A Statistical Exploration of Riemann Zeros using SAGE

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Riemann Hypothesis & Zeros

- Recall: Riemann Zeta Function & its zeros $r_n = ½ + i\, t_n$, $t_1 = 14.1$, … etc. Riemann Hypothesis: Why "½" ?

- Facts

  1) **Riemann Spectrum** $\{t_n\}$ has low entropy (O. Shanker);

  2) $\{Yp\} = \{t_n \log(p)/(2\pi) \bmod Z\}_{n \text{ in } N}$ accumulates around ½ (Rademacher, Ford & Zaharescu)

  3) R-Spec can be recovered from a cofinal subset (R.P. Marco: Statistics of Riemann Zeros).

- Claim: R-Spec has an alg. structure dual to POSet of prime numbers (see LMI: A Partial Order on Prime Numbers)

Main idea: study using Statistics!

- Explore R-Spec $\{t_n\}$ using *statistical tools* …
- Ford & Zaharescu diagrams (p.4): distribution of Yp shows an abundance of Riemann zeros such that Yp are close to ½.
- Histograms of Riemann zeros using SAGE (Xp=exp(iYp)):
    1) N=200000 zeros & resolution 1000;
    2) N=500000 zeros & resolution 2000;
    3) Other diagrams (p=5 & n=3) & big primes

        (Big primes seem to have flat distributions …)
- The **basic histogram** for scaling factor **log(p)** implies the shape and properties of the other for log(p)*k/q

Landau Formula & Average

- **Landau formula** (equivalent to Riemann-Mangoldt exact formula):

$$\text{Sum}_{0<\text{Im}(r)<T}\ x^r = -T/(2\pi)\ \text{Lambda}(x)+O(\log(T)),$$

suggests *Rademacher's remark that* *Xp concentrate about -1*.

- **Landau's Average** 1/T $\text{Sum}_{0<\text{Im}(r)<T}\ p^r$ essentially yields log(p) in terms of Riemann zeros (SAGE exploration):

$$\tfrac{1}{2\pi} \log(p)/p^{1/2} = [\text{Sum}_{t(n)<T}\ p^{i\,t(n)}]/T$$

$$=1/T\ .\ \text{Sum}_{t(n)<T}\ \exp[2\pi i\ t_n/2\pi * \log(p)].$$

- Conj.: Landau's formula leads to Ford & Zaharescu distributions for Xp.

Primes & Riemann Zeros Duality

- **Riemann-Mangoldt exact formula** (interpreted as Poisson summation / trace formula) => primes & Riemann zeros duality.
- Distributional duality (see Mazur & Stein: Primes & RH):
    1) Primes -> Zeros (p.111) & SAGE worksheet;
    2) Zeros -> Primes (p. 119) & SAGE worksheet.
- Mazur & Stein state: *Riemann Spectrum is the key to primes and their deeper structure* … Is it? OR …
- LMI: The **POSet structure of PRIMES** is the *key* to the **structure of the Riemann Spectrum**.

Questions & Research suggestions

A few questions seem a good start studying the above facts:

1) Can one separate the "p-sector" of R-Spec in the duality equation yielding Dirac distribution of log(p)?

2) Is this "p-sector" the dominant part in Landau's Average?

3) Is (2) the reason for the accumulation around ½ observed by Rademacher (exponential approx -1)?

4) Is this related to Gauss sums via +/-1 $p^{1/2}$ =Gauss Sum? Is R-Spec somehow generated by Weil zeros? (R & Qp: Adeles unite reals & p-adic numbers - primes & p=infinity).

… and a statistical exploration is an easy start.

Riemann Spectrum as a Random Variable on the circle: $Xp=p^{it(n)}$

- The Riemann spectrum $\{t(n)\}_{n \text{ in } N}$ (imaginary parts of Riemann zeros $r_n=½ + i\, t_n$) can be investigated as a **statistical ensemble** (Xp is algebraically better suited for study).

- For averages, correlations and convergence purposes, as in Landau's formula yielding log(p), a large sample is needed, e.g. N=100000.

