NOTE TO STUDENTS Fall 2022: These Lecture Notes are based on the last time I taught the course. They may be updated throughout the semester.

DAY 1: SEC 1.2
First Order Equations (p.7–13, 16-17)
p.14: 1, 2(a-c,e-h), 4(a-f), 5, 6, [optional: p. 14: 9 and p. 21: 1-11]
WeBWorK: Review

SKILLS:
- find the order of a differential equation
- verify a given function is a solution to a differential equation (or initial value problem)
  * review of derivatives (trig fcns, product rule, chain rule)
- find all solutions (or solve initial value problem) of given differential equation (straight integration)
  * review of integration techniques

Welcome, Course Policies
WeBWorK due next week

2-minute intros (works best in-person)

STUFF YOU NEED TO REMEMBER:
Algebra! Factoring, fractions, trig identities, and more
All the derivatives! Exponents, exponential functions, logs, trig functions (ALL OF THEM!)
Integrals: substitution, integration by parts

Review of calculus -
- find the derivative of the given function y=f(x)
  A. $y = x^2 + 3x + 8$
  B. $y = \sin x + e^x + \ln x$
  C. y=x^2 cosx
  C. $y = \sqrt{x^3 + 11}$
- find the integral/antiderivative of the given function
  A. $y' = 5x^3 + 7x$
  B. $y' = \sin x + \cos x + e^x$
  C. $y' = 10x^3(x^4 + 7)^6$
  D. $y' = x \ln x$

Defn. A differential equation is an equation that contains one or more derivatives of an unknown function. The **order** of a differential equation is simply the highest derivative that appears in the equation (compare to the *degree* of a polynomial).

Example 1: $y = \frac{x}{2} \cdot y'$

QUESTION: What's the order? Is the function $y = x^2$ a solution?

RECALL NOTATION:
FUNCTION: y, f(x), y(x),
DERIVATIVE: dy/dx, f'(x), y', y'(x)
VARIABLE: Sometimes we use 't' instead of 'x' f(t)=t^3+2t+1

What is a solution? It's a function y that makes the equation work.

Example 2: $y' = x^3$
Find ALL solutions.
Now find one particular solution that satisfies y(1)=2?

When we add one or more conditions on the solution, we call it an **initial value problem**.

BIG IDEA: I'm thinking of a function, y= ______ (<- something involving 'x')
YOU HAVE TO FIGURE IT OUT.

IF YOU KNOW THE DERIVATIVE OF A FUNCTION, CAN YOU TELL ME THE FUNCTION?
HINT 1: $y' = x^3$
HINT 2: y(1)=2

GROUP WORK:

Example 3: Is $y = x \sin x$ a solution to the differential equation
$y \sin x + y' \cos x - 1 = \sin x \cos x$?

Example 4: Find the solution to the differential equation
$y' = x\sqrt{x^2 + 8}$ satisfying $y(1) = 11$.

Example 5: Is $y = \frac{1}{2} + e^{-x^2}$ a solution to the initial value problem

$y' + 2xy = x,\ y(1) = \frac{3}{2}$?

Example 6: Find all solutions to $y' = xe^x$.

Example 7: Find the solution to $y'' = x^5 + \sqrt[3]{x^2} + x^{-2}$ satisfying

$y'(1) = \frac{23}{30}$ and $y(1) = 2$

SOLN:
Ex7: y(x) = (9 x^(8/3))/40 + x^7/42 + x - ln(x) + 631/840

DAY 2: Section 2.1
Linear First Order Equations (p.30–41)
p.41: 1-9 odd, 17-23 odd, 31-37 odd, 38, 40, 42
- find general solutions to a variety of linear first-order differential equations and initial value problems
  - homogeneous and non-homogeneous

A. Solve homogeneous linear first order linear diffy q's
B. Solve nonhomogeneous linear first order linear diffy q's

RECALL: Implicit differentiation - find the derivative $\frac{d}{dx}\ln|y|$

HEADS UP: Very often we use "t" instead of "x" as our independent variable....

DEFINE: ORDER!!
Defn: A first-order differential equation is **linear** if it has the form: $y' + p(x)y = f(x)$
We call such an equation **homogeneous** if $f(x) = 0$, that is: $y' + p(x)y = 0$

Example 1: Solve $y' - x^2y = 0$

Move "y" to RHS, divide by y to separate variables, now integrate. HINT: use "k" as constant!!
ANS: $y = ce^{x^3/3}$

CLASS
Example 2: Solve $xy' + y = 0$. Find the particular solution for which y(1)=3

GENERAL FORMULA FOR SOLVING HOMOGENEOUS LINEAR EQUATIONS
Let's figure out a general method to solve homogeneous linear equations of the form: $y' + p(x)y = 0$ (where p(x) is continuous on some interval (a,b)).
STRATEGY: Rewrite as $\frac{y'}{y} = -p(x)$. Integrate, raise *e* to each side, simplify.

Solution: $y = ce^{-P(x)}$, where $P(x) = \int p(x)dx$ is *any particular* antiderivative of p(x) on (a,b)

NON-HOMOGENOUS EQUATIONS AND VARIATION OF PARAMETERS
What about non-homogenous linear equations? $y' + p(x)y = f(x)$

First, consider the corresponding homogeneous equation (called the **complementary equation**):
$y' + p(x)y = 0$
Suppose we have a solution to the complementary equation - let's call it y_1 - so that
$y_1' + p(x)y_1 = 0$
We're going to modify y_1 to try to find a solution to the original equation.

