

Notes on "Mathematics" from *A Philosopher Looks at Science* by John G. Kemeny

Introduction:

Most high school students never really get the chance to consider the operations of mathematics, but are simply bribed into learning mathematics on the basis of its practical applications. Kemeny proposes to start from scratch

What Mathematics is not:

Mathematics is not self-evident, something every student suspects when their teacher tried to convince them that it was self-evident that the base angles of an isosceles triangle are equal.

As an example of this problem, consider the following famous statement:

"Every even number can be written as the sum of two prime numbers." [A prime number is a number not evenly divided by any number with the exception of 1 and itself]

So, for example $2 = 1 + 1$, $4 = 1 + 3$, etc. But is this true for all prime numbers? Well, H. Goldbach presented this proposition (hunch) years ago, and it has not been proven.
MATHEMATICAL PROPOSITIONS ARE NOT SELF-EVIDENT

Mathematics is also not based upon observation. What kinds of mathematical propositions are there? Well there are four *possible* combinations

Analytic a priori: need only consider the meanings of the terms to determine truth or falsity

Analytic a posteriori: none of these types, because analytic statements do not rest upon experience

Synthetic a priori: these are the Kantian categories, and assume only one possible view of the universe, so they also don't exist

Synthetic a posteriori: factual statements whose truth can only be decided on the basis of observation

Let us take the last view, that propositions depend upon factual observation, as JS Mill argued.

G.S. Hempel points out that if we want to test $3 + 2 = 5$, we could put three microbes on a slide, and then add two. However, if we then found six microbes, we would not consider this a refutation of the mathematical proposition.

How about the logical proposition $a = b$ and $b = c$, therefore $a = c$? If we took one shade of blue, then another just like it, and a third just like the second, and then noticed a difference between one and three, we would conclude that we were wrong about the differences in color. In both examples, we don't rely upon experience to give us the answer.

The upshot of this is that mathematical statements are analytic apriori. Since we have learned this tells us nothing about experience, why bother with them? In order to find out, let us now ask what mathematics is.

What is Mathematics?:

Our starting point should be our last effort to establish certainty, logic: what is the relationship between mathematics and logic? And this inevitably leads us to the issue of "form."

Let us start with an example of form, translating "all bald men are bald" into "all ____ are _____. In this case the lines are used to indicate the same word in the premise and conclusion. The virtue of the statement is that this form can be applied to any sort of proposition and it will be true. *The strength of logic is to find results with far-reaching applications.*

Now, as we discovered in logic, it is easier to substitute the blanks with letters, or "variables." It does not matter what we replace the variables with, the truth remains the same. However, logic cannot function simply by studying form, since it needs to express the relationship of the variables through something called a "constants." "All," "are," "if...then," are examples of logical constants, and are different from the subject-matter constants or variables.

Logic then studies the meaning of logical constants and the way they enter into forms of propositions. The advantage of logic is that it is applicable to any subject matter, because its form is general. While it is more work to examine things in such a general fashion, in the long run it saves time, since one general argument can replace infinitely many specific arguments.

So, what is the relationship of logic to mathematics? According to Russell, "logic is the youth of mathematics, and mathematics is the adulthood of logic." Let us turn to the issue of the identity of the two fields, which was done at the turn of the century. Mathematics is founded on the properties of integers (whole numbers); from them the rest of mathematics is deducible by purely logical arguments.

In the late 19th century, Guiseppe Peano tried to show that the properties of all integers follow from five simple postulates, that is to say, purely in accordance with the rules of logic. The program was incomplete. In the meantime, Mathematical logic (logic strengthened by the use of symbolic methods) was developed. In *Principia Mathematica* Russell and Whitehead showed that Peano's concepts could be defined in terms of logical constants and their properties demonstrated by pure logic. [It did involve two controversial new axioms, infinity and choice]

What did all this prove? That mathematics is a study of the form of arguments and is the most general branch of knowledge. It is, however, entirely devoid of subject matter (this task is left to subject matter constants [variables] which are outside the realm of logic.)

Now we may again ask: what good is mathematics? Well, if you take a simple example the reply may be "not much." It shows that $365-1=364$ implies "-1" *means* going from a number to the previous one, " $=$ " *means* being the same number, and hence *means* the proposition asserts that the number 364 is the number right before 365. All this is true because this is how we named our numbers.

