For repeated multiplication of *square* matrices, you can use the same exponential notation used with real numbers. That is, $A^1 = A$, $A^2 = AA$, and for a positive integer k, A^k is

$$A^k = \underbrace{AA \cdot \cdot \cdot A}_{k \text{ factors}}.$$

It is convenient also to define $A^0 = I_n$ (where A is a square matrix of order n). These definitions allow you to establish the properties

1. $A^jA^k = A^{j+k}$ and 2. $(A^j)^k = A^{jk}$

where j and k are nonnegative integers.

Example **Repeated Multiplication of a Square Matrix**

Find A^3 for the matrix $A = \begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix}$.

Solution $A^3 = \left(\begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix}\right)\begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 6 & -3 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} -4 & -1 \\ 3 & -6 \end{bmatrix}$

Definition:- Transpose of matrix

The transpose of matrix mxn $A = [a_{ij}]$ is define as the nxm matrix $B = [b_{ij}]$ with $b_{ij} = a_{ji}$ for $1 \leq i \leq m$ *and* $1 \leq j \leq n$, and denoted by A^t.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix},$$

Size: $m \times n$

then the transpose, denoted by A^T, is the $n \times m$ matrix below

$$A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} & \cdots & a_{m1} \\ a_{12} & a_{22} & a_{32} & \cdots & a_{m2} \\ a_{13} & a_{23} & a_{33} & \cdots & a_{m3} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & a_{3n} & \cdots & a_{mn} \end{bmatrix}.$$

Size: $n \times m$

Theorem:1(Properties of Transposes)

If A and B are matrices (with size such that the given matrix are defined) .and c is a scalar, then the following properties are true

1- $A=\left(A^{T}\right)^{T}$.
2- $(A+B)^{T}=A^{T}+B^{T}$.
3- $(cA)^{T}=c(A)^{T}$.
4- $(AB)^{T}=B^{T}A^{T}$.

Proof:

1- $A=\left(A^{T}\right)^{T}$

As $A=[aij]$ then $A^{T}=[aji]$ it be $A=[aij]=[aji]^{T}=\left(A^{T}\right)^{T}$

2- $(A+B)^{T}=A^{T}+B^{T}$

$$(A+B)^{T}=([a_{ij}]+[b_{ij}])^{T}=[a_{ij}+b_{ij}]^{T}=[a_{ji}+b_{ji}]=[a_{ji}]+[b_{ji}]=A^{T}+B^{T}$$

3- $(cA)^{T}=c(A)^{T}$. **(H.W)**

4- $(AB)^{T}=B^{T}A^{T}$. **(H.W)**

Example Finding the Transpose of a Product

Show that $(AB)^T$ and B^TA^T are equal.

$$A = \begin{bmatrix} 2 & 1 & -2 \\ -1 & 0 & 3 \\ 0 & -2 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 1 \\ 2 & -1 \\ 3 & 0 \end{bmatrix}$$

Solution

$$AB = \begin{bmatrix} 2 & 1 & -2 \\ -1 & 0 & 3 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & -1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 6 & -1 \\ -1 & 2 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} 2 & 6 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$B^TA^T = \begin{bmatrix} 3 & 2 & 3 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & -2 \\ -2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 6 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$(AB)^T = B^TA^T$$

Remark:-

- We say that the square matrix A is Symmetric if $A^t = A$ such as : If $A = \begin{bmatrix} 2 & 7 \\ 7 & 4 \end{bmatrix}$ then $A^t = \begin{bmatrix} 2 & 7 \\ 7 & 4 \end{bmatrix}$.
- We say that the square matrix A is Orthogonal if

$$A.\ A^T = A^T.A = I$$

Example The Product of a Matrix and Its Transpose

For the matrix

$$A = \begin{bmatrix} 1 & 3 \\ 0 & -2 \\ -2 & -1 \end{bmatrix}$$

Solution

find the product AA^T and show that it is symmetric.

Because

$$AA^T = \begin{bmatrix} 1 & 3 \\ 0 & -2 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 3 & -2 & -1 \end{bmatrix} = \begin{bmatrix} 10 & -6 & -5 \\ -6 & 4 & 2 \\ -5 & 2 & 5 \end{bmatrix}$$

it follows that $AA^T = (AA^T)^T$, so AA^T is symmetric.

Example

- If A and B are two matrices then $(A+B)^t = A^t + B^t$.

Show that by the following example if $A = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix}$ ....(H.W).