# Math 104 Final Review:

## Midterm 1

1. Completeness axiom of real numbers and limit of sequence of real numbers: (Ross 1.1, 1.2, 1.3, 1.4)

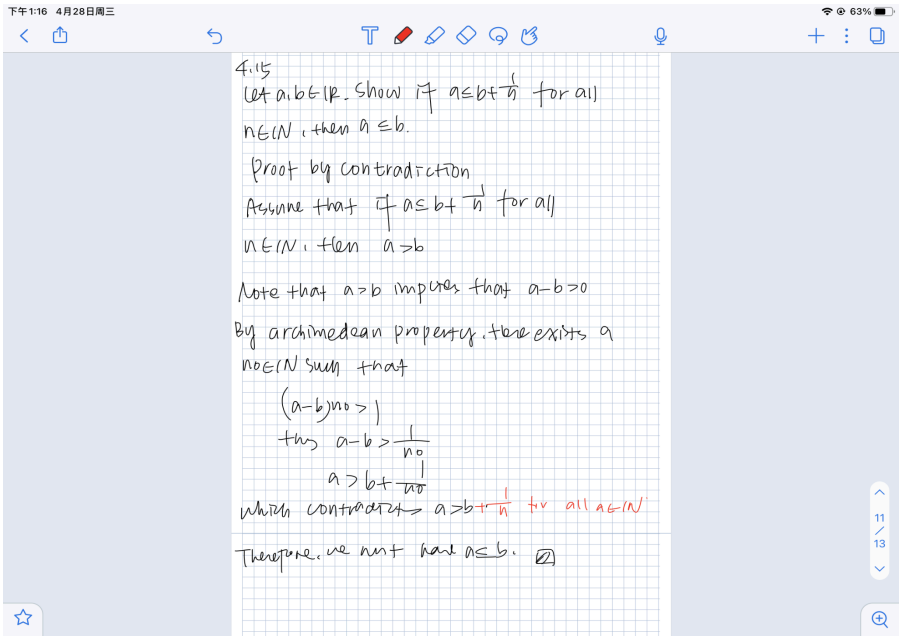
**Lecture 1**:

- **Natural Numbers** $\mathbb{N} = \{1, 2, 3, \dots\}$
- **Integers** $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$
- **Rational Numbers (ratios of integers)** $\mathbb{Q} = \{p/q : p, q \in \mathbb{Z} \text{ and } q \neq 0\}$
- **Complex Numbers** $\mathbb{C} = \{x + iy : x, y \in \mathbb{R}\}$
- **Propositions**: if r is a rational number, (r=c/d, gcd(c,d)=1) and it's a root of the following integer coefficients polynomial: $Cnx^n + Cn-1x^{n-1} + \dots + C0 = 0$, then d divides *Cn, c divides C*0
- **Corollary**: if r =c/d ≠0 is a root of a "monic polynomial", its lead term has coefficient is 1: $x^n + Cn-1x^{n-1} + \dots + C0 = 0$ ( we can apply this to prove that the roots of x^2-2=0 are not rational numbers)

**Lecture 2**:

- **Supremum and Infimum**: Let S be a non-empty subset of R. If S is bounded above (i.e exists an upper bound for S), and S has a least upper bound, then we call it the supremum of S denoted as sup(S). If S is bounded below and S has a greatest lower bound, then we call it the infimum of S, denoted as inf (S).
  NOTE: the difference between sup, inf and max, min, for example, S=(0,1), max(S) doesn't exist, least upper bound(S)=1, greatest lower bound(S)=0
- **Completeness Axiom**: Every non-empty subset $S \subset \mathbb{R}$ that is bounded above has a least upper bound, and if S is bounded from below, then inf S exists.
- **Archimedean Property**: If a>0,b>0 are real numbers, then for $n \in \mathbb{N}$, we have an>b
- **Sequence**: $a1, a2, a3, \dots, an, \in \mathbb{R}$, could be denoted as $(an)n \in \mathbb{N}$
  NOTE: sequence is not a set. Sequence has its element come in order (first, second…). Set is just information about "who is in the set".
- **Some important propositions about inf and sup**:
  1. Suppose that A, B are nonempty sets of real numbers such that $x \leq y$ for all $x \in A$ and $y \in B$. Then $\sup A \leq \inf B$
  2. Suppose that A, B are subsets of R such that $A \subset B$. Then $\sup A \leq \sup B$, and $\inf A \geq \inf B$.
  3. If $A \subset \mathbb{R}$, then $M = \sup A$ if and only if: (a) M is an upper bound of A; (b) for every $M0 < M$ there exists $x \in A$ such that $x > M0$ . Similarly, $m = \inf A$ if and only if: (a) m is a lower bound of A; (b) for every $m0 > m$ there exists $x \in A$ such that $x < m0$ .

