Math 104 Final Review:

Midterm 1

1. Completeness axiom of real numbers and limit of sequence of real numbers: (Ross 1.1, 1.2, 1.3, 1.4)
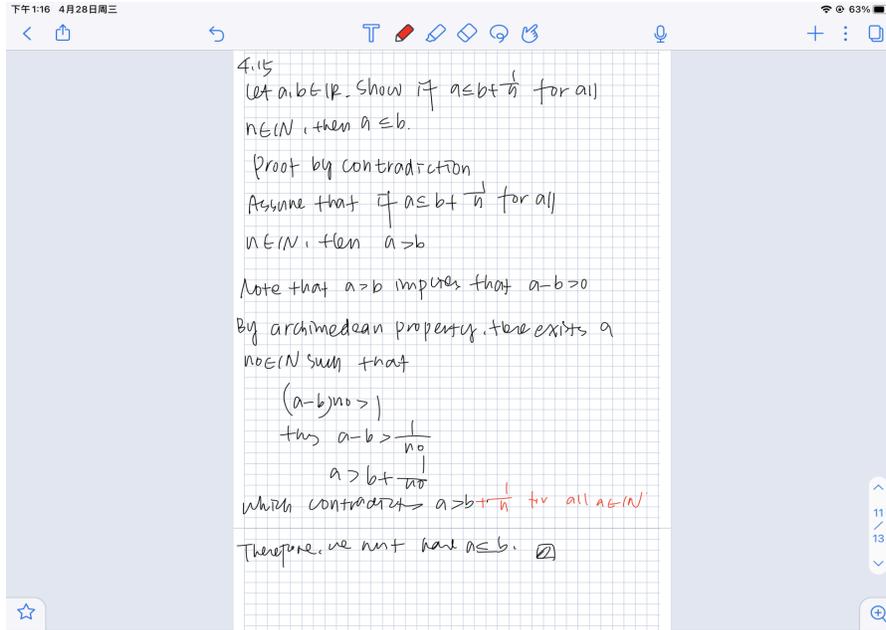
<p align="center">Lecture 1:</p>

- **Natural Numbers** N = {1, 2, 3, . . . }
- **Integers** Z = {. . . , −3, −2, −1, 0, 1, 2, 3, . . . }
- **Rational Numbers (ratios of integers)** Q = {p/q : p, q ∈ Z and q≠0}
- **Complex Numbers** C = {x + iy : x, y ∈ R}
- **Propositions**: if r is a rational number, (r=c/d, gcd(c,d)=1) and it's a root of the following integer coefficients polynomial: $C_n x^n + C_{n-1}x^{n-1}+…+C_0=0$, then d divides *Cn, c divides C0*
- **Corollary**: if r =c/d ≠0 is a root of a "monic polynomial", its lead term has coefficient is 1: $x^n+C_{n-1}x^{n-1}+…+C_0=0$ (we can apply this to prove that the roots of x^2-2=0 are not rational numbers)

<p align="center">Lecture 2:</p>

- **Supremum and Infimum**: Let S be a non-empty subset of R. If S is bounded above (i.e exists an upper bound for S), and S has a least upper bound, then we call it the supremum of S denoted as sup(S). If S is bounded below and S has a greatest lower bound, then we call it the infimum of S, denoted as inf (S).
  <mark>NOTE</mark>: the difference between sup, inf and max, min, for example, S=(0,1), max(S) doesn't exist, least upper bound(S)=1, greatest lower bound(S)=0
- **Completeness Axiom**: Every non-empty subset S ⊂ R that is bounded above has a least upper bound, and if S is bounded from below, then inf S exists.
- **Archimedean Property**: If a>0,b>0 are real numbers, then for n∈ N, we have an>b
- **Sequence**: a1,a2,a3,…,an,∈R, could be denoted as (an)n∈N
  <mark>NOTE</mark>: sequence is not a set. Sequence has its element come in order (first, second…). Set is just information about "who is in the set".
- **Some important propositions about inf and sup**:
  1. Suppose that A, B are nonempty sets of real numbers such that x ≤ y for all x ∈ A and y ∈ B. Then sup A ≤ inf B
  2. Suppose that A, B are subsets of R such that A ⊂ B. Then sup A ≤ sup B, and inf A ≥ inf B.
  3. If A ⊂ R, then M = sup A if and only if: (a) M is an upper bound of A; (b) for every M0 < M there exists x ∈ A such that x > M0 . Similarly, m = inf A if and only if: (a) m is a lower bound of A; (b) for every m0 > m there exists x ∈ A such that x < m0 .

