number of **black links** [because 2-3 trees have the same number of links to every leaf]. LLRBs are therefore balanced.

This should say "a null reference", not "a leaf".



2-3 tree



corresponding red-black BST

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Left-Leaning Red Black Binary Search Tree (LLRB) Properties

Some handy LLRB properties:

- No node has two red links [otherwise it'd be analogous to a 4 node, which are disallowed in 2-3 trees].
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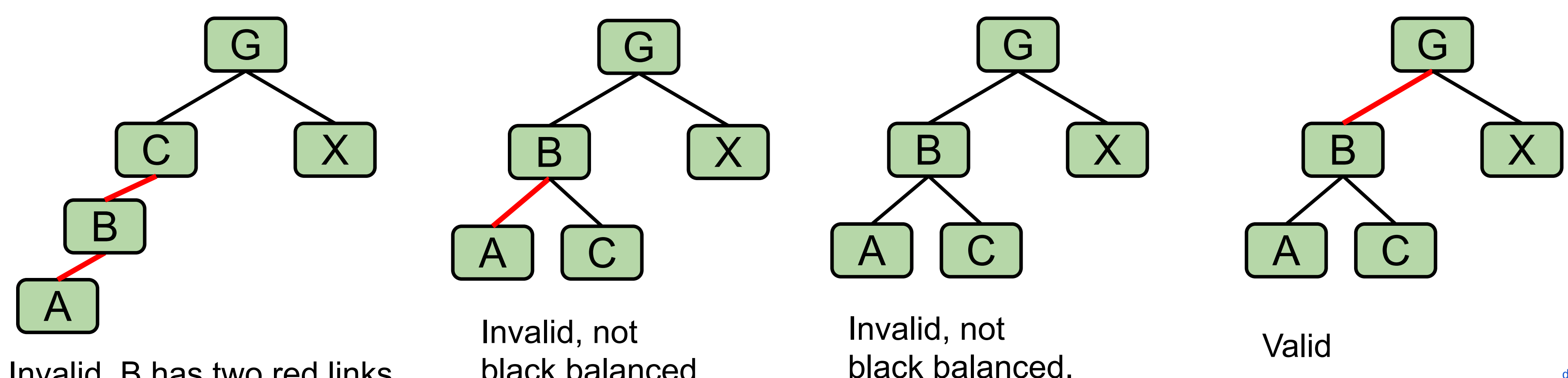

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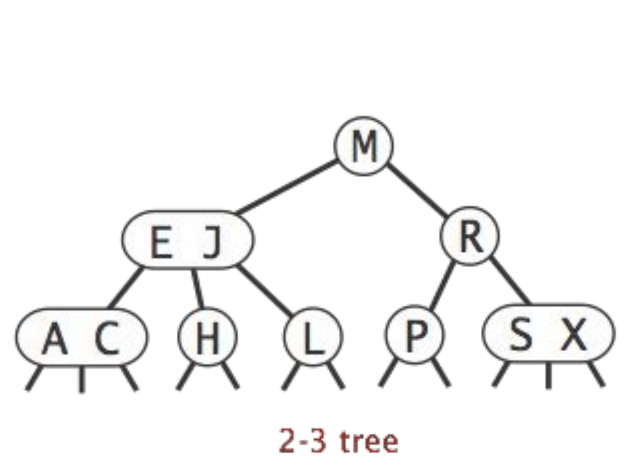
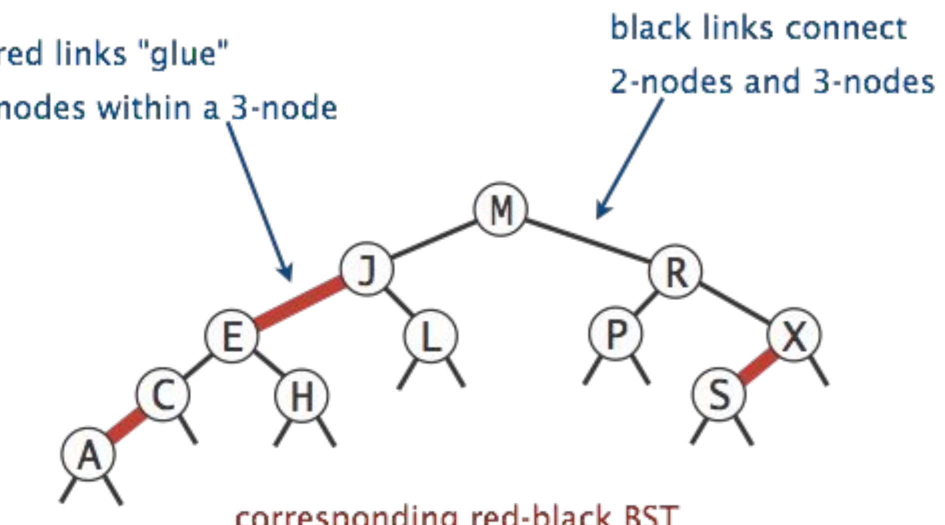
LLRB Construction

One last important question: Where do LLRBs come from?