A TIME-HONORED TRADITION IN DIFFY Qs: "the guess"
Let's guess that our solution has the form "something times y1", so $y = uy_1$, where u is some unknown function of x (we call u an **integrating factor**). Let's try to find u!

Why this guess? Good question - we want to be able to plug into the left side, and have most (but not all) parts cancel out, leaving behind exactly f(x)... playing around a bit shows that uy1 is a good candidate.

Substitute $y = uy_1$ into the equation. For the derivative y', what rule do we use?

$u'y_1 + u(y_1' + p(x)y_1) = f(x)$

Thus $u'y_1 = f(x)$, so $u' = \frac{f(x)}{y_1(x)}$.

Now integrate to find u, then substitute into $y = uy_1$ to find the solution y to the original equation.

Example 3: $y' + 2y = x^3e^{-2x}$

STRATEGY to solve $y' + p(x)y = f(x)$:
STEP 1: Solve the complementary equation $y' + p(x)y = 0$, let $y_1(x)$be a solution.
STEP 2: Look for a solution of the form $y = uy_1$. Substitute into the original equation.
STEP 3: Solve for u by isolating u' and integrating.
STEP 4: Substitute $y_1(x)$, $u(x)$ to find the final solution: $y = uy_1$

Example: a) Find the general solution $y' + (\cot x)y = x \csc x$
b) Solve the initial value problem $y' + (\cot x)y = x \csc x$, $y(\pi/2) = 1$

DAY 3: Section 2.2
Separable Equations (p.45–52)
p.52: 2, 3, 6, 12, 17–27 odd, 28, 35, 37
Solve separable equations
- implicitly vs explicitly (solve for y)
- general vs particular solution/IVP

~~WEBWORK PROBLEMS 1&2: Don't recognize combined constants as a new constant, e.g. solution must be "+7c" instead of "+c", even though 7c is, indeed, a constant~~

Example 1: $y' = -\frac{x}{y}$

COULD DO A DISCUSSION OF: y' giving slope, drawing a slope field, then talk about separable and looking at possible solutions for different values of C??? USE DESMOS SLOPE FIELD: https://www.desmos.com/calculator/p7vd3cdmei

Defn: A first-order differential equation is **separable** if it can be written in the form $h(y) \cdot y' = g(x)$. *That is, we can separate the variables - x on one side, y on the other, with the y side multiplied by y'.*
NOTE: We can solve a separable equation by integrating both sides.

Example 1: a) Solve $y' = -\frac{x}{y}$.
b) Find the particular, explicit solution when y(1)=1. On what interval is the solution valid?
c) Find the particular, explicit solution when y(1)=-2. On what interval is the solution valid?

IMPLICIT SOLUTIONS
Note: Sometimes we cannot find an explicit solution (we can't "solve for y"), but we can still give an **implicit solution:** an equation in y and x that describes the solution.

Example 2. Show that the equation $\frac{dy}{dx} = \frac{x^2}{1-y^2}$ is separable, and then find the solution.
Is your solution implicit or explicit? Can you find an explicit solution?
ANS: $-x^3 + 3y - y^3 = c$

Example 3. Solve the initial value problem explicitly: $y' + y^4\cos(5x) = 0,\ y(0) = 1$. On what interval is the solution valid?
ANS: $y = -\sqrt[3]{\frac{-2}{2+\sin(6x)}}$, -infty<x<infty

DAY 4: Section 2.4
Transformation of Nonlinear Equations into Separable Equations (p.62–68)
p.68: 1–4, 7–11 odd, 15–18, 23–27 odd
- Bernoulli Equations

TODAY'S TRICK: Using substitutions to turn nonlinear, nonseparable equations into separable equations.

HEADS UP: In today's lecture let's use "t" as our independent variable instead of "x"...

Defn: A **Bernoulli Equation** has the form $y' + p(t)y = f(t)y^r$, where r is any real number except 0 or 1.
STEP 1: Let $y_1(t)$ be a solution to the complementary equation $y' + p(t)y = 0$.
STEP 2: Guess a solution of the form $y = uy_1$, where $u(t)$ is some (unknown) function of t.
STEP 3: Substitute into the original equation and separate variables. Then integrate to find u.

Example 1: Solve the initial value problem $y' - y = ty^2$, y(1)=e

ANS (General): $y = \frac{e^t}{e^t(1-t)+c}$ or $y = -\frac{1}{t-1+ce^{-t}}$

ANS (IVP): $y = \frac{e^t}{e^t(1-t)+1}$

Example 2: Solve the IVP $ty' + y = t^4y^4$, $y(1) = 1/2$

ANS: $y = \frac{1}{t(11-3t)^{1/3}}$

Defn: A differential equation is called **homogeneous** if it can be written in the form $y' = f(\frac{y}{t})$.
That is, every occurrence of the variables on the right is in the form of a fraction y/t.
ANNOYING NOTE: Yes, this meaning of "homogeneous" is entirely different than the way we used it previously ("=0").

STEP 1: In this case, we **always** use $y_1 = t$. MEMORIZE IT!
STEP 2: Guess a solution of the form $y = uy_1$ (so $y = ut$).
STEP 3: Substitute into the original equation, rearranging to use $u = \frac{y}{t}$ when necessary. Then integrate to find u.