<p align="center">Homework 1:</p>

4.15
Let $a, b \in \mathbb{R}$. Show if $a \leq b + \frac{1}{n}$ for all $n \in \mathbb{N}$, then $a \leq b$.

Proof by contradiction

Assume that if $a \leq b + \frac{1}{n}$ for all $n \in \mathbb{N}$, then $a > b$

Note that $a > b$ implies that $a - b > 0$

By archimedean property, there exists a $n_0 \in \mathbb{N}$ such that

$$(a-b)n_0 > 1$$
this $a - b > \frac{1}{n_0}$
$$a > b + \frac{1}{n_0}$$
which contradicts $a \leq b + \frac{1}{n}$ for all $n \in \mathbb{N}$

Therefore, we must have $a \leq b$. ☐

2. Limit of bounded monotonic sequence, limit theorem, liminf, limsup (Ross 2.7, 2.9, 2.10)

Lecture 3:

- **Limit of Sequence**: We say a sequence $(a_n)_n$ has limit $\alpha \in \mathbb{R}$, if $\forall \epsilon > 0$, there exists a real number $N > 0$, such that $n > N$, we have $|a_n - \alpha| < \epsilon$, denoted by $\lim_{n \to \infty} a_n = \alpha$
- **Property and tools to find limit**:
    1. All convergent sequences are bounded
    2. If $\lim a_n = \alpha$, and if $k \in \mathbb{R}$, then $\lim(k \cdot a_n) = k \cdot \alpha$
    3. Let a_n, b_n be convergent sequences, $\lim a_n = \alpha, \lim b_n = \beta$, then
        a. $\lim(a_n + b_n) = \lim(a_n) + \lim(b_n) = \alpha + \beta$
        b. $\lim(a_n \cdot b_n) = (\lim a_n) \cdot (\lim b_n) = \alpha\beta$
        c. If $a_n \neq 0$, $\forall n$ and if $\alpha \neq 0$, then, $\lim(1/a_n) = 1/\alpha$

Lecture 4:

- **Continued important theorems:**
    1. $\lim_{n \to \infty} 1/n^p = 0 \quad \forall p > 0$
    2. $\lim a^n = 0$, if $|a| < 1$
    3. $\lim_{n \to \infty} n^{(1/n)} = 1$
    4. $\lim_{n \to \infty} a^{1/n} = 1$ for $a > 0$
- **Cauchy Sequence:** if $\forall \epsilon > 0$, there exists $N > 0$, s.t $\forall n_1, n_2 > N$, we have $|a_{n_1} - a_{n_2}| < \epsilon$, (using graph--oscillation amplitude gets smaller and smaller)
- **Monotone Sequence**: an increasing sequence is such that $a_{n+1} \geq a_n$, a decreasing sequence is such that $a_{n+1} \leq a_n$, they are both called monotone sequences.

- **Theorems**:
    1. All bounded sequences are convergent
    2. Let (an) be a sequence, (an) is cauchy $\Longleftrightarrow an$ converges

Homework 2

3. # Subsequences and Cauchy Sequences: (Ross 2.10, 2.11)

Lecture 5

- **Monotone Sequence and Cauchy**:
    1. limsup: Let (*an*) be a sequence in R, limsup*an*=lim$N\rightarrow\infty$(sup *n>N an*)
    2. liminf :Let (*an*) be a sequence in R, liminf*an*=lim$N\rightarrow\infty$(inf *n>N an*)