- Would not make sense to build a 2-3 tree, then convert. Even more complex.
- Instead, it turns out we implement an LLRB insert as follows:
  - Insert as usual into a BST.
  - Use zero or more rotations to maintain the 1-1 mapping.





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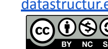
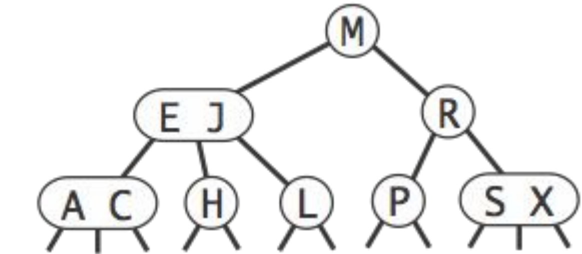


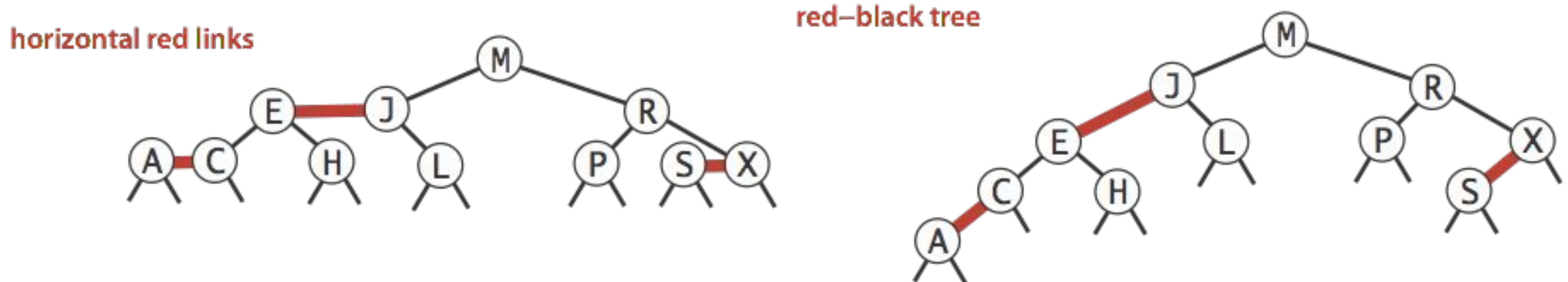
Maintaining 1-1 Correspondence Through Rotations

The 1-1 Mapping

There exists a 1-1 mapping between:

- 2-3 Tree
- LLRB





Implementation of an LLRB is based on maintaining this 1-1 correspondence.

- When performing LLRB operations, pretend like you're a 2-3 tree.
- Preservation of the correspondence will involve tree rotations.

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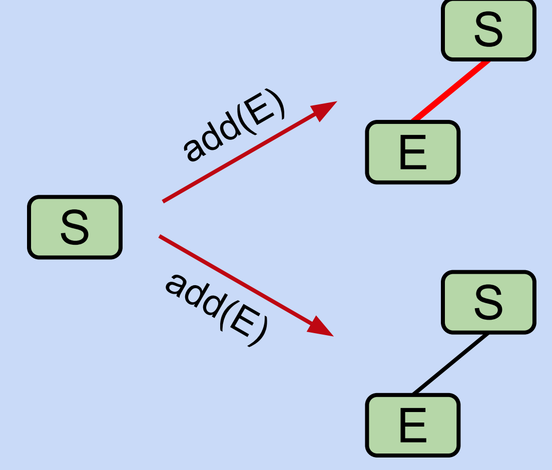
Design Task #1: Insertion Color

Should we use a red or black link when inserting?



