Linear Algebra MAT313 Fall 2022
Professor Sormani
Lesson 15
Proofs with Matrices

Don't forget your team challenge!
Look for 313F22-Oct-Team in googledocs and check your teammates work and add a step of the proof!

You will cut and paste the photos of your notes and completed classwork in a googledoc entitled:

MAT313F22-lesson15-last-first

and share editing of that document with me sormanic@gmail.com. You will also include your homework and any corrections to your homework in this doc.

If you have a question, type QUESTION in your googledoc next to the point in your notes that has a question and email me with the subject MAT313 QUESTION. I will answer your question by inserting a photo into your googledoc or making an extra video.

Today we have two parts:

Part I teaches basic proofs and has six required HW problems

Part II uses sum notation for proofs and has extra credit problems

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Part I: **Watch Playlist 313F20-15-PartI**
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Lesson on Proofs with Matrices

In this lesson we will prove:

Thm Distribution of Matrix Mult over Add

$$A \times (B + C) = A \times B + A \times C$$

Thm Associatity of Matrix Multiplication

$$A \times (B \times C) = (A \times B) \times C$$

Defn: The **zero matrix**, O, is a matrix which has zeroes everywhere.

Thm $A \times O = ?$ and $O \times A = ?$

Defn The **identity matrix, I**, is a is a square matrix with 1's on the diagonal and zeroes elsewhere.

Thm $A \times I = ?$ and $I \times A = ?$

What about

$$A \times B \stackrel{?}{=} B \times A \, ?$$

Defn of Matrix Mult

$A \in \mathcal{M}_{n \times m}$ (m columns, n rows)

$$[A \times B]_{ik} = \sum_{j=1}^{m} a_{ij} b_{jk}$$

$B \in \mathcal{M}_{m \times \ell}$ $\quad A \times B \in \mathcal{M}_{n \times \ell}$

$$A \times B = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1\ell} \\ b_{21} & b_{22} & b_{23} & \cdots & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ b_{m1} & & b_{m3} & & b_{m\ell} \end{pmatrix} = ?$$

j counter goes across → the i^{th} row of A

↓ j counter goes down in the k^{th} column of B

$[A \times B]_{ik}$ = dot product of i^{th} row of A, k^{th} column of B

Defn of Matrix Addition

$$[A + B]_{ik} = [A]_{ik} + [B]_{ik}$$

Example:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+1 & 2+0 \\ 3+0 & 4+1 \end{bmatrix}$$

$A + B \in \mathcal{M}_{n \times m}$

when $A, B \in \mathcal{M}_{n \times m}$.

1:37 AM Sat Sep 26 — Linear Algebra | Linear Algebra 79%

Thm Distribution of Matrix Mult over Add.

$$A \times (B + C) = A \times B + A \times C$$

Proof for $A, B, C \in M_{2\times 2}$:

Statements	Justifications
① LHS $A \times (B+C) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \times \left(\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \right)$	① Given $A, B, C \in M_{2\times 2}$, Defn of $M_{2\times 2}$
② $= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \times \begin{bmatrix} (b_{11}+c_{11}) & (b_{12}+c_{12}) \\ (b_{21}+c_{21}) & (b_{22}+c_{22}) \end{bmatrix}$	② Defn of Matrix Addition
③ $= \begin{bmatrix} a_{11}(b_{11}+c_{11}) + a_{12}(b_{21}+c_{21}) & a_{11}(b_{12}+c_{12}) + a_{12}(b_{22}+c_{22}) \\ a_{21}(b_{11}+c_{11}) + a_{22}(b_{21}+c_{21}) & a_{21}(b_{12}+c_{12}) + a_{22}(b_{22}+c_{22}) \end{bmatrix}$	③ Defn of Matrix Mult
④ $= \begin{bmatrix} a_{11}b_{11} + a_{11}c_{11} + a_{12}b_{21} + a_{12}c_{21} & a_{11}b_{12} + a_{11}c_{12} + a_{12}b_{22} + a_{12}c_{22} \\ a_{21}b_{11} + a_{21}c_{11} + a_{22}b_{21} + a_{22}c_{21} & a_{21}b_{12} + a_{21}c_{12} + a_{22}b_{22} + a_{22}c_{22} \end{bmatrix}$	④ distribution of mult over add for $\mathbb{R}$

Next do right hand side until they match

(5) RHS $A \times B + A \times C =$

$$= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \times \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \times \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

(6)
$$= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix} + \begin{bmatrix} a_{11}c_{11} + a_{12}c_{21} & a_{11}c_{12} + a_{12}c_{22} \\ a_{21}c_{11} + a_{22}c_{21} & a_{21}c_{12} + a_{22}c_{22} \end{bmatrix}$$

(7)
$$= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{11}c_{11} + a_{12}c_{21} & a_{11}b_{12} + a_{12}b_{22} + a_{11}c_{12} + a_{12}c_{22} \\ a_{21}b_{11} + a_{22}b_{21} + a_{21}c_{11} + a_{22}c_{21} & a_{21}b_{12} + a_{22}b_{22} + a_{21}c_{12} + a_{22}c_{22} \end{bmatrix}$$

Now check this matrix

(5) Given $A, B, C \in M_{2\times2}$ and defn of $M_{2\times2}$.