However consider the following: "every number can be written as a product of primes, and in a unique way," or "the number of the primes up to n divided by n and multiplied by natural logarithm of n tends to 1," it becomes clear sometimes the meanings are far from obvious, and it can take centuries to prove them

Here is the "so what" in regard to mathematics: while it brings us nothing new, it brings *us facts that are new to us*, that we did not realize that we possessed. In a mathematical proof, we start with a few propositions which for some reason we accept, and then we come up with entirely new propositions which we must also accept, because they are contained in what we have said before. And, thanks to the complete generality of mathematics, it can do this for any subject whatsoever.

The Relation of Mathematics to Science

The ideal branch to witness mathematics in action is geometry. While it dates back to the Egyptians, we know it from Euclid, who laid down a set of axioms from which all true geometrical propositions followed by pure logic. We now know Euclid fell short on two accounts 1) it was redundant, containing definitions which served no purpose; 2) it made some hidden assumptions that did not follow from the axioms. These were repaired in this century the German mathematician David Hilbert, whose *Plane Geometry* has as undefined terms "point," line, "is on the line," "lies between the points," "are at equal distances," and "are equal angles. These are used to define all other terms, and from these Hilbert derived a list of axioms.

Why not prove the axioms too? Well mathematicians have tried, but found that you can't prove something from nothing, that you always must have a starting point. Euclid found out as much when he attempted to claim as self-evident the axiom that "given a line, and a point not on the line, there is just one line through the point parallel to the line. There was of course no practical test to check whether the two lines never met.

For two thousand years embarrassed mathematicians tried various remedies, and ultimately they decided that this was not an axiom, but a theorem. Geometry must follow from self-evident axioms and if this proposition was not self-evident, then it must follow from the remaining self-evident axioms, which it did not.

The story took a dramatic turn around the beginning of the 18th century, when an ingenious approach was suggested, that of indirect proof: assume that the parallel axiom does not follow from the others--then we will not get a contradiction if we deny it. Let us assume that we can draw more than one parallel to a given line through a given point, and show that this leads to an absurdity." The problem was they detected no absurdity.

In fact, the absurdity was never discovered. However, simultaneously John Bolyai and N.I. Lobachevski, failing to discover the contradiction, succeeded in drawing the conclusion that more than one type of geometry exists, work which was extended by G.F. Gauss and G.F.B. Riemann. They showed that there was a third type of geometry, as well as an infinite number of "mixed" geometries.

The problem that plagues the attempt of mathematics to establish in a purely logical fashion all of its axioms is quite simply that things such as Hilbert's "undefined points" do not belong to the sphere of mathematics or logic. "Point" or "line" means nothing to the geometer, and you could well substitute "dog" for "point," "cat" for line, "dog bites cat" for points on the line and the theorems will follow just as consistently. Of course this is not what mathematicians do. They deal strictly with the skeletal form of the relationship or "x bears a relation R to y" or "xRy." So, no matter what these subject matter constants mean (and they don't mean anything until we give them a meaning) the axioms still follow from the theorems by virtue of their form. *Therefore, if we can find any meaning at all for these six entities which make the axioms true, the theorems will be true under the same interpretation.* This shows both the strength and weakness of mathematics. It can prove perfectly general statements true about all possible applications. On the other hand, it cannot prove the truth of any one statement in an application--that is left to science.

First Sytematic Treatment by	The Three Geometries		
	Bolyai & Lobachevski	Euclid	Reimann
Sum of the angles of a triangle	> 180	180	<180
The Space Curves	out	doesn't	in
Volumes Increase	rapidly	normally	slowly
To a given line, through a Point, the # of parallels	infinite	Cube of radius 1	0
Such a universe would be	infinite	infinite	finite

Behaves like a

saddle

plane

sphere

Ultimately, all geometry tells the scientist is: "Find me x's, y's, R's which make these axioms true, and I will present you with an infinity of true statements--all my theorems. An immediate consequence of this is that mathematics can tell you nothing about the truth of your axioms; as a matter of fact, it is foolish to ask one way or the other the axioms are true. It all depends upon what x, y, and R are. If it is dog bites cat, then it is false, but if it is, for example men liking women, it would be true.