LLRB World

World 2-3

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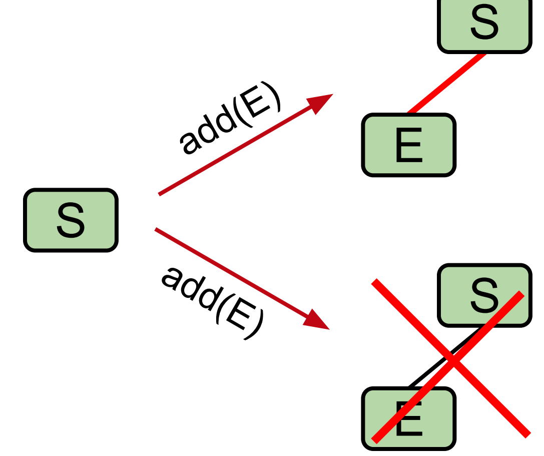
Design Task #1: Insertion Color

Should we use a red or black link when inserting?

- Use red! In 2-3 trees new values are ALWAYS added to a leaf node (at first).



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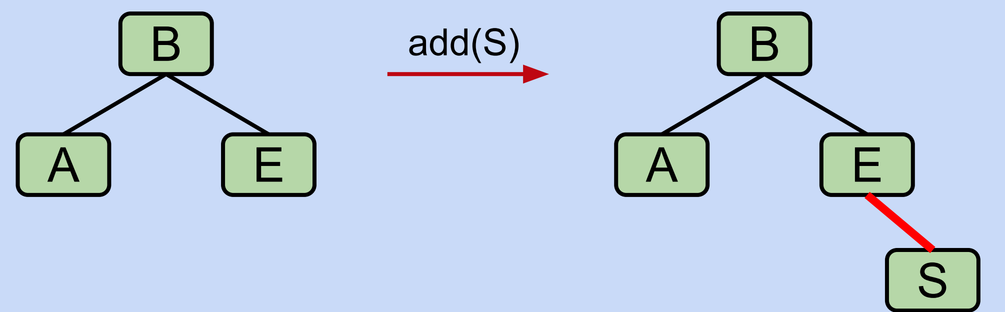
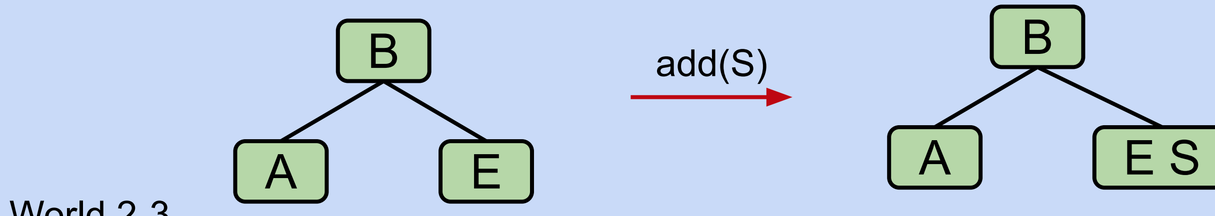
Design Task #2: Insertion on the Right

Suppose we have leaf E, and insert S with a red link. What is the problem below, and what do we do about it?