(6) Defn of Matrix Mult and Order of Operations

(7) Defn of Matrix Addition ✓

matches step 4

QED

Thm Associativity of Matrix Mult

$$A\times(B\times C) = (A\times B)\times C$$

Proof (for $A, B, C \in M_{2\times 2}$):

① LHS $A\times(B\times C) =$

$$= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \times \left(\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \times \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \right)$$

②

$$= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \times \begin{bmatrix} (b_{11}c_{11} + b_{12}c_{21}) & (\ldots) \\ (\ldots) & (\ldots) \end{bmatrix}$$

③

$$= \begin{bmatrix} a_{11}(b_{11}c_{11} + b_{12}c_{21}) + a_{12}(\ldots) \\ a_{21}(b_{11}c_{11} + b_{12}c_{21}) + a_{22}(\ldots) \end{bmatrix}$$

④ Simplify

HW1 Fill in rest of proof

① Given $A, B, C \in M_{2\times 2}$
Defn of $M_{2\times 2}$

② Defn of Matrix Mult.

$$\begin{bmatrix} a_{11}(\ldots) + a_{12}(\ldots) \\ a_{21}(\ldots) + a_{22}(\ldots) \end{bmatrix}$$

③ Defn of Matrix Mult.
④ algebra

2:10 AM Sat Sep 26

Linear Algebra

Linear Algebra

Defn The zero matrix, O, is a matrix which has zeroes everywhere.

Thm $A \times O = O$

Proof: ($A \in M_{2\times 3}$ $O \in M_{3\times 4}$)

we will figure out the answer.

① $A \times O = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

① by given $A \in M_{2\times 3}$ & defn $M_{2\times 3}$
by defn O matrix in $M_{3\times 4}$

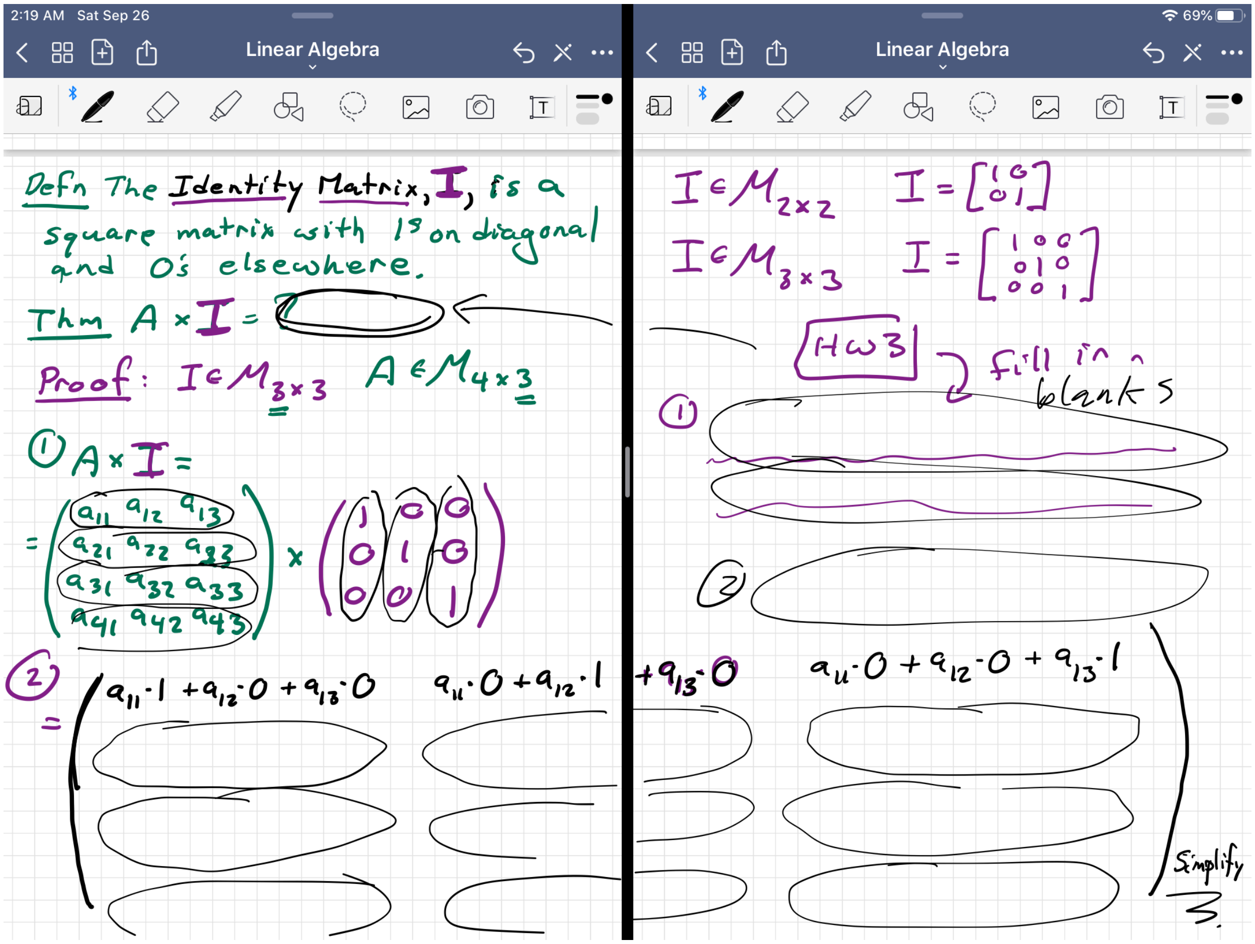
② $= \begin{bmatrix} a_{11}\cdot 0 + a_{12}\cdot 0 + a_{13} 0 & a_{11}0 + \ldots & 0+0+0 & 0 \\ a_{21}\cdot 0 + a_{22}\cdot 0 + a_{23} 0 & 0+0+0 & 0+0+0 & 0 \end{bmatrix}$