- To identify an algebraic structure behind them, smaller N-samples seem appropriate, e.g. N=1000: such samples exhibit non-trivial correlations between Xp & Xq; is this when p & q are correlated, i.e. gcd(p-1,q-1)<>1?

RV Xp: Mean, Deviations & Correlations

- Mean & Correlations of Riemann frequencies as "Random Variables" (basic rational/adelic characters r^{it}):

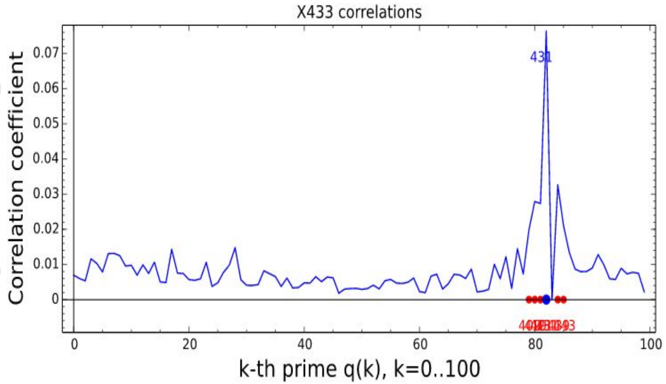
$$Yp=\{t_n \log(p)/2pi \bmod 1\} \quad \text{or} \quad Xp=p^{it(n)}$$

1) Mean of Xp plots (plotting $X_p=p^{it(n)}$, not Yp i.e. "mod 1");
2) Deviations of Xp (StdDev(Xp) for k^{th} prime -> 1);
3) Correlations between Xp & Xq: High Corr., Pdf2
   - Resonances co(Xp,Xq) & *symmetries of primes* gcd(p-1,q-1)
   - Using a small sample of 1000 Xq's shows *resonances*.

… other ideas to explore?

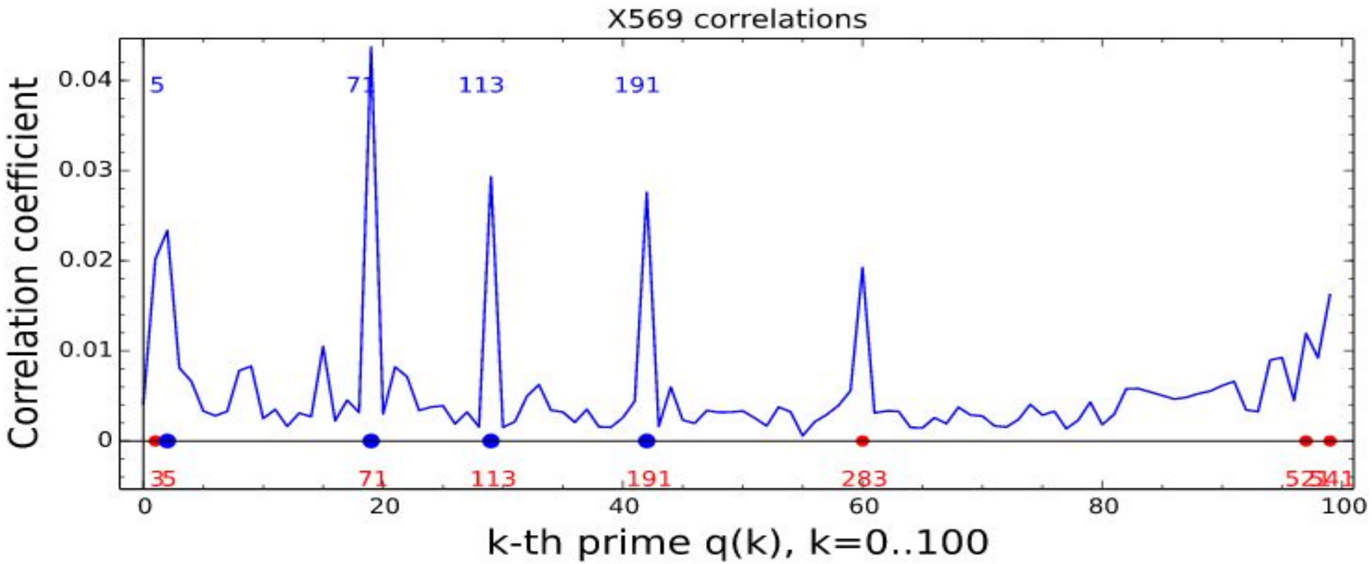
Some observations from the correlations of a fixed Xp with Xq's