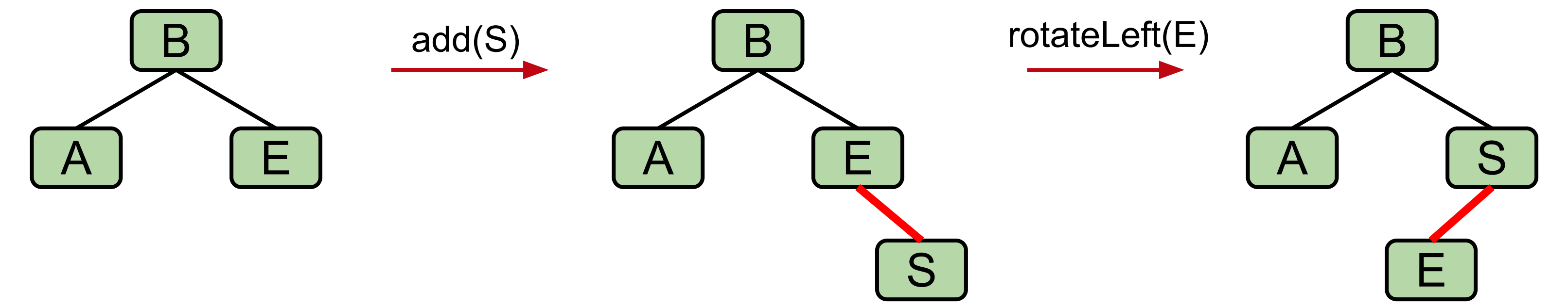
LLRB World



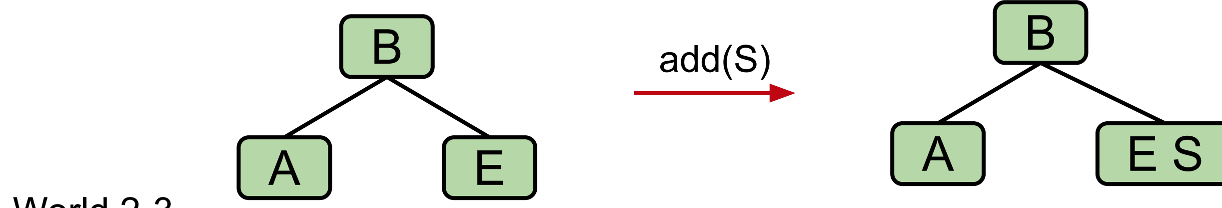
World 2-3

Design Task #2: Insertion on the Right

Suppose we have leaf E, and insert S with a red link. What is the problem below, and what do we do about it: Right links aren't allowed, so rotateLeft(E).



LLRB World



World 2-3

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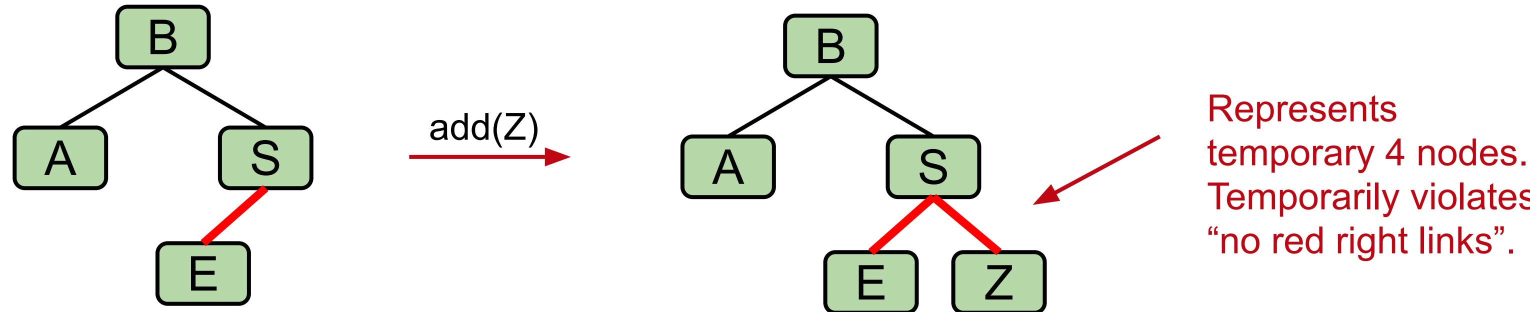
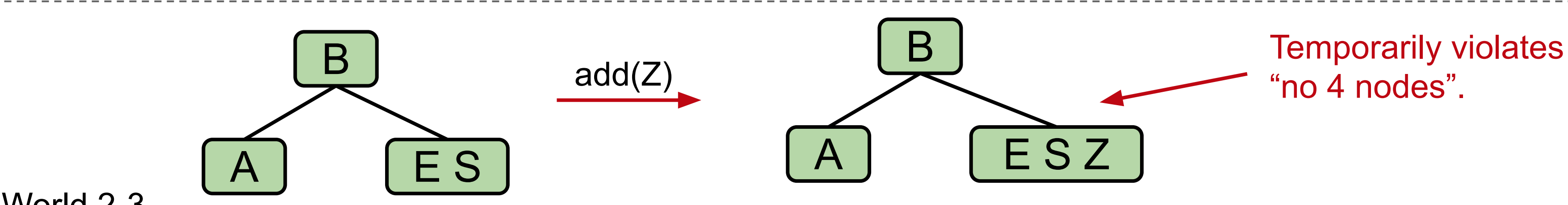
New Rule: Representation of Temporary 4-Nodes

We will represent temporary 4-nodes as BST nodes with two red links.

- This state is only temporary (more soon), so temporary violation of "left leaning" is OK.