    <mark>Remark:</mark> if we allow the notion of lim=+∞ or lim=-∞, then limsup exists in R or =-∞, liminf exists in R or =+∞

    3. A monotone increasing sequence has several properties;
        a. If it's bounded, then its limit exists
        b. If it's unbounded (without upper/lower bounds), then lim*an*=+∞
        c. Strategy: if we want to prove a=b, one way to prove it is $|b-a|<\epsilon$, for any $\epsilon>0$

- **Lemma:**
    a. if (an) is bounded sequence, then its limsup, liminf exists
    b. If (an) is bounded sequence, and α+=limsupan, then for any $\epsilon>0$, $\exists N$ such that $\forall n>N$, we have an<α+ +ϵ

- **Theorem:** let (an) be a bounded sequence. Then lim an exists⟺limsup(an)=liminf(an)

Lecture 6

- **Subsequence:** let (Sn) be a sequence of real numbers. Given a strictly increasing sequence of indices $n1<n2<n3<...nm<....$we define the corresponding subsequence as $tm;$=Snm, (tm) is called a subsequence of (sn).Sometimes, we write (Snk)k for the subsequence.
- **Lemma:**
- a. if (Sn) is convergent, then any subsequence converges to the same point
- b. If α=limSn exists in R, then there exists a subsequence that is also monotone.
- c. let (Sn) be any sequence. Then for any t∈ R, (Sn) has a subsequence converges to t ⟺ $\forall \epsilon>0$, the set $\{n\in N : |Sn\text{-}t|<\epsilon\}$ is infinite.

Lecture 7

- **Theorem:** (Bolzano-Weierstrass Theorem)
    a. Every bounded sequence has a convergence subsequence.
    b. Let (Sn) be a sequence, S be the set of subsequential limits of (Sn), then
        1. S in nonempty
        2. supS=limsup(Sn), infS=liminf(Sn)
        3. S=$\{\alpha\}$⟺ limSn exists and equals to α
    c. Let S be the set of subsequential limits of (sn). Suppose (tn) is a sequence in S∩R and t=lim(tn). Then t∈S

- **Subsequential limit:** let (Sn) be a sequence in R. A subsequential limit is any real number or the +∞,-∞, that is the limit of a subsequence
- **Lemma:** let (Sn) be any sequence. Then there exists a monotone sequence whose limit is limsup(Sn). Similarly, there exists a monotone sequence whose limit is liminf(Sn)

Lecture 8

- **Themorem:**
    a. let sn be a sequence with limit S>0, let tn be any sequence. Then limsup (sn.tn)=S. limsup(tn)
    b. Let (Sn) be a sequence of positive numbers, then we have liminf (Sn+1/Sn)≤liminf(Sn)^(1/n)≤limsup(Sn)^(1/n))≤limsup(Sn+1/Sn)

Homework 3

1.

(a) We define $\limsup \{s_n, s_{n+1}, \dots\} = \limsup s_n$
$\liminf \{s_n, s_{n+1}, \dots\} = \liminf s_n$

We know that $\inf\{s_n, s_{n+1}, \dots\} \le \sup\{s_n, s_{n+1}, \dots\}$ (a)
for each $n \in \mathbb{N}$.

Monotone convergence says that both $\inf\{s_n, s_{n+1}, \dots\}$
and $\sup\{s_n, s_{n+1}, \dots\}$ converge. Since they are bounded
and are monotone

$\sup\{s_n, s_{n+1}, \dots\} - \inf\{s_n, s_{n+1}, \dots\} \ge 0$ for each $n \in \mathbb{N}$

So $\liminf s_n \le \limsup s_n$

(b)

Let $A_N = \sup\{s_n \mid n \ge N\}$
$B_N = \inf\{s_n \mid n \ge N\}$
Then $A_N \ge B_N$.
Since (s_n) is bounded,
hence in A_N and $\lim B_N$ exists. Hence
$\lim A_N \ge \lim B_N$.
Thus $\limsup s_n \ge \liminf s_n$