Example 2: $y' = \frac{y+te^{-y/t}}{t}$

DAY 5: Section 2.5
Equations (p.73–79)
p.79: 1–21 odd, 29, 30, 33, 34

- Identify Exact Equations
- Find solutions to Exact Equations

RECALL: Interval of Convergence

Example 1. Solve: $6x^5 + 2xy^2 + (2yx^2 + 5y^4)\frac{dy}{dx} = 0$ (give the solution implicitly)
Verify that this equation is an implicit solution: $x^6 + x^2y^2 + y^5 = c$

How do we check? Take the derivative, and see if we get the original differential equation. NOTE: Is this a regular derivative (treating y as a function of x), or a partial derivative (treating y as a constant)? CHECK THE DERIVATIVE NOTATION IN THE EXAMPLE!

Note: The left side of the implicit solution will be important - it's a function of x and y, let's call it F(x,y): $F(x, y) = x^6 + x^2y^2 + y^5$
THE BIG QUESTION: How do we find the function $F(x, y) = x^6 + x^2y^2 + y^5$?

Example 2. Find the partial derivatives of the function $F(x, y) = x^6 + x^2y^2 + y^5$
RECALL: How many partial derivatives are there? When we take a partial derivative wrt one variable, how do we treat the other variable?

NOTE: The differential equation $6x^5 + 2xy^2 + (2yx^2 + 5y^4)\frac{dy}{dx} = 0$ can be broken into two parts:
$M(x, y) + N(x, y)y' = 0$
M(x,y) is the result of taking the partial derivative of F with respect to x: $\frac{\partial}{\partial x}F(x, y) = M(x, y)$ *(We also write* F_x*)*
N(x,y) is the result of taking the partial derivative of F with respect to y: $\frac{\partial}{\partial y}F(x, y) = N(x, y)$ *(We also write* F_y*)*

Starting with M, N, how do we find F? Integrate M with respect to x to find F
BEWARE: When we take the partial derivative of F with respect to x, what "disappears"? *Anything that is a pure function of y.* So when we integrate, we have to put back "a function of y" (let's call it $\phi(y)$ -- this plays the role that a constant usually plays in standard integration.

$\int 6x^5 + 2x^2dx = x^6 + x^2y^2 + \phi(y)$.

This is our $F(x, y) = x^6 + x^2y^2 + \phi(y)$. BUT we have to figure out what $\phi(y)$ is. How do we do it?
Take the partial derivative of F with respect to y -- this should equal M.
$F_y = 2x^2y + \phi'(y)$. This had better equal N, so $2x^2y + \phi'(y) = 2yx^2 + 5y^4$, so in this case $\phi'(y) = 5y^4$. Integrate to find $\phi(y) = y^5$.
Now, what is F(x,y)= ?

QUESTION:
Given a differential equation of the form: $M(x, y) + N(x, y)y' = 0$, how do we know that there is a function $F(x, y)$ that will work as in the example above? We have to verify the following condition:
EXACTNESS CONDITION: If such a $F(x, y)$ exists, then $F_{xy} = F_{yx}$, and vice versa. Therefore, to check

if such a F exists, we simply have to check whether $M_y = N_x$.

Let's put all of this together.

SOLVING EXACT EQUATIONS:
Given a differential equation of the form: $M(x, y) + N(x, y)y' = 0$
1. Verify that the equation is **exact** by checking that $M_y = N_x$.
2. Integrate *M* with respect to x to obtain $F(x, y)$. *Treat y as a constant. Don't forget to add a "constant" term* $\phi(y)$.
3. Take the partial derivative F_y and set it equal to *N*, solve for $\phi'(y)$.
4. Integrate $\phi'(y)$ to find $\phi(y)$
5. The general solution to the differential equation is given implicitly by: $F(x, y) = c$.

Example 3. Solve $(y\cos x + 2xe^y) + (\sin x + x^2e^y - 1)y' = 0$
ANS: $y\sin x + x^2e^y - y = c$

Example 4. Solve $(3xy + y^2) + (x^2 + xy)y' = 0$
NOTE: $M_y \neq N_x$*, and so this equation is not exact - this method will not work.*

NOTE: I can realistically do ONE application in a class, and there are 2 days to cover 3 sections. In Spring 2019, I chose to focus on: Day 6: population, and Day 7: Newton's Law of Cooling (skipped Sec 4.3)

DAYS 6&7: Applications
: Section 4.1,4.2
4.1 Growth and Decay (p.130–137) p.138: 1–7 odd, 11, 13, 17
4.2 Cooling and Mixing (p.140–147) p.148: 1–11 odd, 15

4.1 - Radioactive decay - (half-life, initial amount, quantity over time Q(t))
- More applications (bread dough rising, candy consumption, water in a tank) in which change in quantity is related to quantity, plus other factors
- WeBWorK: rabbit populations, falling object with wind resistance
- WeBWorK OPL: radioactive decay, population, investments, drugs in bloodstream
4.2 - cooling problems (an object is moved from one temp to another, temperature over time. Questions about temperature at various times)
- WW OPL: not too many cooling problems - a few asking for numerical answers of various sorts
- mixing problems (tank with certain solution of water+salt, another solution added at a certain rate, mixture is drained at another rate)
- WW OPL: mixing problems.

How do we model real-world situations with differential equations?

FALL 2022: START WITH COOLING?

MATHEMATICAL MODELS
Defn. A differential equation that describes some physical process is often called a mathematical model.

"If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is." --- John von Neumann

Discuss: simplifying assumptions. math describes "perfect world", if we're lucky we can find math model that gives a very good approximation to the "real world"

Example 1: A population of field mice inhabit a rural area. Some owls live in the area. Let's track the population p(t) over time (t in months).