LLRB World



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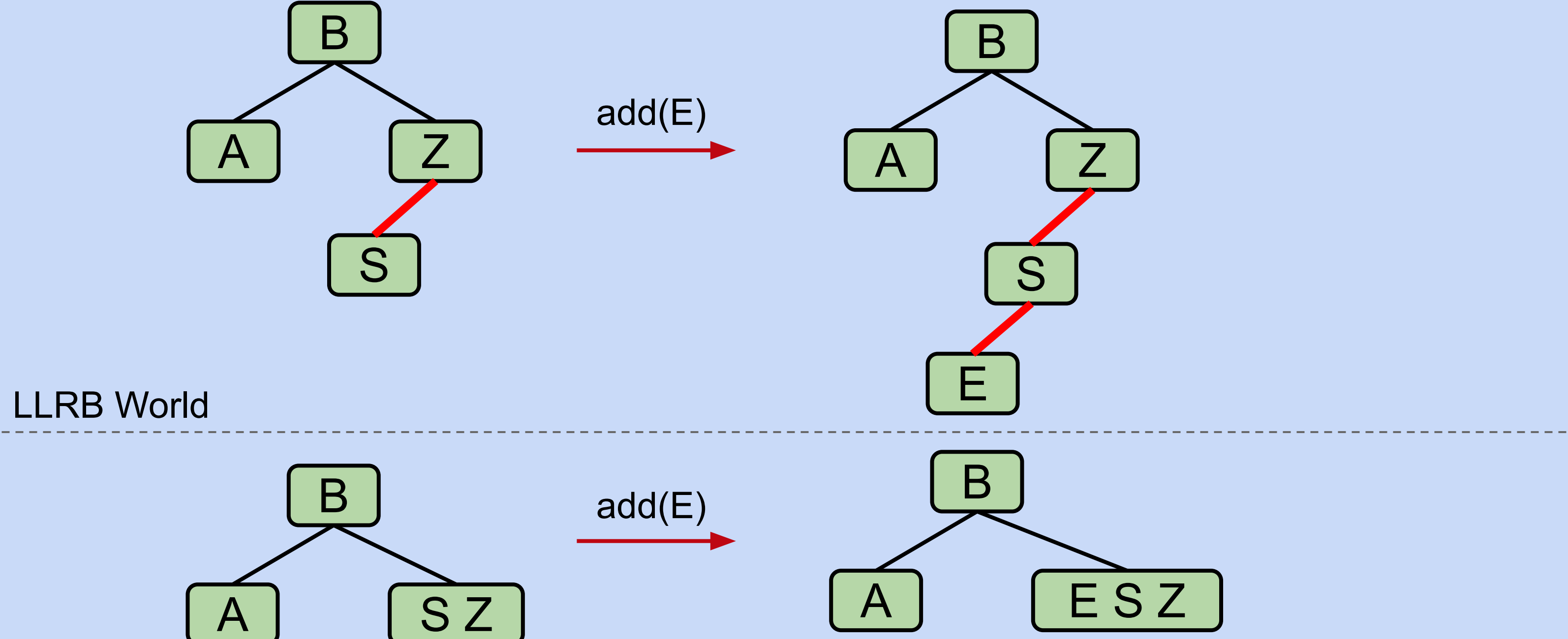
Design Task #3: Double Insertion on the Left

Suppose we have the LLRB below and insert E. We end up with the wrong representation for our temporary 4 node. What should we do so that the temporary 4 node has 2 red children (one left, one right) as expected?



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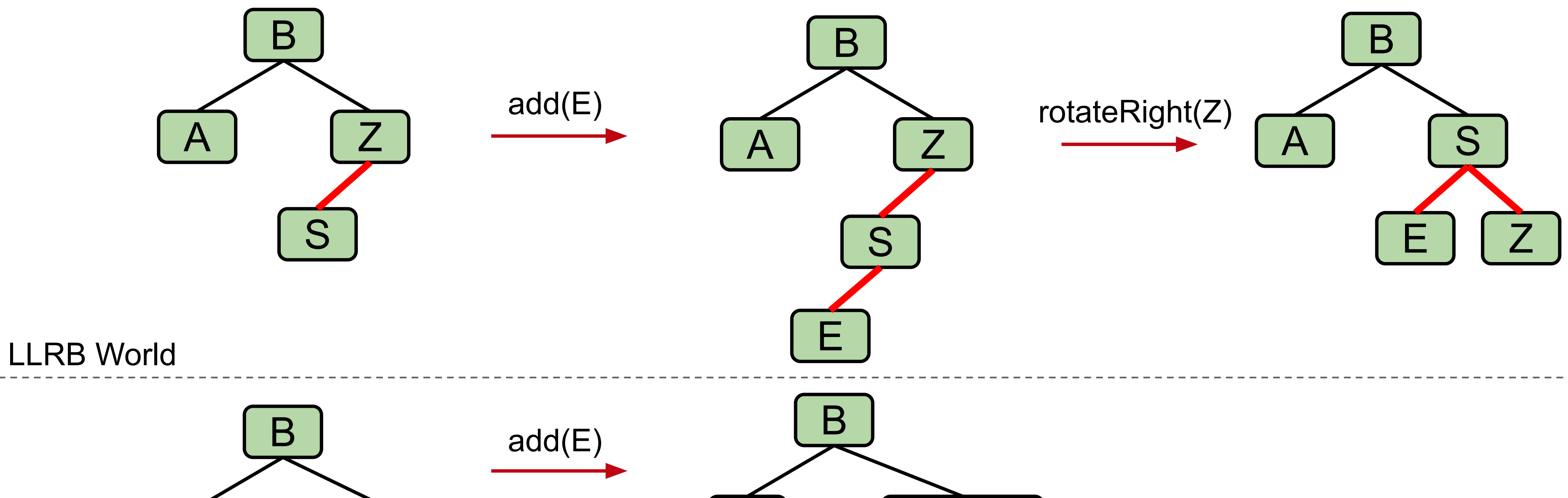
Design Task #3: Double Insertion on the Left