**Homework 1**:

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## 2. Limit of bounded monotonic sequence, limit theorem, liminf, limsup (Ross 2.7,2.9,2.10)

**Lecture 3**:

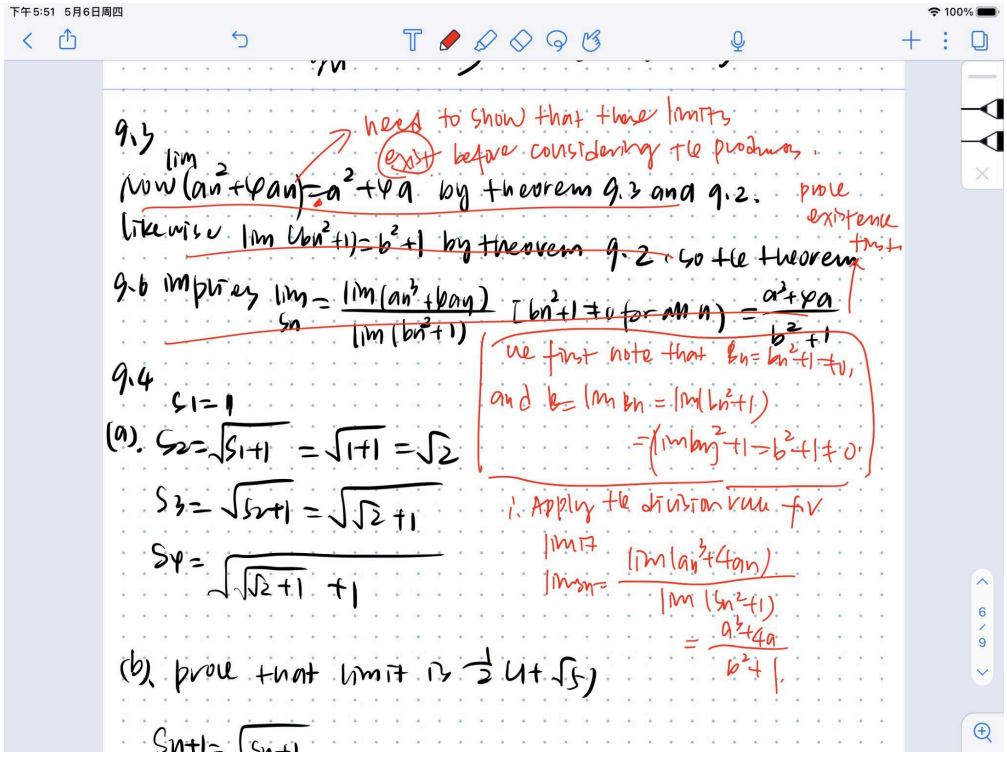
- **Limit of Sequence**: We say a sequence (an)n has limit $\alpha \in R$, if $\forall \epsilon > 0$, there exists a real number N>0, such that n>N, we have $|a_n - \alpha| < \epsilon$, denoted by $\lim_{n\to\infty} a_n = \alpha$
- **Property and tools to find limit**:
  1. All convergent sequences are bounded
  2. If liman=$\alpha$, and if k∈R, then lim(k.an)=k.$\alpha$
  3. Let an, bn be convergent sequences, liman=$\alpha$,limbn=β, then
     a. lim(an+bn)=lim(an)+lim(bn)=$\alpha$+β
     b. lim(an.bn)=(liman).(limbn)=$\alpha$β
     c. If an≠0, ∀n and if $\alpha$≠0, then, lim(1/an)=1/$\alpha$