- every month, each pair of mice produces one new mouse.
- every night, the owls eat 15 mice.

How many mice are killed each month?
How many mice are born each month?
Does the population go up, or down?
Depends on the population!

A. What if we start with 600 mice p(0)=600 - what is the population after one month? Two months?
B. What if we start with 1000 mice p(0)=1000 - how many after one month? Two months?

Let's translate this into a differential equation:
$\frac{dp}{dt} = 0.5p - 450$ "The rate of change of the population is 0.5 times the population minus 450"

Solve for p(t). (Use *c* as multiplicative constant.)

C. What is the particular solution for A. p(0)=800? What happens to this over time?
D. What is the particular solution for B. p(0)=1000? What happens over time?

In the first case, the initial population is too small - so the predation overcomes the reproduction rate, and the mice die off.
In the second case, the population is too large - so the reproduction rate overcomes the predation, and the mice population grows out of control.

Graph using slope field: https://bluffton.edu/homepages/facstaff/nesterd/java/slopefields.html

GRAPH: 0.5y-450
X: 0, 10
Y: 0,1200

Is there any "middle ground", an initial population for which the predation and reproduction are balanced and the population stays steady?

The *equilibrium solution* is p(0)=900.
The equilibrium solution can be obtained by setting the derivative $\frac{dp}{dt} = 0$ and solving for p.

GENERAL SETUP: $\frac{dp}{dt} = rp - k$

- Where t is time (measured in months), and p(t) is the population.
- r is the rate constant or growth rate (the rate at which reproduction occurs)
- k is the predation rate (the number killed each month by predators)

Soln: p(t) = c e^(0.5 t) + 900

NEWTON'S LAW OF COOLING
You take a hot cup of coffee outside in the middle of winter. What happens to the temperature T(t) of the coffee over time?

Does it continue to cool forever?
Does it cool more quickly if the temperature outside is 22℉ (New York) or -40℉ (Alaska)?

"The rate of change of temperature is proportional to the DIFFERENCE between the temperature T of the object and the temperature T_m of the environment (or medium)"

NEWTON'S LAW OF COOLING:
$T' = -k(T - T_m)$

- T(t) is the temperature at time t

- T_m is the temperature of the medium (environment)
- k is a positive quantity, the *temperature decay constant* (depends on surface area of object and various properties of the environment)

NOTE: If T_0 is the initial value of T, then the general solution is $T = T_m + (T_0 - T_m)e^{-kt}$

Resource on coffee temperatures: https://driftaway.coffee/temperature/
Example (update): A n extra-hot cup of coffee at 180℉ is carried outside in 22℉ weather. After 5 minutes, the temperature of the coffee has dropped to 160℉. How long does it take the coffee to reach the perfect temperature of 130℉?

T=22+(158)e^(-0.027068t)
t=14.0559 minutes

Example (OLD): A cup of coffee at 200℉ is carried outside in 22℉ weather. After 5 minutes, the temperature of the coffee has dropped to 120℉. How long does it take the coffee to reach 75℉?
T=22+178e^(-0.11936t)
t=10.1499 minutes

WeBWorK:

When a hot object is placed in a water bath whose temperature is $25°\mathrm{C}$, it cools from $100°\mathrm{C}$ to $50°\mathrm{C}$ in 160s. In another bath, the same cooling occurs in 140s. Find the temperature of the second bath.

The temperature of the second bath = ☐ $°\mathrm{C}$

ANS: 19.0414

SPRING 2019: Skipping Section 4.3, as I already spent 2 days on applications (population, and cooling).

~~**Day 7: Section 4.3**~~
~~4.3 Elementary Mechanics (p.151–160)~~
~~p.160: 3, 5, 7, 10~~
~~-objects in motion subject to gravity + air resistance~~

~~FALLING OBJECTS (turn to a friend, DISCUSS these three questions with them for 5 minutes, come up with some answer to each)~~

~~**Example 1:** Suppose that an object is falling in the atmosphere near sea level. Formulate a differential equation that describes the motion.~~

- ~~What kind of function are we trying to find? Position? Velocity? Acceleration?~~
- ~~Any guess as to what the solution will look like? Will it increase or decrease over time?~~
- ~~What are different factors that might influence the motion of the object?~~

SKETCH SOME OPTIONS - label v-axis, t-axis, etc

Thinking question: What are the different factors that might influence the motion of the object?

- Newton: $F = ma$ (let us measure mass in kg, acceleration in m/s^2, force in Newtons)
- Relationship between a(t) and v(t)? $a = \frac{dv}{dt}$
- $F = m\left(\frac{dv}{dt}\right)$
- Force #1: Gravity pulls down with force mg (g=acceleration due to gravity)
- Force #2: Air resistance pushes up with force proportional to velocity γv (γ= constant = drag coefficient)
- Combine: $m\left(\frac{dv}{dt}\right) = mg - \gamma v$

$m\left(\frac{dv}{dt}\right) = mg - \gamma v$
m = mass (kg)
v = velocity (m/s)
g = acceleration due to gravity = $9.8 m/s^2$
γ= gamma = drag coefficient

Example 2: Suppose the mass of the object in example 1 is 10kg, and the drag coefficient has been determined to be $\gamma = 2\ kg/s$. Write the differential equation describing the object's motion in the form $\frac{dv}{dt} =$ (that is, isolate $\frac{dv}{dt}$ on one side).