The bottom line is that statements of Mathematics are only forms and their truth can only be considered if we fill in the blanks, giving them and interpretation. [This is the distinction between pure and applied mathematics]

The question occurs "How can you say that geometrical axioms are only forms, neither true nor false, when science makes such frequent use of them? The answer is that when science uses them, they automatically fill in the blanks: they have very definite meanings in mind for each of the six terms. But this is a factual question which mathematics cannot answer. The mathematician can only say that if the universe is such that the axioms (so interpreted) happen to be true, then all the theorems (similarly interpreted) will also be true.

In respect to geometry, the answer about the nature of the universe was given by Einstein, a physicist, who argued the universe was a mixture of different types, varying from place to place, according to the distribution of matter in it. Since we currently believe that the universe as a whole is finite, though endless, curving back on itself, it is apparent that it is not Euclidean. Speaking of the relationship of science to math, Einstein remarked "As far as the laws of mathematics refer to reality, they are not certain, and as far as they are certain, they do not refer to reality."

Then why is mathematics so vital for science? Because *it is the ideal language in which to formulate our scientific theories, and it tells us just what our theories imply.*

Contrary to popular belief, we rarely test a scientific theory directly. Rather, we test its logical or mathematical consequences. In the case of Euclid's parallel axiom, there is no method for testing it directly. However, one of the consequences of this axiom along with the others is that the sum of the angles of a triangle is 180 degrees. In the other two types of geometry we get sums of more or less than 180 degrees. This can be tested. The catch in comparing these forms of geometry is the real world problem of scale. For small triangles with sides no more than a few billion miles, the difference between the three is negligible, and so Euclidean geometry in most cases suffices. However, on a celestial scale, it would not yield accurate results.

This can be more clearly seen in the formula for volumes. For Euclid volume increases as the cube of the distance. In Riemann geometry this happens more slowly, while in

Bolyai and Lobachevski's geometry it increases more rapidly. Applying this, our present observations indicate that the number of the nebulae (large group of stars) increases more slowly than the cube of the distance, thus--assuming they are evenly distributed (which we do)--it seems as though the universe curves in. However, recent data casts doubt on this conclusion. If this is false, we search for another mathematical formulation to describe the phenomena.

Pure vs. Applied Mathematics:

In order to understand how mathematics itself operates, let us construct a simple mathematical system. We will begin by stating it in its pure form—with just the variables—and then—to make it easier to follow—we will supply subject matter to the axioms.

Axiom 1: There is an x such that no other x bears the relation R to it

Axiom 2: No x has more than one x bearing the relation R to it

Axiom 3: Each x bears the relation R to just one other x , and never bears this relationship to itself

Now that we have stated the system in its pure form, let us interpret the x 's as dogs and the relation R as biting.

Axiom 1': There is a dog who is not bitten

Axiom 2': No dog gets bitten by more than 1 dog

Axiom 3': Each dog bites one other dog, but never bites itself

It is clear that these axioms are not in fact true: the second is false and probably the first part of Axiom 3'. But the mathematician does not care about this; he is only claiming that if the axioms are true, then certain other facts will also follow. Let us now derive one consequence from this mathematical system, using the above interpretation to make it easier to follow.

Theorem: There are infinitely many dogs

Proof: Using the indirect method, we can assume that there are a finite number of dogs in order to get a contradiction that proves the theorem (there are only two options—finite or infinite).

1) From Axiom 1' we know that there is a dog who is not bitten (1st dog)

2) From Axiom 3' we know that the 1st dog bites another dog, which we can call the 2nd dog, and then he bites the 3rd dog, and so on.

3) Now let us consider the last dog: whom does it bite? It must bite some other dog (Axiom 3'), he can't bite himself (Axiom 3', second part), so it must bite an earlier dog. However, it can't be the 1st dog (Axiom 1') and all the other dogs on the list got bitten by the dog before them, and hence they can't be bitten by the last dog (Axiom 2').

Conclusion: there is no dog for the last dog to bite, which contradicts Axiom 3'. Thus there must be infinitely many dogs.

Now this is a perfectly rigorous mathematical proof, an excellent example of a theorem which, while it contains nothing new, nevertheless is capable of furnishing us with information we previously did not have. For example, if we know that there are only a finite number of dogs in the world, we can and must reject some of the axioms. Does this make the axioms useless? No, for there could be entirely different interpretations which make the axioms true. For Example

Axiom 1": There is a year A.D. which is not preceded by any other year A.D.