Since $A_N \ge A_{N+1}$ and A_N is a bounded sequence,
hence $\lim A_N = \inf\{A_N \mid N \in \mathbb{N}\}$.
We can also prove it directly. Let $u = \inf\{A_N \mid N \in \mathbb{N}\}$.
Then $A_N \ge u$, $\forall N$. And for any $\varepsilon > 0$, $\exists N$, s.t.
$u + \varepsilon > A_N$, otherwise $u+\varepsilon$ would be a bigger
lower bound of $\{A_N\}$, contradict the def. of $\inf$.
$\Rightarrow \forall n > N$, we have
$u + \varepsilon > A_N \ge A_n \ge u \Rightarrow |A_n - u| < \varepsilon$
Hence, A_n converges to u.

2.

Define $A_k = \sup\{a_n : n \ge k\}$, $B_k = \sup\{b_n : n \ge k\}$, so
by definition $\limsup\limits_{n \to \infty} a_n = \lim\limits_{k \to \infty} A_k$ and similarly
for b_k is $\limsup\limits_{n \to \infty} b_n$

We consider $\lim\limits_{k \to \infty} C_k = \limsup\limits_{n \to \infty}(a_n + b_n)$

Then for all $n \ge k$, we have $a_n + b_n \le A_k + B_k$, because we extract
a_n by the supremum of all terms of (a_n) with $n \ge k$ and likewise
for the b_n.

$C_k = \sup\{a_n + b_n : n \ge k\} \le A_k + B_k$

This holds for all k, so we take $\lim$ on both sides as $k \to \infty$.

$\lim\limits_{k \to \infty} C_k = \lim\limits_{k \to \infty}(A_k) + \lim\limits_{k \to \infty}(B_k)$ $a_n = \begin{cases} 1 & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$ $b_n = \begin{cases} 0 & n \text{ even} \\ 1 & n \text{ odd} \end{cases}$

Thus $\limsup\limits_{n \to \infty}(a_n + b_n) = \limsup\limits_{n \to \infty} a_n + \limsup\limits_{n \to \infty} b_n$ ∅
$a_n + b_n = 1$
$\therefore \limsup(a_n + b_n) = 1$

e.g. a_n and b_n be sequences repeat in cycle of 4 $\limsup(b_n) = \limsup(b_n) = 1$

$a_n = (0,1,2,1,0,1,2,1,\dots)$ $b_n = (2,1,2,1,0,2,1,2,1,0,\dots)$
$a_n + b_n = (2,3,4,1,2,3,4,1,\dots)$ ⟵ not a strict example

$\limsup(a_n + b_n) = 4$

$\limsup a_n = 2$ $\limsup b_n = 2$ $\therefore \limsup a_n + \limsup b_n = 2+2 = 4$

$\therefore \limsup(a_n + b_n) = \limsup a_n + \limsup b_n$.

an example that $\limsup(a_n + b_n) < \limsup a_n + \limsup b_n$.

$|s_{n+1} - s_n| = \frac{1}{n+1} < \frac{1}{n}$ for each n, but (s_n) is not bounded.