ANS: $\frac{dv}{dt} = 9.8 - \frac{v}{5}$

Example 3: Solve this differential equation - that is, find a formula for v(t).

solve for v: $v = 49 + ce^{-t/5}$

LOOK AT A SLOPE FIELD! What do you see?

What happens if c is: positive, negative, 0?

Example 4: Is there a solution in which the velocity is constant? What is it?

RULE: to find it exactly, set the derivative to 0.

Defn. A constant solution to a differential equation is called an **equilibrium solution**.
In the case of a falling object, it is often referred to as **terminal velocity**.

Example 6: Find the specific solution describing the the motion of an object if the initial velocity is $v_0 = 35\ m/s$:

Day 8: EXAM 1
Day 9: Section 3.1
3.1 Euler's Method (p.96–106)
p.106: 1–7 odd, 11–13, 15–19 odd, 20–22

- compute solutions to various IVPs using Euler's method. Compare approximate solution to actual value where possible.

NOTE: The movie Hidden Figures features a mention of "Euler's Method" in the key "chalkboard scene": https://www.youtube.com/watch?v=v-pbGAts_Fg While I'm not sure of the exact problem they are solving, I did find this publication about the re-entry problem (technical): https://www.faa.gov/other_visit/aviation_industry/designees_delegations/designee_types/ame/media/Section%20III.4.1.7%20Returning%20from%20Space.pdf

Example 1: Suppose y(x) is a solution to the initial value problem dy/dx=3-2x-0.5y, y(0)=1. Find the value of the function y at x=1 (find y(1)).
How do we do it?

Discuss
OPTION A:
1. Solve the differential equation.
2. Use the initial value y(0)=1 to find "c" and obtain the particular solution
$y = 14 - 4x - 13e^{-0.5x}$ *or y=14-4x-13e^(-0.5x)*
3. Substitute x=1 into the particular solution to find the value y(1).
NOTE: y(1)=2.11510

What if we can't solve the differential equation?
OPTION B:
Try to approximate the answer - numerical methods.
BENEFITS: It always works, even if we can't solve the differential equation!
DRAWBACKS: It only gives an approximate answer, not an exact answer.

Sketch.

Let's divide the x-interval up into 4 equal pieces.
How wide is each piece?
ANS: This important quantity is called the **step size** h. Here, h=0.25.
What are the x-coordinates?
x_0 = 0
x_1 = 0.25
x_2 = 0.5
x_3 = 0.75
x_4 = 1

Find (x_1,y_1)

See Desmos project here to watch approximation in action

Now, to find the y-value as we move from x_0 to x_1, we will use the slope of the function to determine whether we move up or down. What is the slope of the solution y at the point (0,1)?

ANS: y'=2.5

Are we moving up or down? *Sketch line. This is the **tangent line** to the function at the point (0,1).*

QUES: Could we find the equation of the tangent line, if we wanted?

ANS: YES - need a point and the slope, then use point-slope form y-y1 = m(x-x1)

How far up do we move, as we go from x_0=0 to x_1 =0.25? *Think about slope m = rise/run. If we know the "run" (the x-distance), we can calculate the "rise" (the y-distance) by multiplying run * slope.*

Y-distance is 0.25 * (2.5) = 0.625

The new y-coordinate is: old y-coordinate + y-distance = 1+0.625 = 1.625

Find (x_2,y_2)

x_2 = 0.5

y_2 = y_1 + h * (slope at (0.25, 1.625))

Continue, until you find the point (1,y_n). What is the final value of y? This is an approximation of y(1).

i	h	x_i	y_i	k = f(x_i,y_i)	y_(i+1)
0	0.25	0	1	2.5	1.625
1	0.25	0.25	1.625	1.6875	2.046875
2	0.25	0.5	2.046875	0.9765625	2.291015625
3	0.25	0.75	2.291015625	0.3544921875	2.379638672
4	0.25	1	**2.379638672**		

How close is our approximation to the "real" value? We would have to know the "real" answer to compare.

Here is the real answer: y(1) = 2.1151...

How do we make it better? Use more points / use a smaller step size!

What if h=0.1. How many points? Final approximation for y(1)?

Use spreadsheet here:

https://docs.google.com/spreadsheets/d/1DelFRN2CKpNk4oZ6UrUppaEFeyJlcj4fHbN_PsqaUMo/edit?usp=sharing

Euler's Method. Given the differential equation y' = f(x,y) with initial condition y(a)=b, find an approximate value of the solution at x=c using step size *h*.

- Find a sequence of points (x_0,y_0), (x_1,y_1), (x_2,y_2)...(x_n,y_n)
    - NOTE: the number of steps *n* is related to the step size *h* by: h = (c-a)/n
- The first point (x_0,y_0) is given by the initial condition y(a)=b, so x_0=a and y_0=b
- x-coordinate: x_(i+1)=x_i + h
  You can write down the x-coordinates immediately, since h is the difference between successive x-values:
- y-coordinate: y_(i+1) = y_i + h*f(x_i,y_i)
  To approximate the value of y at x_(i+1), we use the slope of y at x_i

I recommend making a table:
n x_n y_n f(x_n,y_n) y_(n+1)

SAVE THIS EXAMPLE:
Example2:
dy/dx=-2y+x^3 e^(-2x)
Solve

Example 3:
dy/dx = x^2-sin(x)*y^2, y(0.6)=3.5, estimate the value of y(1.8) using Euler's method with a step size of h=0.2.