Suppose we have the LLRB below and insert E. We end up with the wrong representation for our temporary 4 node. What should we do?

- Rotate Z right.



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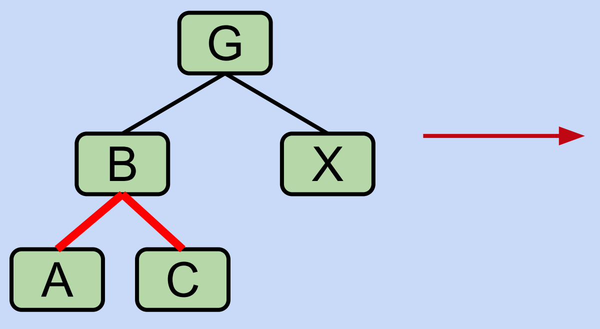
Design Task #4: Splitting Temporary 4-Nodes

Suppose we have the LLRB below which includes a temporary 4 node.

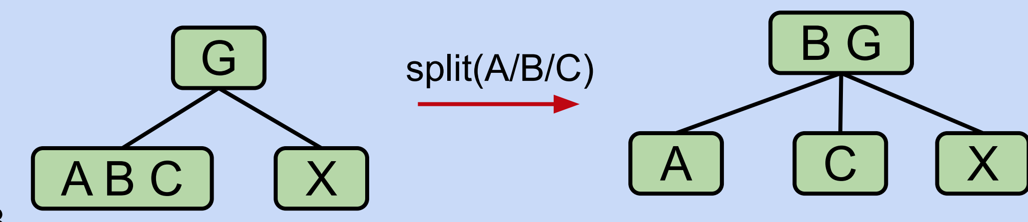
What should we do next?

- Try to figure this one out! It's a particularly interesting puzzle.



Hint: Ask yourself "What Would 2-3 Tree Do?" WW23TD?

LLRB World



World 2-3

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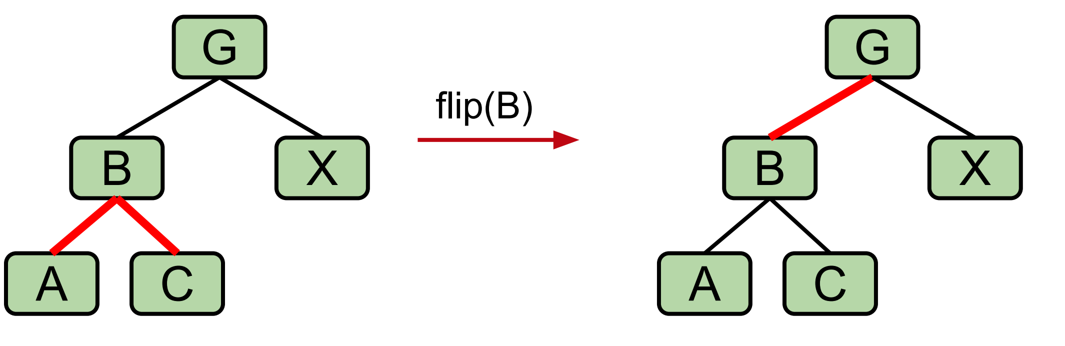
Design Task #4: Splitting Temporary 4-Nodes

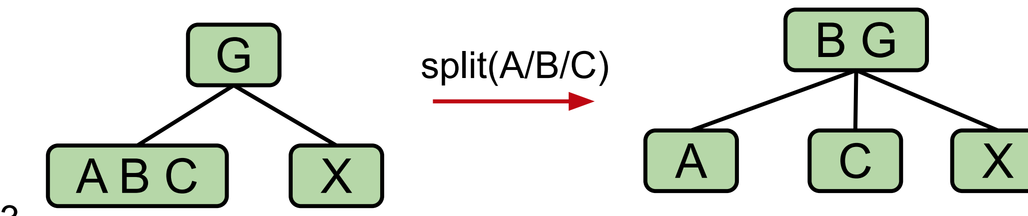
Suppose we have the LLRB below which includes a temporary 4 node.
What should we do next?