② by defn of matrix mult

③ $= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \in M_{2\times 4}$

③ by arithmetic

HW2 $O \times A = O$ when $A \in M_{2\times 3}$ $O \in M_{4\times 2}$

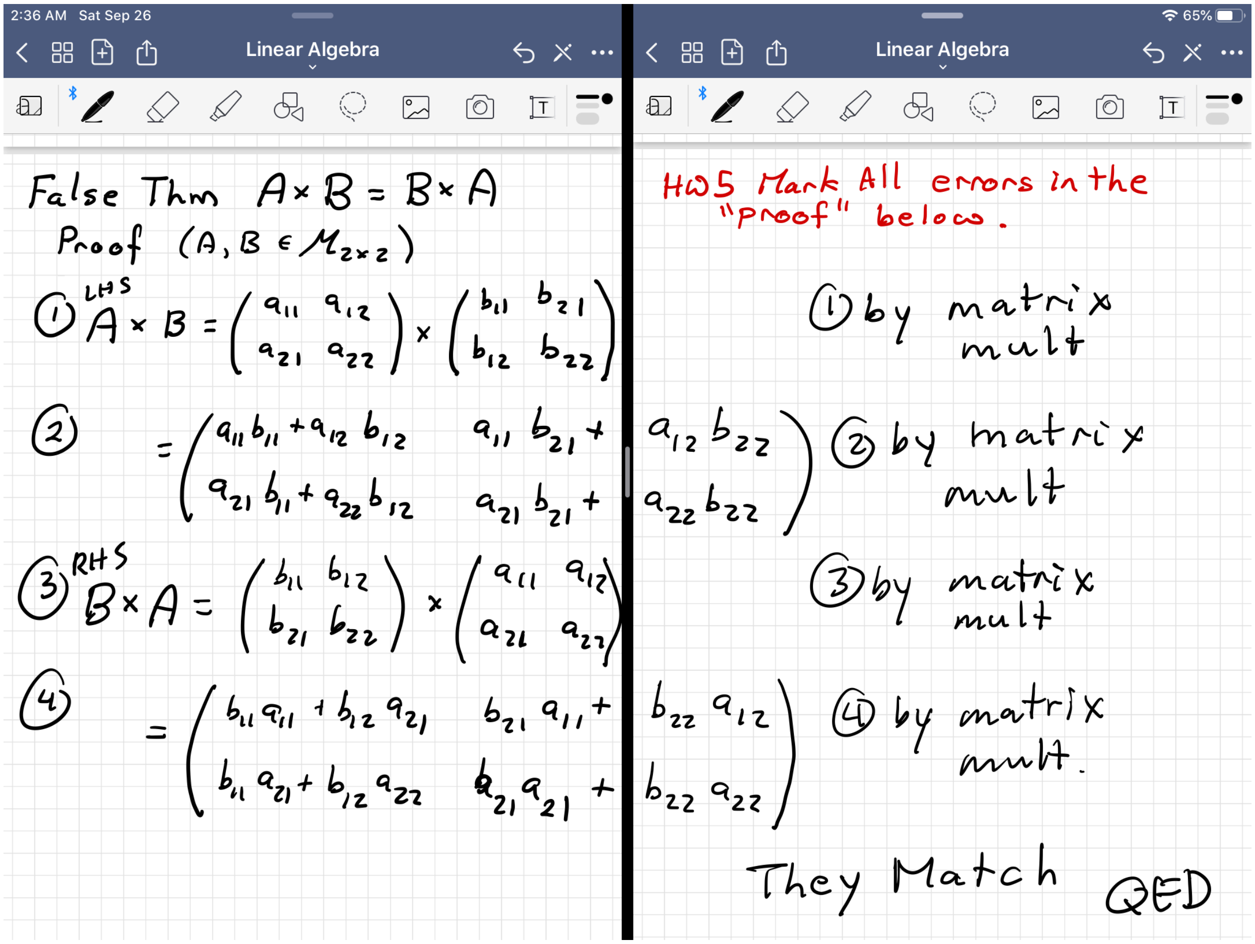


HW1-HW3 are described above.

HW4 **Prove IxB=B when B is 2x3 and I is the correct choice of identity matrix.**

HW5 **is to find the errors in the incorrect proof below which has many errors so find all of the errors. Then either fix the proof or find a pair of specific 2x2 matrices A and B for**

which this fails.



HW6: Prove that if A and B are 2x2 matrices and v is a vector then
A(Bv)=(AxB)v
Hint: Bv is a vector so on the LHS you first use matrix times vector and then matrix times vector again. On the RHS you need to use matrix times matrix to find (AxB) and then matrix times vector to multiply the answer times v.

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Part II: Watch [Playlist 313F20-15-PartII] which is Extra Credit highly recommended for math majors and has three extra credit problems.
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2:44 AM Sat Sep 26 — Linear Algebra