"CORRECT" ANS: y(1.8) = 1.60733

i	h	x_i	y_i	k = f(x_i,y_i)	y_(i+1)
0	0.2	0.6	3.5	-6.556870299	2.18862594
1	0.2	0.8	2.18862594	-2.796195579	1.629386824
2	0.2	1	1.629386824	-1.234022515	1.382582321
3	0.2	1.2	1.382582321	-0.3416242858	1.314257464
4	0.2	1.4	1.314257464	0.2578596019	1.365829385
5	0.2	1.6	1.365829385	0.6953055316	1.504890491
6	0.5	1.8	1.504890491	1.03453176	2.022156371

WHAT DO YOU NEED TO BE ABLE TO DO?
1. Implement Euler's method by hand (with calculator) on the exam
2. Implement Euler's method and other numerical methods using your choice of technology

Day 8: Section 3.2
3.2 The Improved Euler Method and related Methods (p.109–116)
p.116: 1–7 odd, 11–13, 15–19 odd, 20–22
- compute solutions to various IVPs using Improved Euler's method. Compare approximate to actual where possible

Since yesterday's initial example ran long, I started today with a "Big Picture" - here's the setup that numerical methods can help with, and here's the 1-2-3 steps for Euler's method (and showed them using Example 1). Then did the same for Improved Euler's.

SETUP: y'=f(x,y), y(a)=c, approximate y(b) using step size h.
Find sequence of points, (x_0,y_0)=(a,c), (x_1, y_1), ... (x_n, y_n)
n = number of steps
h = (b-a)/n
EULERS METHOD
Start with (x_i, y_i)
- x_(i+1)=x_i + h
- y_(i+1) = y_i + h*f(x_i,y_i)

End with (x_(i+1), y_(i+1))
I recommend making a table:
i x_i y_i f(x_i,y_i) y_(i+1)

IMPROVED EULERS METHOD
Start with (x_i, y_i)
- x_(i+1) = x_i + h
- k1=f(x_i, y_i)
- z_(i+1) = y_i + h*k1
- k2 = f(x_(i+1), z_(i+1))
- y_(i+1)=y_i+h * [k1+k2]/2

End with (x_(i+1), y_(i+1))

Example 1: Suppose y(x) is a solution to the initial value problem $y' = 3 - 2x - 0.5y,\ \ y(1) = 0.6$. Find an approximate value of y(2) using Euler's method with step size h=0.5.

i	h	x_i	y_i	k = f(x_i,y_i)	y_(i+1)
0	0.5	1.0	0.6	0.7	.95
1	0.5	1.5	0.95	-0.475	0.7125
2	0.5	2.0	**0.7125**		

y_(i+1) = y_i + k*h = 0.6 + 0.7*0.5=.95

y2=y1+kh=0.95 + (-0.475)(0.5)=0.7125

ANS: y(2)==0.7125

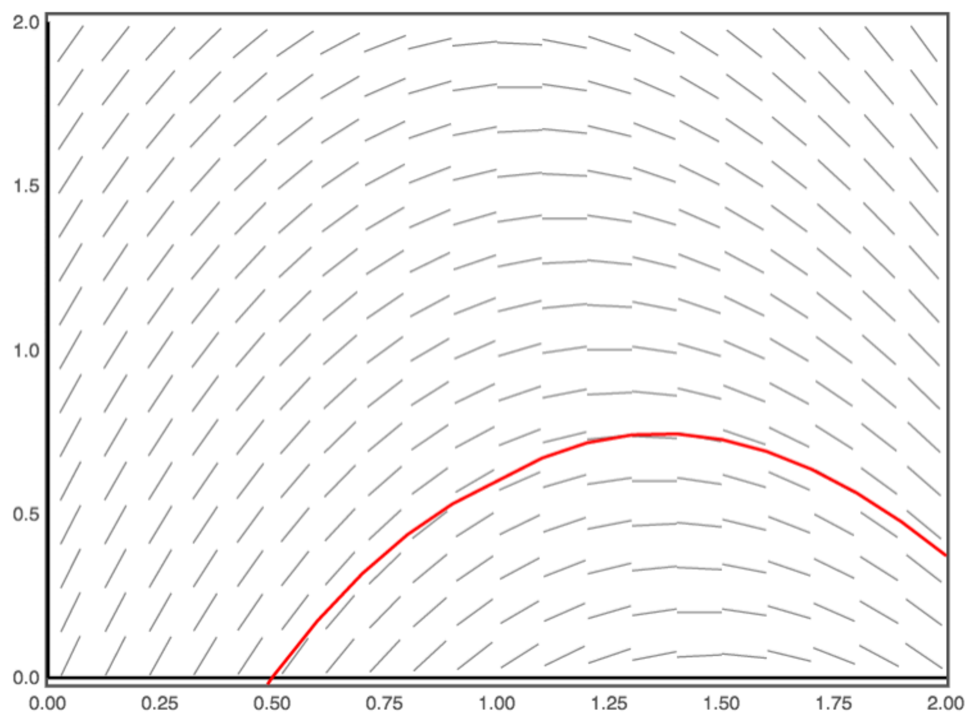
QUESTIONS: How close is this approximation to the correct answer?
How can we improve our approximation?

ACTUAL SOLUTION

NOTE: The solution to this initial value problem is:
y(x) = -4 x - 15.498 e^(-0.5 x) + 14
"CORRECT" ANS: ***y(2) = 0.298612***

PROBLEM: In Euler's method, when moving one point (x_i,y_i) to the next (x_(i+1),y_(i+1)) we use the slope at the first point to approximate the curve on the entire interval. If the slope varies across the interval, this may be inaccurate.

