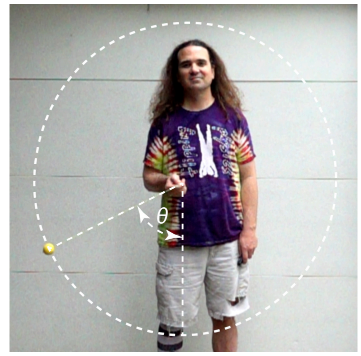
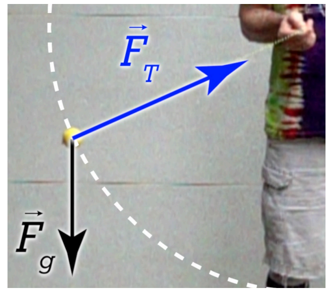
Flipping Physics Lecture Notes:

Nonuniform Circular Motion - Force of Tension in a Rope
http://www.flippingphysics.com/non-uniform-circular-motion-tension.html



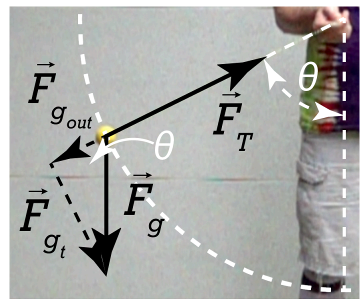
Example: A ball of mass *m* on the end of a string of length *L* moves in a vertical circle with an angular speed *ω*. The string forms an angle *θ* with the vertical as shown. Determine the tension in the rope in terms of *m*, *L*, *ω, θ*, and known constants.



Knowns: m, L, ω, θ □ F_T = ?

We start by drawing the Free Body Diagram.

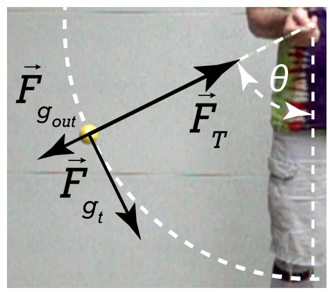
Then we break forces into components in the in direction and the tangential direction.



Notice the angle *θ* we use to determine the components of the force of gravity is the same angle *θ* defined in the problem.

$$\sin\theta = \frac{O}{H} = \frac{F_{g_t}}{F_g} \Rightarrow F_{g_t} = mg\sin\theta$$

$$\cos\theta = \frac{A}{H} = \frac{F_{g_{out}}}{F_g} \Rightarrow F_{g_{out}} = mg\cos\theta$$



Redraw the Free Body Diagram.

Sum the forces in the tangential direction.

$$\sum F_t = F_{g_t} = ma_t \Rightarrow mg\sin\theta = ma_t \Rightarrow a_t = g\sin\theta$$

We now know the value of the tangential acceleration of the ball, however, this does not help us solve for the force of tension in the string. ☺

Sum the forces in the in-direction, remembering that inward is positive and outward is negative.

$$\sum F_{in} = F_T - F_{g_{out}} = ma_c \Rightarrow F_T = F_{g_{out}} + ma_c = mg\cos\theta + mr\omega^2 \Rightarrow F_T = mg\cos\theta + mL\omega^2$$

This answer matches what we previously showed visually.[1]

But notice we can also solve for the minimum angular speed to keep the ball moving in a circle. When the ball is at the top of the circle, at that minimum speed the force of tension is reduced down to zero and the angle is 180°.

$$\Rightarrow 0 = mg\cos(180) + mL\omega^2 \Rightarrow -mg\cos(-1) = mL\omega^2 \Rightarrow g = L\omega^2 \Rightarrow \omega_{min} = \sqrt{\frac{g}{L}}$$

Which is the same answer we got for the “Minimum angular speed necessary to keep water in a vertically revolving bucket”![2]

[1] http://www.flippingphysics.com/non-uniform-circular-motion-ball.html
[2] https://www.flippingphysics.com/water-bucket-minimum-speed.html