**Lecture 4**:

- **Continued important theorems:**
  1. $\lim_{n\to\infty} 1/n^p = 0 \quad \forall p>0$
  2. lima^n=0, if $|a|<1$
  3. $\lim_{n\to\infty} n^{(1/n)} = 1$
  4. $\lim_{n\to\infty} a^{1/n} = 1$ for a>0
- **Cauchy Sequence:** if $\forall \epsilon > 0$, there exists N>0, s.t ∀n1,n2>N, we have $|a_{n1} - a_{n2}| < \epsilon$, (using graph--oscillation amplitude gets smaller and smaller)
- **Monotone Sequence**: an increasing sequence is such that an+1≥an, a decreasing sequence is such that an+1≤an, they are both called monotone sequences.

- **Theorems**:
    1. All bounded sequences are convergent
    2. Let (an) be a sequence, (an) is cauchy $\Leftrightarrow a_n$ converges

**Homework 2**




## 3. Subsequences and Cauchy Sequences: (Ross 2.10, 2.11)

**Lecture 5**

- **Monotone Sequence and Cauchy**:
    1. limsup: Let (*an*) be a sequence in R, limsup$an$=lim$N\to\infty$(sup $n>N$ $an$)
    2. liminf :Let (*an*) be a sequence in R, liminf$an$=lim$N\to\infty$(inf $n>N$ $an$)

    Remark: if we allow the notion of lim=+∞ or lim=-∞, then limsup exists in R or =-∞, liminf exists in R or =+∞
    3. A monotone increasing sequence has several properties;
        a. If it's bounded, then its limit exists
        b. If it's unbounded (without upper/lower bounds), then lim*an*=+∞
        c. Strategy: if we want to prove a=b, one way to prove it is $|b-a|<\epsilon$, for any $\epsilon>0$

- **Lemma:**

    a. if (an) is bounded sequence, then its limsup, liminf exists

    b. If (an) is bounded sequence, and $\alpha_+$=limsupan, then for any $\epsilon>0$, $\exists N$ such that $\forall n>N$, we have an$<\alpha_+ +\epsilon$

- **Theorem:** let (an) be a bounded sequence. Then lim an exists⇔limsup(an)=liminf(an)

## Lecture 6

- **Subsequence:** let (Sn) be a sequence of real numbers. Given a strictly increasing sequence of indices $n1<n2<n3<...nm<....$we define the corresponding subsequence as $t_{m;}$=Snm, ($t_{m)}$ is called a subsequence of (sn).Sometimes, we write (Snk)k for the subsequence.
- **Lemma:**
- a. if (Sn) is convergent, then any subsequence converges to the same point
- b. If $\alpha$=limSn exists in R, then there exists a subsequence that is also monotone.
- c. let (Sn) be any sequence. Then for any t∈ R, (Sn) has a subsequence converges to t ⇔ $\forall \epsilon$>0, the set $\{n \in N : |Sn\text{-}t| < \epsilon\}$ is infinite.

## Lecture 7

- **Theorem:** (Bolzano-Weierstrass Theorem)
    a. Every bounded sequence has a convergence subsequence.
    b. Let (Sn) be a sequence, *S* be the set of subsequential limits of (Sn), then
        1. S in nonempty
        2. supS=limsup(Sn), infS=liminf(Sn)
        3. S=$\{\alpha\}$⇔ limSn exists and equals to $\alpha$

c. Let S be the set of subsequential limits of (sn). Suppose (tn) is a sequence in S∩R and t=lim(tn). Then t∈S

- **Subsequential limit:** let (Sn) be a sequence in R. A subsequential limit is any real number or the +∞,-∞, that is the limit of a subsequence
- **Lemma:** let (Sn) be any sequence. Then there exists a monotone sequence whose limit is limsup(Sn). Similarly, there exists a monotone sequence whose limit is liminf(Sn)

## Lecture 8

- **Themorem:**
    a. let sn be a sequence with limit S>0, let tn be any sequence. Then limsup (sn.tn)=S. limsup(tn)
    b. Let (Sn) be a sequence of positive numbers, then we have liminf (Sn+1/Sn)≤liminf(Sn)^(1/n)≤limsup(Sn)^(1/n))≤limsup(Sn+1/Sn)

## Homework 3

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1.