Axiom 2": No year A.D. is preceded by more than one such year

Axiom 3": Each year A.D. precedes one other such year, but never precedes itself

These are clearly true: the 1st Axiom states the existence of the year 1 A.D., the 2nd Axiom that there is always at most one previous year, and the 3rd that there is always a next year. Since all of the axioms so interpreted are true, all theorems so interpreted must be true. From this we learn that it is true that time is endless, which is also true.

This approach not only shows how mathematical systems operate to prove things, it clearly shows the difference between pure and applied mathematics, a very important distinction in mathematics. Pure mathematics states that any entities x and relation R which happen to satisfy the three axioms must also satisfy the theorems. It says nothing about what x and R are, and nothing about the truth of the axioms, which are so far only forms and hence cannot be said to be true or false. We pass to applied mathematics when we interpret the variables, by stating what x and R are. Then it becomes a matter of fact whether the axioms are true, and the problem is taken out of the hands of pure mathematics and put in the hands of science.

Applied mathematics belongs to science, and this distinction provides the final answer as to the status of mathematical propositions. Mathematical propositions are analytic a priori, while the applied mathematical propositions of science are in fact synthetic a posteriori

To put this another way, the x 's, y 's, and R 's in mathematics are undefined: they can be used in turn to define other words, but they form part of the basic vocabulary which is undefined. For example, in Geometry angles are defined in terms of lines, but the lines are never defined *explicitly*. However, while we don't define them, we do offer an implicit definition of lines, by stating that they are the kinds of things that have the properties stated in the three axioms, to use the example above. Now this "relationship" is of interest to a scientist, for when he wants to examine some phenomenon, he looks for a mathematical systems whose axioms and terms have the sort of relationship as that found in the subject matter which he is studying. So, if a scientist has some entities that he or she wants to study, together with the relationships that are found between the entities, he searches pure mathematics for a system that displays this same relationship. This is called "forming a theory."

The discussion of forming a theory not only distinguishes between pure and applied mathematics, it also shows why pure mathematics is of vital importance. One never knows what mathematical systems will ultimately turn out to be useful. For example, the alternate Geometries described earlier were only “applied” much later, to meet the new requirements of our views of the universe set forth by the General Theory of Relativity.

How does science “interpret” mathematical propositions? A scientist takes a certain mathematical propositions and substitutes for the variable certain constants, expressing concepts that can be directly observed or measured. In this manner they take mere forms and, by filling in the blanks, turn them into statements of facts that he or she can check. For example, if we interpret the geometrical axiom that two points determine a line by the Euclidean interpretation, we get the factual statement that there is just one possible light path through any two points in space. That is open to scientific experimentation.

Can All Sciences Use Mathematics?

In fact, all sciences *must* use mathematics. When scientists state theories, they are practicing mathematics. In essence a scientist chooses a mathematical system with the proper relationship between the variables, and replaces the blanks or variables with the subject matter.

Take as an example, a scientific theory that does not seem to be explicitly mathematical, such as the biological theory that the development of the individual in the early stage mirrors the development of the race. We would begin by arranging both embryos and the human race into “stages”, and therefore get two sets of series. We would then state that the stages in one series correspond to the stages in the other in such a way that wherever one of these properties shows up in the embryo, the same property showed up in the race at the corresponding stage. Mathematically speaking, we have x ’s (stages of embryos) y ’s (stages of the race) a relation R between them of correspondence, and certain properties in each stage that can be investigated.

How would we solve this mathematical problem? We would first have two statements that the x ’s and y ’s form two series. This would give us, for each series, axioms like those used in the examples above. Then we would have to state that R is a correspondence, for axioms, the axioms for which are well known in mathematics, and that the properties always belong to the corresponding stages. This what the mathematician means when he or she says that they have two isomorphic series (isomorphic with respect to the given properties) The study of isomorphism belongs to the area of mathematics called Topology, one of the many branches of mathematics that does not use numbers.

In the example above, we see that all of science uses mathematics, and often it borrows from pure mathematics for its tools. The development of scientific theory depends upon the development of mathematics for its ability to analyze the world. Mathematics then is

the irreplaceable language of science, even though it by itself never provides any new information. Yet it must be used constantly for us to realize what we already have. It never ceases to amaze us that its seemingly simple statements have consequences that we have never dreamed of.