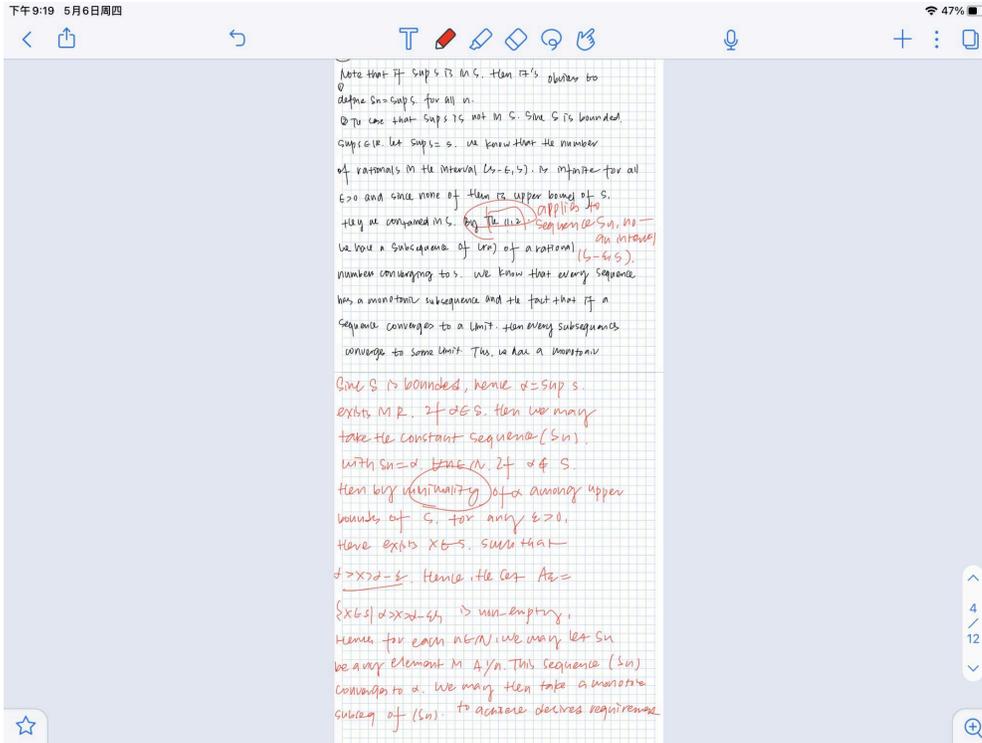So (s_n) is not a Cauchy sequence and thus not convergent.

(10.7) ✗

Let $u = \sup S$, for each $\varepsilon > 0$. $\exists s \in S$. s.t. $u - \varepsilon < s \leq u$.

Hence, $\forall n \in \mathbb{N}$. Let $\varepsilon = \frac{1}{n}$, we can choose $s_n \in S$.

s.t.

$u - \frac{1}{n} < s_n \leq u$. Thus (s_n) forms a sequence in S convergent to u.

(10.8) $\sigma_n = \frac{1}{n}(s_1 + s_2 + \cdots + s_n)$. prove (σ_n) is an increasing sequence. (s_n) be sequence of positive numbers

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7. Let (s_{nk}) converges to t. Then $\forall \varepsilon > 0$. $\exists k > 0$.

s.t. $|s_{nk} - t| < \varepsilon$, $\forall k > k$.

If $t \in S$, then the constant sequence $s_n = t$ would work. If $t \notin S$, then let $t = 0.b_1b_2\cdots$

then $\{n_k | k > k\}$ is an infinite set contained in $\{n | |s_n - t| < \varepsilon\}$. be a decimal expansion of t. (if the expansion is finite we add trailing zeros).

$\Leftarrow$ We are going to build n_k iteratively.

let $\varepsilon_k = \frac{1}{k}$. for $k = 1, 2, 3, \cdots$

Then let $n_1 \in \{n | |s_n - t| < \varepsilon_1\}$

Assume $n_1, \cdots, n_{k-1}$ are constructed. s.t. $s_n = 0.b_1 \cdots b_n 3$.

$n_1 < n_2 \cdots < n_{k-1}$, $|s_{n_i} - t| < \varepsilon_i$

we can construct n_k as follo

$\therefore |s_n - t| = |0.\cdots 03 - 0.b_1b_{n+1}\cdots|$

Then define.

$\leq |0.\cdots 3| + |0.\cdots 0 b_{n+1} b_{n+2} \cdots|$

$\leq 10^{-n} + 10^{-n} = 2 \cdot 10^{-n}$

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pick any $n_k \in \{n | n > n_{k-1}, |s_n - t| < \varepsilon_k\}$

By induction, we get a seq $n_1 < n_2 < n_3 \cdots$

Since $|s_{nk} - t| < \varepsilon_k = \frac{1}{k}$. We have $s_{nk} \to t$.
as $k \to \infty$

Since $2 \cdot 10^{-n} \to 0$. as $n \to \infty$. we have $s_n \to t$.