(a) We define $\limsup\{s_n, s_{n+1}, \dots\} = \lim a_n$

$\liminf\{s_n, s_{n+1}, \dots\} = \lim A_n$

We know that $\inf\{s_n, s_{n+1}, \dots\} \le \sup\{s_n, s_{n+1}, \dots\}$ (a)

for each $n \in \mathbb{N}$.

Monotone convergence says that both $\inf\{s_n, s_{n+1}, \dots\}$ and $\sup\{s_n, s_{n+1}, \dots\}$ converge since they are bounded and are monotone.

$\sup\{s_n, s_{n+1}, \dots\} - \inf\{s_n, s_{n+1}, \dots\} \ge 0$ for each $n \in \mathbb{N}$.

So $\liminf s_n \le \limsup s_n$

Let $A_N = \sup\{s_n \mid n \ge N\}$, $B_N = \inf\{s_n \mid n \ge N\}$

Then $A_N \ge B_N$.

Since $(s_n)$ is bounded, hence $\forall N$, $A_N$ and $\lim A_N$ exists. Hence $\lim A_N \ge \lim B_N$.

thus $\limsup s_n \ge \liminf s_n$

(b).

Since $A_N \ge A_{N+1}$ and $A_N$ is a bounded sequence, hence $\lim A_N = \inf\{A_N \mid N \in \mathbb{N}\}$.

we can also prove it directly: Let $u = \inf\{A_N \mid N \in \mathbb{N}\}$.

Then $A_N \ge u$, $\forall N$. And for any $\varepsilon > 0$, $\exists N$, s.t.

$u + \varepsilon > A_N$, otherwise $u + \varepsilon$ would be a bigger lower bound of $\{A_N\}$, contradict the def of $\inf$.

$\Rightarrow \forall n > N$, we have

$u + \varepsilon > A_N \ge A_n \ge u \Rightarrow |A_n - u| < \varepsilon$

Hence, $A_n$ converges to $u$.

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2.

Define $A_k = \sup\{a_n : n \ge k\}$, $B_k = \sup\{b_n : n \ge k\}$, so

by definition $\limsup_{n\to\infty} a_n = \lim_{k\to\infty} A_k$ and similarly

for $B_k$ in $\limsup_{n\to\infty} b_n$

we consider $\lim_{k\to\infty} C_k = \limsup_{n\to\infty}(a_n + b_n)$

then for all $n \ge k$, we have $a_n + b_n \le A_k + B_k$, because we estimate $a_n$ by the supremum of all terms of $(a_n)$ with $n \ge k$ and likewise for the $b_n$.

$C_k = \sup\{a_n + b_n : n \ge k\} \le A_k + B_k$

Then holds for all $k$, so we take lim on both sides as $k \to \infty$.

$\lim_{k\to\infty} C_k = \lim_{k\to\infty}(A_k) + \lim_{k\to\infty}(B_k)$

Thus $\limsup_{n\to\infty}(a_n + b_n) = \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$ ∎

$a_n = \begin{cases} 1 & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$ $b_n = \begin{cases} 0 & n \text{ even} \\ 1 & n \text{ odd} \end{cases}$