Discuss using slope field generator here:
*https://bluffton.edu/homepages/facstaff/nesterd/java/slopefields.html?flags=0&ODE=x,y&SYS=t,x,y&dydx=3-2x-0.5y&dxdt=x+y&dydt=x*y-1&x=0,2,20&y=0,2,15&method=euler&h=0.1&pts0=%5B1,0.6%5D*
or this screenshot:



We can improve the accuracy by using *the average of the slopes at the two points* (x_i,y_i) and (x_(i+1),y_(i+1)).

BASIC IDEA: Recall that the slope at a point is given by f(x,y). In Euler's method, we compute the next y-value by: y_(i+1)=y_i + h*f(x_i,y_i)

To improve this, we want to use: y_(i+1)=y_i + h* [f(x_i,y_i)+f(x_(i+1),y(i+1)]/2
PROBLEM: We don't yet know the y-value of the next point, y_(i+1). So instead, we use Euler's method to get an initial approximation of y_(i+1) -- call it z_(i+1) -- and then use the slope at (x_(i+1), z_(i+1)) in our calculation.

IMPROVED EULER FORMULA: yn+1=yn+f(tn,yn)+f(tn+h, yn+hfn)2h
NOTE: This is a "two step" method -- first we calculate z_(i+1) = y_i+h f(x_i,y_i), then we use that value to plug in and calculate y_(i+1).

IMPROVED EULER METHOD Given a point (x_i,y_i), how do we find the next point, (x_(i+1),y_(i+1)) Calculate: Find x_(i+1) = x_i + h Find k1=f(t_i, y_i) Find z_(i+1) = y_i + h*k1 Find k2 = f(t_(i+1), z_(i+1)) Find y_(i+1)=y_i+h * [k1+k2]/2 Now we have (x_(i+1),y_(i+1))

Example 2: Suppose y(x) is a solution to the initial value problem dy/dx=3-2x-0.5y, y(1)=0.6. Find an approximate value of y(2) using the Improved Euler's method with step size h=0.5.
"CORRECT" ANS: y(2) = 0.298612

i	h	x_i	y_i	k1	z_(i+1)	k2	y_(i+1)
0	0.5	1	0.6	0.7	0.95	-0.475	0.65625
1	0.5	1.5	0.65625	-0.328125	0.4921875	-1.24609375	0.2626953125
2	0.5	2	**0.2626953125**				

ROUND 1:
k1 = f(1,0.6)=3-2x-0.5y = 3-2(1)-0.5(0.6) =
z = y_i + k1*h=0.6+0.7*0.5 = 0.95 ← this is a temporary y-value
k2 = f(1.5,0.95)=3-2(1.5)-0.5(.95) = -0.475 ← slope at right side of interval
y_(i+1)=y_i + (k1+k2)/2 * h = 0.6 + (0.7 -0.475)/2 *(0.5) = 0.65625

ROUND 2:
k1 = f(1.5,0.65625) = 3-2(1.5)-0.5(.65625) = -0.328125
z = 1.5 + -0.328125 * (0.5) = 0.4921875
k2 = f(2,0.4921875) = 3-2(2)-0.5(0.4921875)=-1.24609375
y_(i+i)=0.65625 + (-0.328125+-1.24609375)/2 *(0.5)= 0.2626953125

ANS: according to Improved Euler's Method, **y(2) == 0.2626953125**

COMPARE:
Euler's: **y(2)==0.7125**

Improved Euler's: **y(2) == 0.2626953125**
Actual Value: ***y(2) = 0.298612***

i	h	x_i	y_i	k1	z_(i+1)	k2	y_(i+1)
0	0.5	1	0.6	0.7	0.95	-0.475	0.65625
1	0.5	1.5	0.65625	-0.328125	0.4921875	-1.24609375	0.2626953125
2	0.5	2	**0.2626953125**				

Example 3:
dy/dx = x^2-sin(x)*y^2, y(0.6)=3.5, estimate the value of y(1.8) using Improved Euler's method with a step size of h=0.2.

"CORRECT" ANS: y(1.8) = 1.60733

i	h	x_i	y_i	k1	z_(i+1)	k2	y_(i+1)
0	0.2	0.6	3.5	-6.556870299	2.18862594	-2.796195579	2.564693412
1	0.2	0.8	2.564693412	-4.07851894	1.748989624	-1.574030043	1.999438514
2	0.2	1	1.999438514	-2.363994307	1.526639652	-0.7322369764	1.689815385
3	0.2	1.2	1.689815385	-1.221415276	1.44553233	-0.0991600018 6	1.557757858
4	0.2	1.4	1.557757858	-0.431301719 3	1.471497514	0.3956183461	1.55418952
5	0.2	1.6	1.55418952	0.1455248986	1.5832945	0.7987378463	1.648615795
6	0.2	1.8	**1.648615795**				

Day 8: Section 3.3
The Runge-Kutta Method (p.119–124)
p.124: 1–7 odd, 11–13, 15–19 odd, 20–22

- compute solutions to various IVPs using the Runge-Kutta method. Compare approximate solution to actual value where possible.

Today we are going to consider one final method. This method is the most complicated, but comes with a corresponding increase in precision - the solutions "get better quickly" as the step size decreases. This method is powerful enough to be used in many modern numerical methods applications - the Runge-Kutta Method.

BASIC IDEA. We will *not* carefully develop this method from scratch. However, I want to give you a flavor of the idea.