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$$A \times (B \times C) = (A \times B) \times C$$

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Thm $A \times O = ?$ and $O \times A = ?$

Defn The **identity matrix**, I, is a is a square matrix with 1's on the diagonal and zeroes elsewhere.

Thm $A \times I = ?$ and $I \times A = ?$

What about

$$A \times B \stackrel{?}{=} B \times A ?$$

Linear Algebra

Part II Proofs using Σ notations

Defn of Matrix Mult

$$[A \times B]_{ik} = \sum_{j=1}^{m} a_{ij} b_{jk}$$

i^{th} row of A: $(a_{i1} \; a_{i2} \cdots a_{im})$ $\cdot$ $\begin{pmatrix} b_{1k} \\ b_{2k} \\ \vdots \\ b_{mk} \end{pmatrix}$ k^{th} column of B

Defn of Addition

$$[A + B]_{ik} = [A]_{ik} + [B]_{ik}$$
$$= a_{ik} + b_{ik}$$

Thm Distribution of Matrix Mult over Add

$$A \times (B + C) = A \times B + A \times C$$

Proof using Σ notation

for $A \in M_{n \times m}$, $B, C \in M_{m \times \ell}$:

① LHS $[A \times (B+C)]_{ik} = \sum_{j=1}^{m} [A]_{ij} [B+C]_{jk}$

① by defn of matrix mult

② $= \sum_{j=1}^{m} [A]_{ij} \left([B]_{jk} + [C]_{jk} \right)$

② defn of matrix addition

③ $= \sum_{j=1}^{m} \left([A]_{ij} [B]_{jk} + [A]_{ij} [C]_{jk} \right)$

③ distribution of mult over addition of reals.