$a_n + b_n = 1$

$\therefore |\limsup(a_n + b_n)| = 1$

$|\limsup(a_n)| = |\limsup(b_n)| = 1$

eg. $a_n$ and $b_n$ be sequences repeat in cycle of 4

$a_n = (0, 1, 2, 1, 0, 1, 2, 1 \dots)$ $b_n = (2, 1, 2, 0, 2, 1, 2, 0 \dots)$

$a_n + b_n = (2, 2, 4, 1, 2, 2, 4, 1 \dots)$ ↳ not a strict example

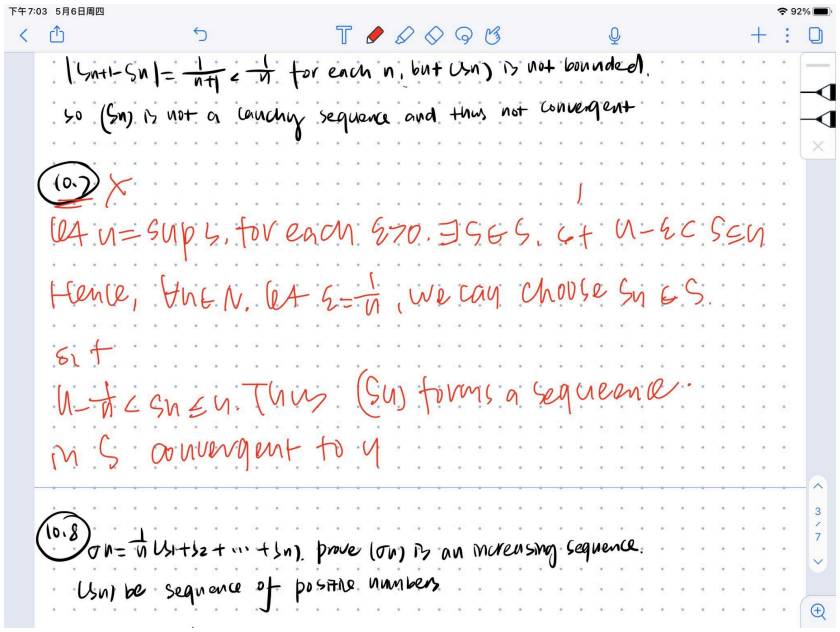
$\limsup(a_n + b_n) = 4$

$\limsup a_n = 2$ $\limsup b_n = 2$ $\therefore \limsup a_n + \limsup b_n = 2 + 2 = 4$

$\therefore \limsup(a_n + b_n) = \limsup a_n + \limsup b_n$

an example that $\limsup|a_n + b_n| < |\limsup a_n| + |\limsup b_n|$

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$|s_{n+1}-s_n| = \frac{1}{n+1} < \frac{1}{n}$ for each $n$, but $(s_n)$ is not bounded.

so $(s_n)$ is not a cauchy sequence and thus not convergent

(10.7) ✗

Let $u = \sup S$, for each $\varepsilon > 0$. $\exists s \in S$, s.t. $u - \varepsilon < s \le u$

Hence, $\forall n \in \mathbb{N}$. let $\varepsilon = \frac{1}{n}$, we can choose $s_n \in S$.

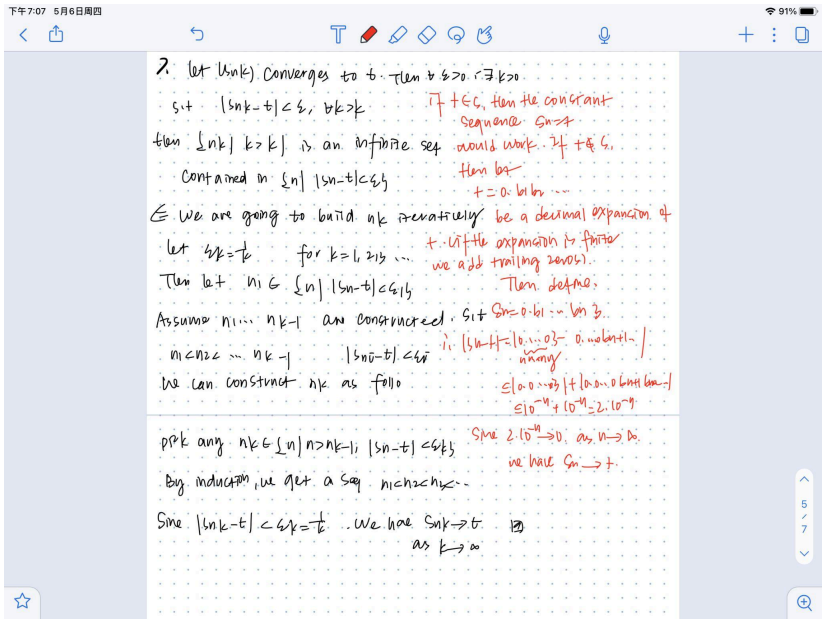
s.t.

$u - \frac{1}{n} < s_n \le u$. Thus $(s_n)$ forms a sequence.

in $S$ convergent to $u$

(10.8) $\sigma_n = \frac{1}{n}(s_1 + s_2 + \cdots + s_n)$. prove $(\sigma_n)$ is an increasing sequence.