- There is a "main resonance" around p, e.g. for p=433 there is a sharp peak at q=431, and lower resonances around it. This is a manifestation of "continuity" with respect to the variable q.
- But there are lower resonances ("harmonics"), which need to be explained; for ex. p=



“Harmonics” of X_{569}

With the main peak out of the picture, we see the lower “modes” (“Fourier harmonics”): q=3, 5, 71, 113, 191, 283.



At 521 & 541 the correlation increases towards the peak.

Symmetry Structure of Fp & Fq's

- The corresponding "toroidal structure" of F_p^x (multiplicative characters) corresponds to the factorization of p-1 & q-1:

p=569: $p-1=|Aut_{Ab}(Z/pZ,+)|=2^3 . 71$

q=3: $q-1=2$

q=5: $q-1=2^2$

q=71: $q-1=2.5.7$

q=113: $q-1=2^3.7$

q=191: $q-1=2.5.19$

q=283: $q-1=2.3.47$

Note: F_{71} symmetry cycle of Fp yields a correlation with X_{71}.

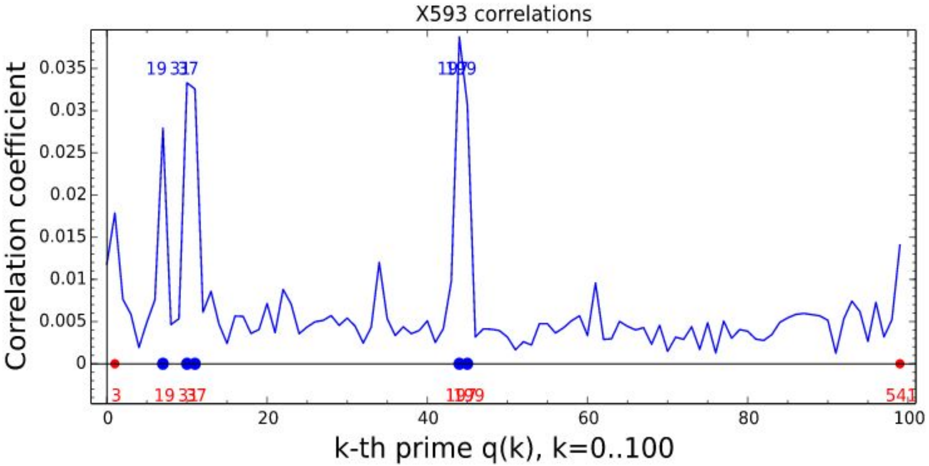
37 cycle of F_{593} & X_{37} resonance of X_{593}

- But why also at

  $q=19$, $q^*=(q-1)/2=3^2$,

  $q=197$, $q^*=2.7^2$

and at $q=199$, $q^*=3^2.11$?



- For 19 & 37 by continuity?

but NOT at $q=17$, $q^*=2^3$! … there should be an *additional reason* … and for 197 & 199 by continuity and ?? … Maybe 2nd level of structure (Aut(Aut(Fp)): $37-1=2^2.3^2$, and 3^2 gives a resonance for q=199, and 197 happens to be “close” to 199 (continuity) … (we are in a “pre-Keplerian phase” … no “laws” yet ...)

X_{743}, p*=7.53 & 53 cycle of 107*=53

- Big resonance at q=5 …?