④ $= \sum_{j=1}^{m} \left([A]_{ij} [B]_{jk} \right) + \sum_{j=1}^{m} [A_{ij}][C_{jk}]$ ④ reg. Σ for

Linear Algebra

Part II Proofs using Σ notations

Defn of Matrix Mult

$$[A \times B]_{ik} = \sum_{j=1}^{m} [A]_{ij} [B]_{jk}$$

i^{th} row of A: $a_{i1} \; a_{i2} \cdots a_{im}$ $\cdot$ k^{th} column of B: $b_{1k}, b_{2k}, \ldots, b_{mk}$

Defn of Addition

$$[A + B]_{ik} = [A]_{ik} + [B]_{ik}$$
$$= a_{ik} + b_{ik}$$

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Thm Distribution of Matrix Mult over Add

$$A \times (B + C) = A \times B + A \times C$$

Proof using Σ notation

for $A \in M_{n \times m}$ $B, C \in M_{m \times \ell}$:

① LHS $[A \times (B + C)]_{ik} = \sum_{j=1}^{m} [A]_{ij} [B + C]_{jk}$

① by defn of matrix mult

② $= \sum_{j=1}^{m} [A]_{ij} \left([B]_{jk} + [C]_{jk}\right)$

② defn of matrix addition

③ $= \sum_{j=1}^{m} \left([A]_{ij} [B]_{jk} + [A]_{ij} [C]_{jk}\right)$

③ distribution of mult over addition of reals.

④ $= \sum_{j=1}^{m} \left([A]_{ij} [B]_{jk}\right) + \sum_{j=1}^{m} [A_{ij}][C_{jk}]$ ④ reals, Σ ...

⑤ RHS

$[A \times B + A \times C]_{ik} =$

$= [A \times B]_{ik} + [A \times C]_{ik}$

⑤ defn of matrix add.

⑥ $= \sum_{j=1}^{m} [A]_{ij} [B]_{jk} + \sum_{j=1}^{m} [A]_{ij} [C]_{jk}$

⑥ by def matrix Mult.

Steps 4 + 6 Match

QED

EC $A, B \in M_{n \times m}$ $C \in M_{m \times \ell}$

$$(A + B) \times C = A \times C + B \times C$$

Lesson 15 Proofs with Matrices

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Defn: The **zero matrix**, O, is a matrix which has zeroes everywhere.

Thm $A\times O = O$ and $O\times A = O$

Defn The **identity matrix**, I, is a is a square matrix with 1's on the diagonal and zeroes elsewhere.

Thm $A\times I = A$ and $I\times A = A$

What about

$$A\times B \overset{?}{=} B\times A \ ?$$

FALSE

Thm: If $A\in M_{n\times m}$ and $O\in M_{m\times \ell}$

then $A\times O = O\in M_{n\times \ell}$

Proof:

① $[A\times O]_{ik} = \sum_{j=1}^{m} [A]_{ij}[O]_{jk}$ — ① by defn matrix mult

② $= \sum_{j=1}^{m} [A]_{ij}\cdot 0$ — ② by defn of zero matrix

③ $= \sum_{j=1}^{m} 0$ — ③ by $a\cdot 0 = 0$ for any $a\in\mathbb{R}$

④ $= 0$ — ④ $0+0=0$

⑤ $= [O]_{ik}$. — ⑤ Defn of zero matrix QED

EC $O\times A = O$

($O \in M_{n\times m}$, $A \in M_{m\times \ell}$, $O \in M_{n\times \ell}$)

Lesson 15 Proofs with Matrices

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Thm $A\times O = O$ and $O\times A = O$

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Thm $A\times I = A$ and $I\times A = A$

What about

$$A\times B \stackrel{?}{=} B\times A \; ?$$

FALSE

Thm Given $A\in M_{n\times m}$, $I\in M_{m\times m}$, then $A\times I = A$

(EC3) $I\times A = A$, $I\in M_{m\times m}$, $A\in M_{m\times \ell}$

Thoughts:

$$[I]_{ij} = \begin{cases} 1 & \text{if } i=j \text{ (diagonal)} \\ 0 & \text{if } i\neq j \text{ (elsewhere)} \end{cases}$$

Defn of Identity Matrix

Proof:

(1) $[A\times I]_{ik} = \sum_{j=1}^{m} [A]_{ij}[I]_{jk}$ — (1) by defn matrix mult

(2) $= [A]_{i1} I_{1k} + [A]_{i2} I_{2k} + \dots + [A]_{ik} I_{kk} + \dots + [A]_{im} I_{mk}$ — (2) defn of Σ

(3) $= [A]_{i1}\cdot 0 + [A]_{i2}\cdot 0 + \dots + [A]_{ik}\cdot 1 + \dots + [A]_{im}\cdot 0$ — (3) by defn of identity

(4) $= 0+0+\dots+0+[A]_{ik}+0+\dots+0$ — (4) algebra

(5) $= [A]_{ik}$ — (5) by $0+a=a$ for $a\in\mathbb{R}$

QED

Extra Credit: Prove for arbitrary matrices A and B and vector v using sum notation that
A(Bv)=(AxB)v

Extra Credit: Prove for arbitrary matrices A, B, and C using sum notation that
Ax(BxC)=(AxB)xC