$(s_n)$ be sequence of positive numbers



7. let $(s_{n_k})$ converges to $t$. Then $\forall \varepsilon > 0$. $\exists k > 0$

s.t. $|s_{n_k} - t| < \varepsilon$, $\forall k > k$

then $\{n_k \mid k > k\}$ is an infinite set

contained in $\{n \mid |s_n - t| < \varepsilon\}$

∈ we are going to build $n_k$ iteratively

let $\varepsilon_k = \frac{1}{k}$ for $k = 1, 2, 3 \ldots$

Then let $n_1 \in \{n \mid |s_n - t| < \varepsilon_1\}$

Assume $n_1, \ldots, n_{k-1}$ are constructed. s.t $n_1 < n_2 < \cdots < n_{k-1}$ $|s_{n_{i-1}} - t| < \varepsilon_{i-1}$

we can construct $n_k$ as follo

pick any $n_k \in \{n \mid n > n_{k-1}, |s_n - t| < \varepsilon_k\}$

By induction, we get a seq $n_1 < n_2 < n_3 < \cdots$

Since $|s_{n_k} - t| < \varepsilon_k = \frac{1}{k}$. we have $s_{n_k} \to t$ as $k \to \infty$ ∎

if $t \in \mathbb{Q}$, then the constant sequence $s_n = t$ would work. If $t \notin \mathbb{Q}$, then let $t = a_0.b_1b_2\ldots$ be a decimal expansion of $t$. (if the expansion is finite we add trailing zeros).

Then define:

$s_n = a_0.b_1 \ldots b_n$ for $n$.

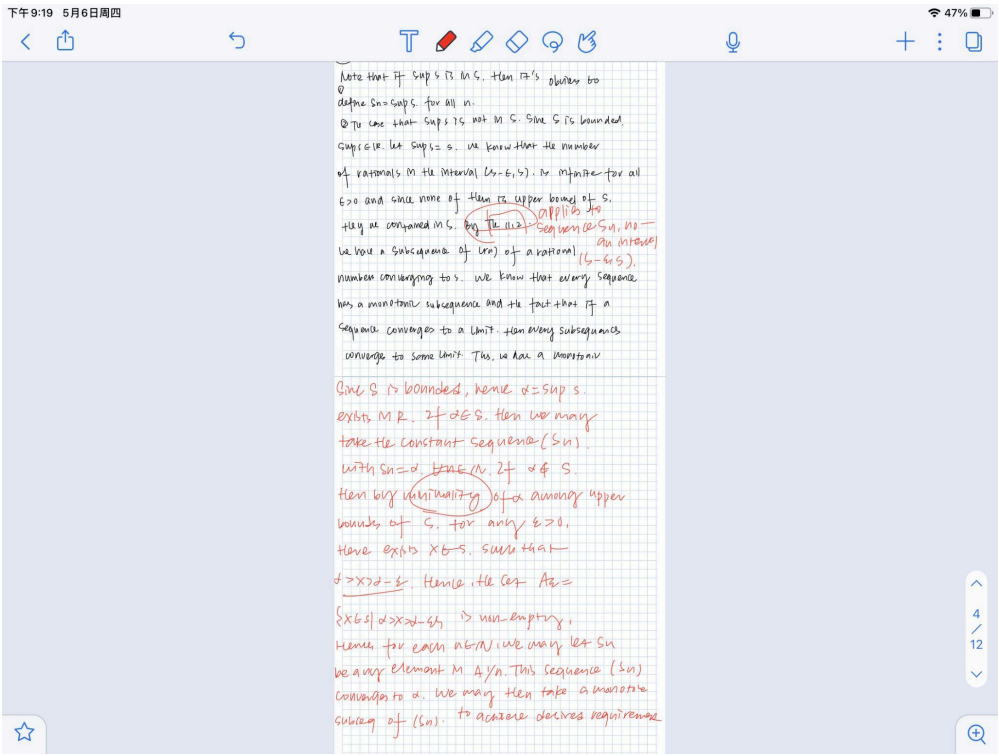
i.e. $|s_n - t| = |0.\ldots 0\underbrace{b_{n+1}\ldots}_{\text{many}}| \le |0.\ldots 0 b_{n+1} b_{n+2}\ldots|$

$\le 10^{-n} + 10^{-n} = 2 \cdot 10^{-n}$

Since $2 \cdot 10^{-n} \to 0$. as $n \to \infty$.

we have $s_n \to t$.