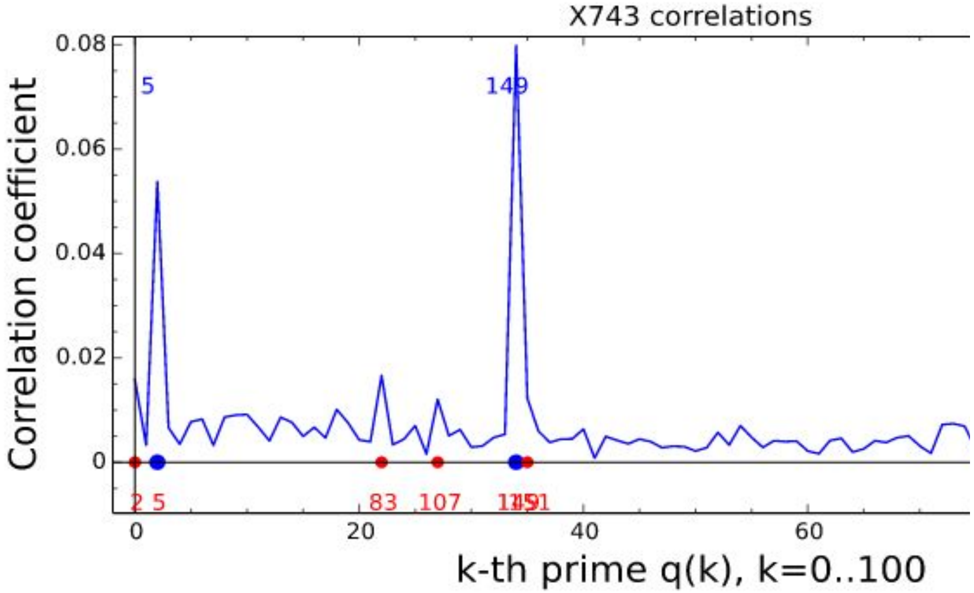
(Is b/c 5 is “close” to 7?)

- At q=107 is “too small”!

    q*=53   (53*=2.13)

- Why at q=149 so big?

    q*=2 . 37   ($37^*=2.3^2$)



(The size of the sample

N=1000 is “blurring” the data, and correlations only approximate some coefficients of algebraic relations between Xp’s & Xq’s …).

Other interesting pairs (p,q)

p=787, p*=3.131, resonance at q=131

p=809 with resonance at q=101|p*

p=863 w. peak at 431|863*=431 …

… Time to “conjecture something” :) [Keplerian phase]

Conjecture & Research ideas

Let p, q be odd primes. If q|p then c(Xp,Yq) is "high" (to be refined …)*

<u>Goal</u>: study how the *Riemann characters* q^{it} correlate non-trivially with the *symmetries* Fp^x of *finite strings* (Z/pZ,+):

$$Aut_{Ab}(Z/pZ,+) \quad \leftrightarrow \quad q^{it},\ t \text{ in } R\text{-}Spec.$$

Further Statistics Study: suggestions

- Cluster analysis …
  - Centroid based clustering
  - SAGE plot of RV Xp (see averages plots)

… the pots are similar, so cluster analysis of Xp data might be useful.

- You (statisticians) name it!

Exploring POSet P & R-Spec duality

- SAGE investigation of correlations between the "random variables" Xp & the symmetry content of the *finite strings* Fp:

$$F_p^{\times} \to Aut_{Ab}(Z/pZ,+)$$

corresponding to the factorization of p-1:

1) Plotting correlation coefficients
2) … and prime correlations.

(… no definite conclusions yet …)

Conclusions and Questions

- **Riemann zeros** have an **algebraic structure:** the *Riemann characters* & *symmetries of finite fields* are correlated!

- One may **use Statistical Tools** to explore the Riemann spectrum and “extract” more information (enriching the relevant R-zeros, relative to a prime p etc.).

- Is there a **p-sector of the Riemann spectrum**? Can one recovering log(p) from a “minimal” subset?

(to be continued … by some “Newton” :)

Thank you!

Bibliography (short)

1) K. Ford and Zaharescu, On the distribution of imaginary parts of Riemann zeros

2) L. M. Ionescu, A partial Order on the set of primes, http://arxiv.org/abs/1407.6659

3) R. P. Marco, Statistics of Riemann zeros, http://arxiv.org/abs/1112.0346

4) L. M. Ionescu, A study of the Riemann zeros, NSF GP, 2014, and references therein.

5) O. Shanker, Entropy of Riemann zeros

Notes (2016)

- Riemann “angular” frequencies: Xp={t/(2pi)}

    R. zeros = $\frac{1}{2} + 2\pi i \nu$

[periods = 1/Nu etc.]