

ASSIGNMENT NO 1 : MTH 623

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Question no 1 :

If $n=2$ write out explicitly the triple summation $C_{rst}x^r y^s z^t$

SOLUTION :

The out explicitly the triple summation

$$\sum_{r=0}^n \sum_{s=0}^{n-r} \sum_{t=0}^{n-r-s} C_{rst} x^r y^s z^t$$

Given:

- o $r=0$ to $r=2$
- o $s=0$ to $s=2-r$
- o $t=0$ to $s=2-r-s$

Step 1: when $r=0$

- **$s=0$, t goes from 0 to 2 :**

$$\begin{aligned} & C_{000}x^0 y^0 z^0 + C_{001}x^0 y^0 z^1 + C_{002}x^0 y^0 z^2 \\ &= C_{000} + C_{001}z + C_{002}z^2 \end{aligned}$$

- **$s=1$, t goes from 0 to 1 :**

$$\begin{aligned} & C_{010}x^0 y^1 z^0 + C_{011}x^0 y^1 z^1 \\ &= C_{010}y + C_{011}yz \end{aligned}$$

- $s = 2$, t goes from 0 to 0 :

$$C_{020}x^0y^2z^0 = C_{020}y^2$$

Step 2 : when $r = 1$

- $s = 0$, t goes from 0 to 2 :

$$\begin{aligned} & C_{100}x^1y^0z^0 + C_{101}x^1y^0z^1 \\ &= C_{100}x^1 + C_{101}x^1z^1 \\ &= C_{100}x + C_{101}xz \end{aligned}$$

- $s = 1$, t goes from 0 to 0:

$$C_{110}x^1y^1z^0 = C_{110}xy$$

Step 3 : when $r = 2$

- o $s = 0$ to $t = 0$

$$C_{200}x^2y^0z^0 = C_{200}x^2$$

The full expanded expression is :

$$C_{000} + C_{001}z + C_{002}z^2 + C_{010}y + C_{011}yz + C_{020}y^2 + C_{100}x + C_{101}xz + C_{110}xy + C_{200}x^2$$

Question no 2 :

Use the Kronecker delta to calculate the partial derivative if a_{ij} are constant such that $a_{ij} = a_{ji}$.

$$\frac{\partial}{\partial x_k}(a_{ij}x_i x_j)$$

SOLUTION :

Using the product rule and the Kronecker delta , Differentiate $x_i x_j$ w.r.t x_k

$$\begin{aligned}\frac{\partial}{\partial x_k}(a_{ij}x_i x_j) &= a_{ij} \frac{\partial}{\partial x_k}(x_i x_j) \\ \frac{\partial}{\partial x_k}(a_{ij}x_i x_j) &= a_{ij} \left(\frac{\partial x_i}{\partial x_k} \cdot x_j + x_i \cdot \frac{\partial x_j}{\partial x_k} \right) \\ \frac{\partial}{\partial x_k}(a_{ij}x_i x_j) &= a_{ij} (\delta_{ik} x_j + x_i \delta_{jk}) \\ \frac{\partial}{\partial x_k}(a_{ij}x_i x_j) &= a_{ij} \delta_{ik} x_j + a_{ij} x_i \delta_{jk}\end{aligned}$$

By putting $i = k$, $j = k$ and

$$\begin{aligned}a_{ij} &= a_{ji}, a_{ik} = a_{jk} \\ \frac{\partial}{\partial x_k}(a_{ij}x_i x_j) &= a_{ik} x_j + a_{jk} x_i \\ \frac{\partial}{\partial x_k}(a_{ij}x_i x_j) &= a_{ik} x_j + a_{ik} x_i \\ \frac{\partial}{\partial x_k}(a_{ij}x_i x_j) &= 2a_{ik} x_i\end{aligned}$$

Question no 3 :

Given a point P in Cartesian coordinates as P (x , y , z) =(3 , 4 ,5),
convert this point into

- i. The cylindrical coordinates (r, θ, z)
- ii. The parabolic cylindrical coordinates (u , v ,z) , defined by :

$$x = \frac{1}{2}(u^2 - v^2), y = uv, z = z$$

SOLUTION :

- i. The cylindrical coordinates (r, θ, z) .

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$z = z$$

- Calculate r:

$$X = 3, y = 4$$

$$r = \sqrt{x^2 + y^2}$$

$$r = \sqrt{(3)^2 + (4)^2}$$

$$r = \sqrt{9+16}$$

$$r = \sqrt{25}$$

$$r = 5$$

- Calculate θ :

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\theta = \tan^{-1}\left(\frac{4}{3}\right)$$

$$\theta = \tan^{-1}(1.3334)$$

$$\theta = 0.927$$

- **z-coordinate:**

The z-coordinates remains the same so, $z=5$

The cylindrical coordinates of P are :

$$(r, \theta, z) = (5, 0.927, 5)$$

ii. **The parabolic cylindrical coordinates (u, v, z),**

In parabolic cylindrical coordinates (u, v, z), the relations are:

$$x = \frac{1}{2}(u^2 - v^2)$$

$$y = uz$$

$$z = z$$

Step 1: set up Equation for u and v:

Given $x = 3$, $y = 4$ and $z = 5$

$$x = \frac{1}{2}(u^2 - v^2)$$

$$3 = \frac{1}{2}(u^2 - v^2)$$

$$3 \times 2 = u^2 - v^2$$

$$6 = u^2 - v^2$$

Therefore,

$$Y = uv$$

$$4 = uv$$

So,

$$z = z$$

$$z = 5$$

Step 2: Solve for u and v

$$y = uv$$

$$4 = uv$$

$$v = \frac{4}{u}$$

Substitute values in the equation

$$x = \frac{1}{2}(u^2 - v^2)$$

$$3 = \frac{1}{2} \left((u^2)^2 - \left(\frac{4}{u} \right)^2 \right)$$

$$3 * 2 = u^4 - \frac{16}{u^2}$$

$$6 = u^4 - \frac{16}{u^2}$$

$$6 = \frac{u^4 - 16}{u^2}$$

$$6u^2 = u^4 - 16$$

$$u^4 - 6u^2 - 16 = 0$$

Let $w = u^2$ and $w^2 = u^4$

$$w^2 - 6w - 16 = 0$$

$$a = 1, b = -6, c = -16$$

Quadratic equation formula:

$$w = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

By putting the values

$$w = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(-16)}}{2(1)}$$

$$w = \frac{6 \pm \sqrt{36 + 64}}{2}$$

$$w = \frac{6 \pm \sqrt{100}}{2}$$

$$w = \frac{6 \pm 10}{2}$$

$$w = \frac{6+10}{2}, w = \frac{6-10}{2}$$

$$w = \frac{16}{2}, w = \frac{-4}{2}$$

$$w = 8, w = -2 (w = u^2 \geq 0)$$

So,

$$u^2 = 8$$

Square on both sides

$$\begin{aligned}\sqrt{u^2} &= \sqrt{8} \\ u &= \sqrt{2 \times 2 \times 2} \\ u &= \sqrt{2^2 \times 2} \\ u &= 2\sqrt{2}\end{aligned}$$

Step 3: calculate v :

$$y = uv \quad \therefore y = 4$$

$$4 = uv \quad \therefore u = 2\sqrt{2}$$

$$v = \frac{4}{2\sqrt{2}}$$

$$v = \frac{1}{\sqrt{2}}$$

$$v = \sqrt{2} \quad \therefore \frac{1}{\sqrt{2}} = \sqrt{2}$$

The parabolic cylindrical coordinates of P are :

$$(u, v, z) = (2\sqrt{2}, \sqrt{2}, 5)$$

Question no 4 :

Consider the toroidal coordinate system defined by the coordinates

(σ, τ, φ) , where σ and τ are parameters related to the distance from

the origin , with $\sigma \in [0, \infty)$ and $\tau \in [0, 2\pi)$. φ is the azimuthal angle around

the z-a. The coordinates are defined as follows:

$$x = \frac{\sigma \sinh \phi \cos \varphi}{\cosh \sigma - \cos \tau}, y = \frac{\sigma \sinh \sigma \sin \varphi}{\cosh \sigma - \cos \tau}, z = \frac{\sigma \sinh \tau}{\cosh \sigma - \cos \tau}$$

Where :

$$x = \frac{\sigma \sinh \phi \cos \varphi}{\cosh \sigma - \cos \tau}, y = \frac{\sigma \sinh \sigma \sin \varphi}{\cosh \sigma - \cos \tau}, z = \frac{\sigma \sinh \tau}{\cosh \sigma - \cos \tau}$$

**Derive the scale factors for the toroidal coordinates system. Using
These scale factors , express the infinitesimal volume element dV
In toroidal coordinates.**

SOLUTION :

The Cartesian coordinates (x ,y, z) in terms of the toroidal coordinates
(σ, τ, ϕ) are given by :

$$x = \frac{\sigma \sinh \phi \cos \varphi}{\cosh \sigma - \cos \tau}, y = \frac{\sigma \sinh \sigma \sin \varphi}{\cosh \sigma - \cos \tau}, z = \frac{\sigma \sinh \tau}{\cosh \sigma - \cos \tau}$$

Where :

a. $\sigma \in [0, \infty)$

b. $\tau \in [0, 2\pi)$

c. $\phi \in [0, 2\pi)$

Compute the scale factors:

$$h_{\sigma} = \sqrt{\left(\frac{\partial x}{\partial \sigma}\right)^2 + \left(\frac{\partial y}{\partial \sigma}\right)^2 + \left(\frac{\partial z}{\partial \sigma}\right)^2}$$

$$h_{\tau} = \sqrt{\left(\frac{\partial x}{\partial \tau}\right)^2 + \left(\frac{\partial y}{\partial \tau}\right)^2 + \left(\frac{\partial z}{\partial \tau}\right)^2}$$

$$h_{\phi} = \sqrt{\left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2 + \left(\frac{\partial z}{\partial \phi}\right)^2}$$

Final form of scale Factors:

$$h_{\sigma} = \frac{a}{\cos \sigma - \cos \tau}$$

$$h_{\tau} = \frac{a}{\cos \sigma - \cos \tau}$$

$$h_{\phi} = \frac{a \sinh \sigma}{\cos \sigma - \cos \tau}$$

The Infinitesimal Volume Element dV in toroidal coordinate is :

$$dV = h_{\sigma} h_{\tau} h_{\phi} d\sigma d\tau d\phi$$

$$dV = \left(\frac{a}{\cos \sigma - \cos \tau}\right) \left(\frac{a}{\cos \sigma - \cos \tau}\right) \left(\frac{a \sinh \sigma}{\cos \sigma - \cos \tau}\right) d\sigma d\tau d\phi$$

$$dV = \frac{a^3 \sinh \sigma}{(\cos \sigma - \cos \tau)^3} d\sigma d\tau d\phi$$