# Homework 4



## 4. Metric Space and Topology (Ross 2.13)

**Lecture 9**

- **Metric Space**: A metric space is a set S, together with a distance function $d:S\times S\to R$, such that 1. $d(x,y)\geq 0$, and $d(x,y)=0\Leftrightarrow x=y$
  2. $d(x,y)=d(y,x)$
  3. $d(x,y)+d(y,z))\geq d(x,z)$
- **Cauchy Sequence in a metric space (S,d):** A sequence (Sn) in S is cauchy if $\forall \epsilon>0$, there exists a N>0, such that $\forall$ n,m>N, $d(Sn,Sm)<\epsilon$
- **Convergence in a metric space (S,d):** A metric space (S,d) is complete, if every Cauchy sequence is convergent
- **Induced distance function:** if (S,d) is a metric space, and $A\subset S$ is any subset, then $(A,d\,|\,A\times A)$is a metric space
- **Completeness:** A metric space (S,d) is complete if every Cauchy sequence is convergent.
- **Bolzano-Weierstrass Theorem for R^n:** Every bounded sequence $(sm)m\in R^n$ has a convergent subsequence.
- **Topology:** let S be a Set. A topological structure on S is the data of a collection i of subset S, if $U\subset S$, and $U\in I$

**Homework 5**

- Prove well-defined first

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Ex 13.3 13.4 13.6 13.7)

Since $\{x_n\}$ and $\{y_n\}$ bounded
$\Rightarrow \{|x_n - y_n|\}$ is a bounded seq, hence $\sup\{|x_n - y_n|\}$ exists in $\mathbb{R}$. Hence $d(x,y) \in \mathbb{R}$. $\forall |x_n|, |y_n| < \infty$ in $\mathbb{R}$

13.3 Let $B$ be the set of all bounded sequences $x = (x_1, x_2, \dots)$ and define $d(x,y) = \sup\{|x_j - y_j| : j = 1, 2, \dots\}$

(a). Show $d$ is a metric for $B$

* need to show that $d$ is well-defined

Check

(i) $d(x,x) = \sup\{|x_j - x_j| : j = 1, 2, \dots\} = 0$ for all $x \in S$

$d(x,y) = \sup\{|x_j - y_j| : j = 1, 2, \dots\} > 0$

(ii) $d(x,y) = \sup\{|x_j - y_j| : j = 1, 2, \dots\} = \sup\{|y_j - x_j| : j = 1, 2, \dots\} = d(y,x)$ for all $x, y \in S$

(iii) $|x_j - z_j| \le |x_j - y_j| + |y_j - z_j|$.

$\therefore \sup\{|x_j - z_j| : j = 1, 2, \dots\} \le \sup\{|x_j - y_j| : j = 1, 2, \dots\} + \sup\{|y_j - z_j| : j = 1, 2, \dots\}$

$\therefore d(x,z) \le d(x,y) + d(y,z)$ for all $x, y, z \in S$

(b). (i) $d^*(x,x) = \sum_{j=1}^{\infty} |x_j - x_j| = 0$ for all $x \in S$.

$d^*(x,y) = \sum_{j=1}^{\infty} |x_j - y_j| = |x_1 - y_1| + |x_2 - y_2| + \cdots |x_j - y_j| > 0$.



13.4) Prove (ii) and (iv) in Discussion 13)

Assume include $\infty$ open sets

(ii) Union of a collection of open set is open

Let $U$ be a collection of open sets ~~$U_1, U_2, \dots U_n$~~.

① Let $\{U_\alpha\}_{\alpha \in A}$ be a collection of open sets

$X = \bigcup_{U_i \in U} (U_i)$.

Let $x \in \bigcup_{i \in N} (U_i)$. Then $x \in U_i$ for some $i \in N$.

If $p \in \bigcup U_\alpha$, then $\exists U_{\alpha_0}$ s.t. $p \in U_{\alpha_0}$.