- Flip the colors of all edges touching B.
  - Note: This doesn't change the BST structure/shape.



BST, the magic was inside of you all along.

LLRB World



World 2-3

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… and That's It!

Congratulations, you just invented the red-black BST.

- When inserting: Use a red link.
- If there is a *right leaning "3-node"*, we have a **Left Leaning Violation.**
  - Rotate left the appropriate node to fix.
- If there are *two consecutive left links*, we have an **Incorrect 4 Node Violation.**
  - Rotate right the appropriate node to fix.
- If there are any *nodes with two red children*, we have a **Temporary 4 Node.**
  - Color flip the node to emulate the split operation.

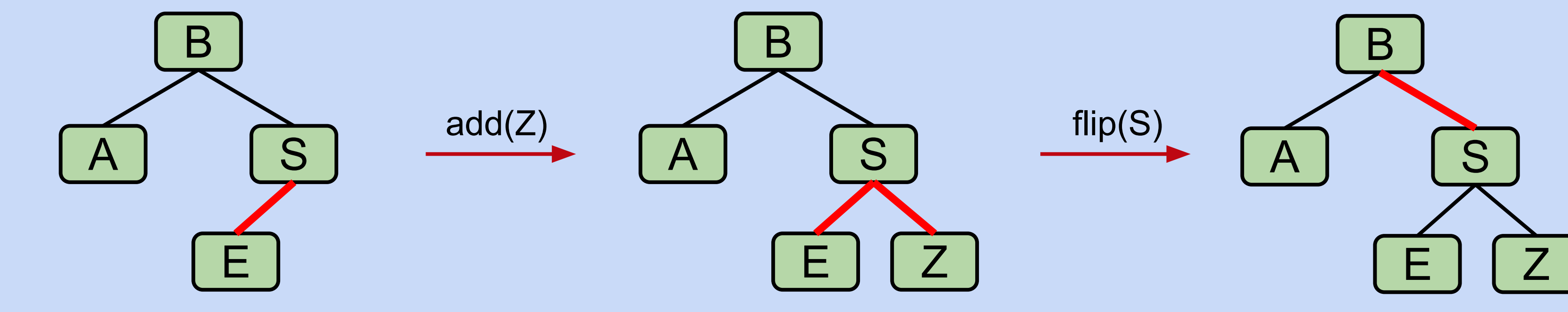
One last detail: Cascading operations.

- It is possible that a rotation or flip operation will cause an additional violation that needs fixing.

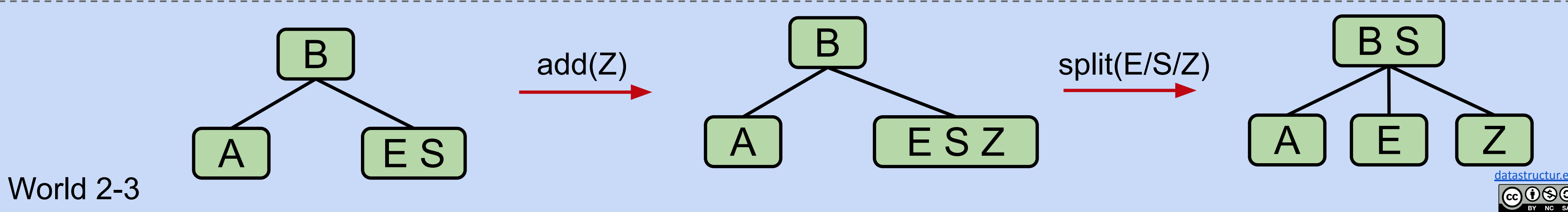
Cascading Balance Example

Inserting Z gives us a temporary 4 node.

- Color flip yields an invalid tree. Why? What's next?