Note that if sup S B in S, then it's obvious to
define $S_n = \sup S$ for all n.
② In case that sup S is not in S. Since S is bounded,
$\sup S \in \mathbb{R}$. let $\sup S = s$. We know that the number
of rationals in the interval $(s-\epsilon, s)$ is infinite for all
$\epsilon > 0$ and since none of them is upper bound of S,
they are contained in S. By Th. 11.2 ⟶ applies to Seq (a_Sn, no
we have a subsequence of (rn) of a rational an interval
number converging to s. We know that every sequence $(s-\epsilon, s)$,
has a monotone subsequence and the fact that if a
sequence converges to a limit, then every subsequence
converges to some limit. Thus, we have a monotonic

Since S is bounded, hence $\alpha = \sup S$.
exists M.R. If $\alpha \in S$, then we may
take the constant sequence (S_n)
with $S_n = \alpha$. $\forall n \in \mathbb{N}$. If $\alpha \notin S$.
then by minimality of α among upper
bounds of S. for any $\epsilon > 0$,
there exists $x \in S$, such that
$\alpha > x > \alpha - \epsilon$. Hence, the set $A_\epsilon =$
$\{ x \in S | \alpha > x > \alpha - \epsilon \}$ is non-empty.
Hence, for each $n \in \mathbb{N}$ we may let S_n
be any element in $A_{1/n}$. This sequence (S_n)
converges to α. We may then take a monotone
subseq of (S_n). to achieve desires requirement

4. Metric Space and Topology (Ross 2.13)

Lecture 9

- **Metric Space**: A metric space is a set S, together with a distance function d:S×S→R, such that 1. d(x,y)≥0, and d(x,y)=0⟺x=y
  2. d(x,y)=d(y,x)
  3. d(x,y)+d(y,z))≥d(x,z)
- **Cauchy Sequence in a metric space (S,d):** A sequence (Sn) in S is cauchy if $\forall \epsilon > 0$, there exists a N>0, such that $\forall$ n,m>N, d(Sn,Sm)<ϵ
- **Convergence in a metric space (S,d):** A metric space (S,d) is complete, if every Cauchy sequence is convergent
- **Induced distance function:** if (S,d) is a metric space, and A⊂S is any subset, then (A,d|A×A)is a metric space
- **Completeness:** A metric space (S,d) is complete if every Cauchy sequence is convergent.
- **Bolzano-Weierstrass Theorem for R^n:** Every bounded sequence (sm)m∈R^n has a convergent subsequence.
- **Topology:** let S be a Set. A topological structure on S is the data of a collection i of subset S, if U⊂S, and U∈I

Homework 5

- Prove well-defined first

Math 104 HW 5
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Ex 13.3 13.4 (13.6 13.7)

Since $\{x_n\}$ and $\{y_n\}$ bounded thus (x_n, y_n) is a bounded seq. hence $\sup\{|x_n - y_n|\}$ exists in $\mathbb{R}$. Hence $d(x,y)\in\mathbb{R}$, $d(x,y)$ well-defined

13.3 Let B be the set of all bounded sequences $x = (x_1, x_2, \cdots)$ in $\mathbb{R}$ and define $d(x,y) = \sup\{|x_j - y_j| : j = 1, 2, \cdots \}$

(a) Show d is a metric for B

※ need to show that d is well-defined

Check

(i) $d(x,x) = \sup\{|x_j - x_j| : j = 1, 2, \cdots \} = 0$. for all $x \in S$.
$d(x,y) = \sup\{|x_j - y_j| : j = 1, 2, \cdots \} > 0$

(ii) $d(x,y) = \sup\{|x_j - y_j| : j = 1, 2, \cdots \} = \sup\{|y_j - x_j| : j = 1, 2, \cdots \} = d(y,x)$ for all $x, y \in S$

(iii) $|x_j - z_j| \leq |x_j - y_j| + |y_j - z_j|$.
$\therefore \sup\{|x_j - z_j| : j = 1, 2, \cdots \} \leq \sup\{|x_j - y_j| : j = 1, 2, \cdots \} + \sup\{|y_j - z_j| : j = 1, 2, \cdots \}$

$\therefore d(x,z) \leq d(x,y) + d(y,z)$ for all $x, y, z \in S$

(b) (i) $d^*(x,x) = \sum_{j=1}^{k} |x_j - x_j| = 0$ for all $x \in S$.