$\because U_i$ is open. $r > 0$

$\therefore \{s \in U_i : d(x,s) < r\} \subseteq U_i \subseteq \bigcup_{i \in N} (U_i)$

Hence: $\exists \delta > 0$. $B_\delta(p) \subset U_{\alpha_0}$.

Since $U_{\alpha_0}$ is open

$\therefore B_\delta(p) \subset U_{\alpha_0} \subset \bigcup U_\alpha$.

Therefore $\bigcup_{i \in N} (U_i)$ is open

(iv) The intersection of finitely many open set is again an open set.

$A = \bigcap_{i=1}^{k} U_i$, $k \in N$. $U_i$ are open sets

Let $x \in \bigcap_{i=1}^{k} U_i$. Then $x \in U_i$ for $i \in \{1, 2, \dots k\}$

② If $p \in \bigcap_{i=1}^{N} U_i : U_i$ open $\Rightarrow \exists \delta_i > 0$. $\forall i \in \{1, \dots N\}$ s.t. $B_{\delta_i}(p) \subset U_i$. Let $\delta = \min\{\delta_i\}$

For each $U_i$, there exist $r_i$ such that

$\{s \in U_i : d(s,x) < r_i\} \subseteq U_i$

Let $r = \min\{r_1, \dots r_k\}$.

$\{s : d(x,s) < r\} \subseteq \{s : d(x,s) < r_i\}$ for $i \in \{1, 2, \dots k\}$

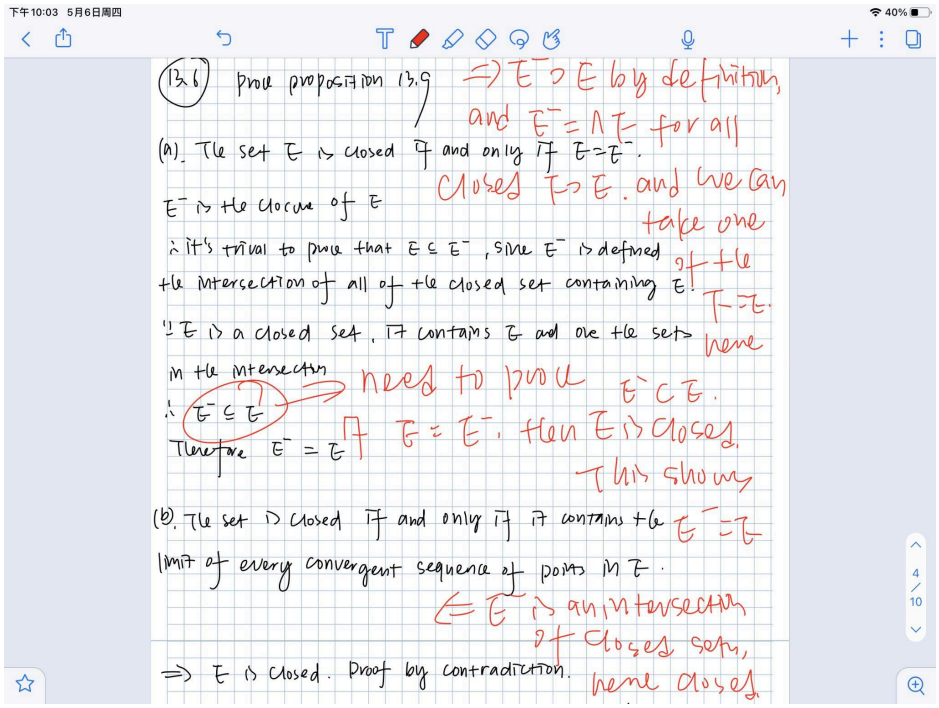
$\therefore \{s : d(x,s) < r\} \subseteq U_i$ for all $i \in \{1, 2, \dots k\}$

Then $B_\delta(p) \subset U_i$ $\forall i$

implies $\{s : d(x,s) < r\} \subseteq \bigcap_{i=1}^{k} U_i$.

$B_\delta(p) \subset \bigcap_{i=1}^{N} U_i$

Therefore $\bigcap_{i=1}^{k} U_i$ is an open set.




**Lecture 10**
**Midterm 1**