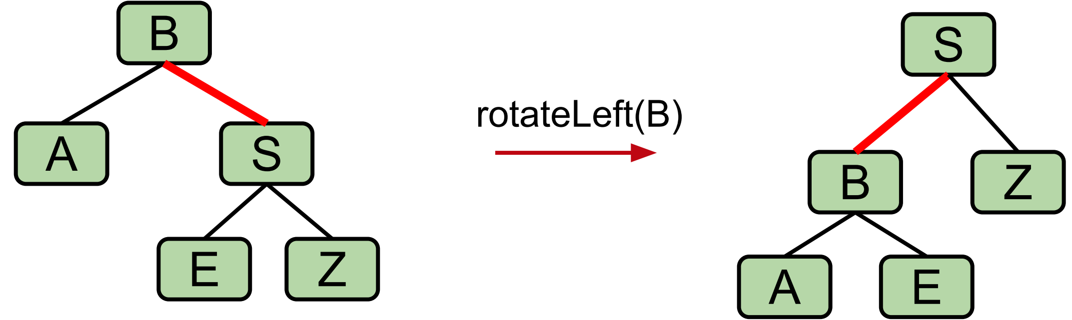
LLRB World



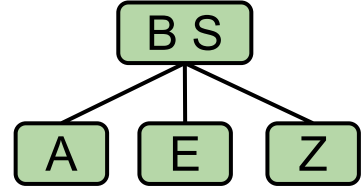
Cascading Balance Example

Inserting Z gives us a temporary 4 node.

- Color flip yields an invalid tree. Why? What's next?
- We have a right leaning 3-node (B-S). We can fix with rotateLeft(b).



LLRB World



World 2-3

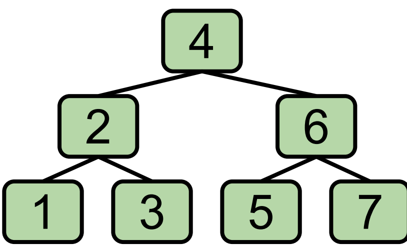
Optional Exercise

Insertion of 7 through 1

To get an intuitive understanding of why all this works, try inserting the 7, 6, 5, 4, 3, 2, 1, into an initially empty LLRB.

- You should end up with a perfectly balanced BST!

To check your work, see this demo.



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LLRB Runtime and Implementation

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LLRB Runtime

The runtime analysis for LLRBs is simple if you trust the 2-3 tree runtime.

- LLRB tree has height O(log N).
- Contains is trivially O(log N).
- Insert is O(log N).
  - O(log N) to add the new node.
  - O(log N) rotation and color flip operations per insert.

We will not discuss LLRB delete.

- Not too terrible really, but it's just not interesting enough to cover. See optional textbook if you're curious (though they gloss over it, too).

LLRB Implementation

Amazingly, turning a BST into an LLRB requires only 3 clever lines of code.

- Does not include helper methods (which do not require cleverness).

```
private Node put(Node h, Key key, Value val) {
    if (h == null) { return new Node(key, val, RED); }

    int cmp = key.compareTo(h.key);
   if (cmp < 0)        { h.left  = put(h.left,  key, val); }
   else if (cmp > 0) { h.right = put(h.right, key, val); }
   else              { h.val   = val;                     }

    if (isRed(h.right) && !isRed(h.left))      { h = rotateLeft(h);  }
    if (isRed(h.left)  &&  isRed(h.left.left)) { h = rotateRight(h); }
    if (isRed(h.left)  &&  isRed(h.right))     { flipColors(h);      }

    return h;
}
```

just 3 lines needed to balance

https://algs4.cs.princeton.edu/33balanced/RedBlackBST.java.html

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Search Tree Summary

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Search Trees

In the last 3 lectures, we talked about using search trees to implement sets/maps.

- **Binary search trees** are simple, but they are subject to imbalance.
- **2-3 Trees (B Trees)** are balanced, but painful to implement and relatively slow.
- **LLRBs** insertion is simple to implement (but delete is hard).
  - Works by maintaining mathematical bijection with a 2-3 trees.
- Java's TreeMap is a red-black tree (not left leaning).
  - Maintains correspondence with 2-3-4 tree (is not a 1-1 correspondence).
  - Allows glue links on either side (see Red-Black Tree).
  - More complex implementation, but significantly (?) faster.

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… and Beyond

There are many other types of search trees out there.

- Other self balancing trees: AVL trees, splay trees, treaps, etc. There are at least hundreds of different such trees.

And there are other efficient ways to implement sets and maps entirely.

- Other linked structures: Skip lists are linked lists with express lanes.
- Other ideas entirely: Hashing is the most common alternative. We'll discuss this very important idea in our next lecture.

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