$d^*(x,y) = \sum_{j=1}^{k} |x_j - y_j| = |x_1 - y_1| + |x_2 - y_2| + \cdots |x_k - y_k| > 0$.

13.4 Prove (iii) and (iv) in Discussion 13.7

(iii) Union of a collection of open set is open

U be a collection of open sets $\{u_1, u_2, \cdots u_n\}$

$X = \bigcup_{i \in \mathbb{N}} (u_i)$.

Let $x \in \bigcup_{i \in \mathbb{N}} (u_i)$. Then $x \in u_i$ for some $i \in \mathbb{N}$.

$\because u_i$ is open. $r > 0$

$\therefore \{s \in u_i : d(x,s) < r \} \subseteq u_i \subseteq \bigcup_{i \in \mathbb{N}} (u_i)$.

Therefore $\bigcup_{i \in \mathbb{N}} (u_i)$ is open

(iv) The intersection of finitely many open set is again an open set.

$A = \bigcap_{i=1}^{k} u_i$, $k \in \mathbb{N}$. u_i are open sets

Let $x \in \bigcap_{i=1}^{k} u_i$. Then $x \in u_i$ for $i \in \{1, 2, \cdots k\}$

For each u_i, there exist r_i such that

$\{s \in u_i : d(s,x) < r_i\} \subseteq u_i$

Let $r = \min\{r_1, \cdots r_k\}$.

$\{s : d(x,s) < r\} \subseteq \{s : d(x,s) < r_i\}$

$\therefore \{s : d(x,s) < r\} \subseteq u_i$ for all $i \in \{1, 2, \cdots k\}$

implies $\{s : d(x,s) < r\} \subseteq \bigcap_{i=1}^{k} u_i$

Therefore $\bigcap_{i=1}^{k} u_i$ is an open set.

Assume include 3 open sets

① Let $\{u_\alpha\}_{\alpha \in A}$ be a collection of open sets

if $p \in \bigcup_{} u_\alpha$, then $\exists \alpha_0 \in A$ s.t $p \in u_{\alpha_0}$.

Hence $\exists \varepsilon > 0$, $B_\varepsilon(p) \subset u_{\alpha_0}$, since u_{α_0} is open

$\therefore B_\varepsilon(p) \subset u_{\alpha_0} \subset \bigcup_{} u_\alpha$.

② if $p \in \bigcap_{i=1}^{N} u_i$, u_i

open, then $\exists \delta_i > 0$, $u_i \in \{1, \cdots N\}$

s.t $B_{\delta_i}(p) \subset u_i$. Let $\delta = \min\{\delta_i\}$, $i \in \{1, 2, \cdots k\}$

Then $B_\delta(p) \subset u_i$ $\forall i$

$B_\delta(p) \subset \bigcap_{i=1}^{N} u_i$

(13.6) Prove proposition 13.9

⟹ $E^- \supset E$ by definition,
and $E^- = \cap E_i$ for all

(a). The set E is closed if and only if $E = E^-$.

E^- is the closure of E

closed $E_i \supset E$. and we can
take one

∴ it's trivial to prove that $E \subseteq E^-$, since E^- is defined
the intersection of all of the closed set containing E!

of the
$E_i = E$.
hence

∵ E is a closed set, it contains E and one the sets
in the intersection

∴ $E^- \subseteq E$

need to prove $E^- \subseteq E$.

Therefore $E^- = E$

If $E = E^-$, then E is closed.

This shows
$E^- = E$

(b). The set is closed if and only if it contains the
limit of every convergent sequence of points in E.

⟸ E is an intersection
of closed sets,
hence closed

⟹ E is closed. Proof by contradiction.

Lecture 10
Midterm 1