Optimization

Dalton H Bermudez BEng.
PhD student in Medical Physics

Notes adapted from Prof Diego Hernando

Optimization Application in Medical Physics/BME

- Machine learning
- CT and MRI undersampled reconstruction
- Map across different imaging modalities (e.g MRI to CT image)
- Dose optimization in radiotherapy
- Image registration & segmentation

o Generally post as a mathematical problem to find the best possible solution x* for some problems with set constraints

Optimization mathematical formulation

$$\omega^* = argmin_\omega \sum_{i=1}^{n} \log\left(1 + e^{-y_i \omega^T x_i}\right)$$

- n: number of data points
- Xi : ith data point
- Yi: ground truth of ith data point
- w*: optimal nth dimension vector perpendicular to the plane that linearly separates the two groups

o In a general sense the formulation for an optimization problem is:

$$\bar{x}^* = argmin_{\bar{x}} f(\vec{x})$$

$$\text{such that } g_k(\vec{x}) \leq b_k, for\ k = 1, ..k$$

$$\text{such that } h_p(\vec{x}) = c_l, \text{ for k = 1, ... k}$$

Vector norms Conditions

1) $|\vec{x}| \geq 0 \; always, and \; |\vec{x}| = 0 \; if \; \vec{x} = 0$
2) $\left||c\vec{x}|\right| = |c|\left||\vec{x}|\right|, for \; any \; scalar \; c$
3) $||\vec{x} + \vec{y}|| < \left||\vec{x}|\right| + \left||\vec{y}|\right| \; (triangle \; inequality)$

Important norms:

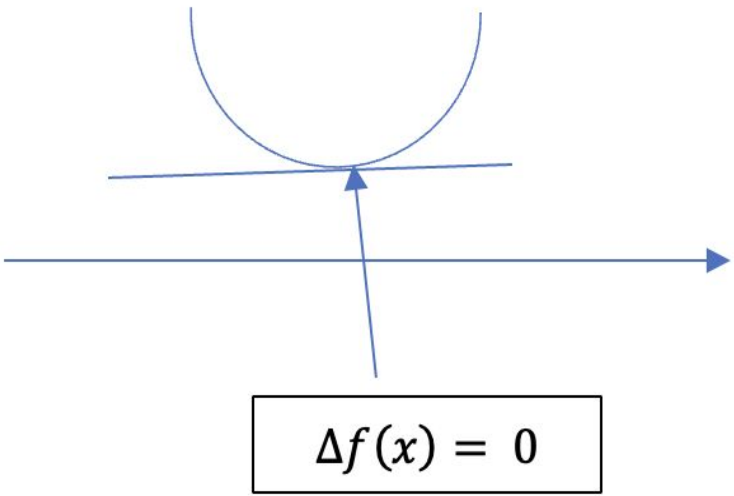
$$l_1 \; norm: ||\vec{x}_1|| = \sum_{m=1}^{N} |x_n|$$

$$l_2 \; norm: ||\vec{x}||_2 = \sqrt{\sum_{m=1}^{N} |x_n|^2} \; \text{(Euclidean)}$$

l2-norm (Closed form solution)

$$f(\vec{x}) = ||A\vec{x} - \vec{y}||_2^2$$
$$\Delta f(\vec{x}) = 2A^T(A\vec{x} - \vec{y}) = 0$$
$$A^T A\vec{x}^* = A^T\vec{y}$$
$$\vec{x}^* = (A^T A)^{-1}A^T\vec{y}$$

If A^TA is invertible



Constrained vs UnConstrained Optimization

Unconstrained

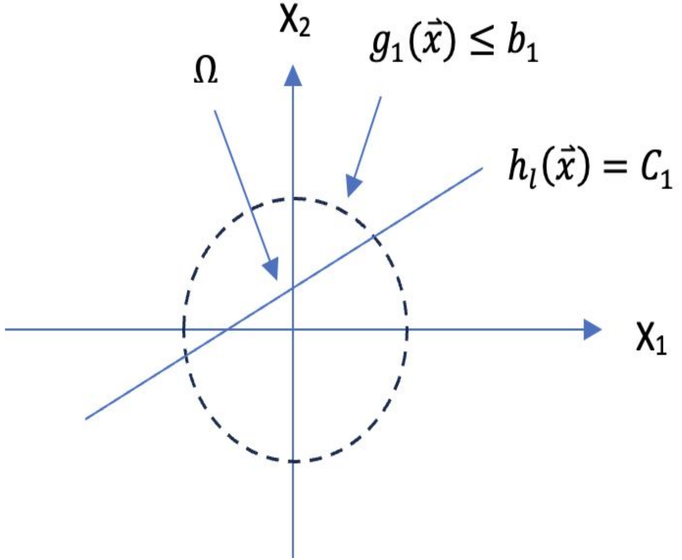
$$\bar{x}^* = argmin_{\bar{x}}\, f(\vec{x}), \text{ for } \vec{x} \epsilon R^n$$

Constrained

$$\bar{x}^* = argmin_{\bar{x}}\, f(\vec{x})$$

such that $g_k(\vec{x}) \leq b_k, for\ k = 1, .. k$

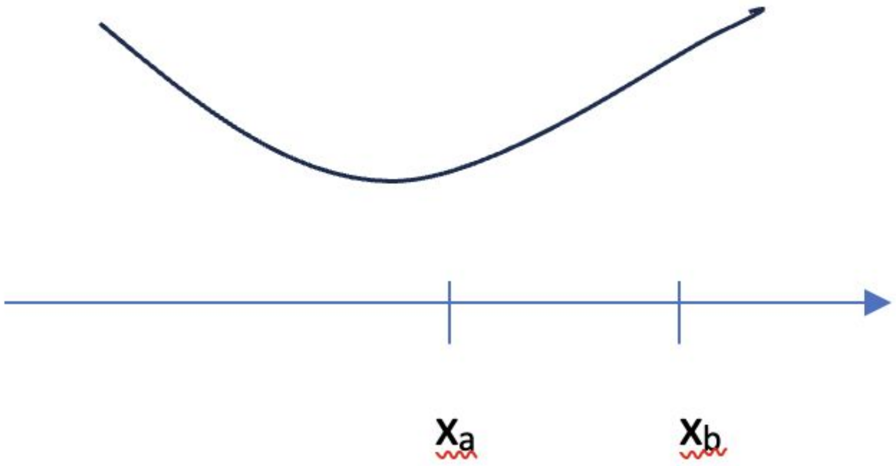
such that $h_p(\vec{x}) = c_l$, for k = 1, ... k



Definition of Convexity

Definition: $f(\vec{x})\ R^N \rightarrow R$ is convex if for any two $\vec{x}_a, \vec{x}_b$ and a scalar t$\epsilon(0,1)$. The following holds:

$$f(t\vec{x}_a + (1-t)\vec{x}_b) = tf(\vec{x}_a) + (1-t)f(\vec{x}_b)$$



- If f(x) is convex and g(y) is convex and monotonically increasing than g(f(x)) is also convex.
- If $||x||_2$ is convex and g(y) = y^2 is convex and monotonically increasing for y>0, then $||x||_2$^2 is convex.