COMPARE: To approximate the solution curve y(x) on an interval between points, from x_i to x_(i+1):

- The basic idea of Euler's Method is to approximate the solution curve y(x) with a straight line. *SKETCH*
- The basic idea of the Runge-Kutta Method is to approximate the solution curve y(x) with a parabola instead. *SKETCH.*

FACT ABOUT PARABOLAS: On an interval [a,b], the slope of the secant line through the endpoints is equal to what? (any guesses?)

- NOT the average of the slopes at each end, but
- Use the slope at three points - the two endpoints, and the point in the middle. BUT weight the slope in the middle more heavily - count it 4 times.
- If the three slopes are m1, m2, and m3, then the slope of the secant line is:
  m = (m1 + 4m2 + m3)/6
- This is the idea in Runge-Kutta.
- PROBLEMS: We don't actually know the three points - only the first one. So we make a series of approximations, two in the middle and one on the right, and combine the slopes at these approximations in the correct way.

The classic Runge-Kutta method uses a weighted average of slopes to compute the next y-value:
y_(i+1)=y_i + h (k1+ 2*k2 + 2*k3 + k4)/6 $y_{n+1} = y_n + h\left(\frac{k_1+2k_2+2k_3+k_4}{6}\right)$

NOTE: When I presented this in class in Spring 2017, I simplified the formulas by first finding each "approximate y-value" used in the subsequent evaluation of f(x,y). I labeled these z2,z3,z4 to match the corresponding k2,k3,k4 - here's the version I used in class:

RUNGE-KUTTA METHOD:
Start with: (x_i, y_i).

- x_(i+1) = x_i + h

- k1 = f(x_i, y_i)
- z2 = y_i + 0.5h*k1
- k2 = f(x_i + 0.5h, z2)
- z3 = y_i + 0.5h*k2
- k3 = f(x_i + 0.5h, z3)
- z4 = y_i + h*k3
- k4 = f(x_i + h, z4)
- y_(i+1)=y_i + h (k1+ 2*k2 + 2*k3 + k4)/6

End with (x_(i+1), y_(i+1))

RUNGE-KUTTA METHOD:
Start with: (x_i, y_i).

- x_(i+1) = x_i + h
- k1 = f(x_i, y_i)
- k2 = f(x_i + 0.5h, y_i + 0.5h*k1)
- k3 = f(x_i + 0.5h, y_i + 0.5h*k2)
- k4 = f(x_i + h, y_i + h*k3)
- y_(i+1)=y_i + h (k1+ 2*k2 + 2*k3 + k4)/6

End with (x_(i+1), y_(i+1))

Example 1: Consider the initial value problem y'+2y=x^3 e^(-2x), y(0)=1. Approximate the value of y(0.6) using a step size of 0.3.
(link to spreadsheet)

Example 1:	y'+2y=x^3 e^(-2x), y(0)=1		find y(0.6) using 2 steps					
i	**h**	**x_i**	**y_i**	**k1 = f(x_i,y_i)**	**k2=f(x_i+.5h,y_i+.5hk1)**	**k3=f(x_i+.5h, y_i+.5hk2)**	**k4=f(x+h,y+hk3)**	**Runge-Kutta y_(i+1) = y_i + h*(k1+2k2+2k3+k4)/6**
0	0.3	0	1	-2	-1.397499739	-1.578249817	-1.038232196	0.5505134347
1	0.3	0.3	0.5505134347	-1.086208955	-0.7381155225	-0.8425435523	-0.5304427882	0.31161494
2	0.3	0.6	**0.31161494**					

Table 3.3.1. Numerical solution of $y' + 2y = x^3e^{-2x}$, $y(0) = 1$, by the Runge-Kuttta method and the improved Euler method.

x	$h = 0.1$	$h = 0.05$	$h = 0.1$	$h = 0.05$	Exact
0.0	1.000000000	1.000000000	1.000000000	1.000000000	1.000000000
0.1	0.820040937	0.819050572	0.818753803	0.818751370	0.818751221
0.2	0.672734445	0.671086455	0.670592417	0.670588418	0.670588174
0.3	0.552597643	0.550543878	0.549928221	0.549923281	0.549922980
0.4	0.455160637	0.452890616	0.452210430	0.452205001	0.452204669
0.5	0.376681251	0.374335747	0.373633492	0.373627899	0.373627557
0.6	0.313970920	0.311652239	0.310958768	0.310953242	0.310952904
0.7	0.264287611	0.262067624	0.261404568	0.261399270	0.261398947
0.8	0.225267702	0.223194281	0.222575989	0.222571024	0.222570721
0.9	0.194879501	0.192981757	0.192416882	0.192412317	0.192412038
1.0	0.171388070	0.169680673	0.169173489	0.169169356	0.169169104
	Improved Euler		Runge-Kutta		Exact

Day 9: Chapter 5 - Second Order Equations
Section 5.1 - 5.1 Homogeneous Linear Equations (p.194–203) - p.203: 1–5 odd, 9–21 odd

NOTE: I'm glossing over Sec 5.1, in part because we are running behind this semester. Instead, I'll give a quick intro to second order equations and then focus on constant coefficients

Section 5.2
Constant Coefficient Homogeneous Equations (p.210–217) - p.217: 1–17 odd, 18–21

NOTE: Today I'll cover only equations with two distinct real roots

DISCUSS: Second-order differential equations
Defn. A **linear second-order differential equation** can be written in the form: $y''+p(x)y'+q(x)y=f(x)$

NOTE: If the equation cannot be written in the form above, then it is nonlinear. Nonlinear second order equations are generally hard!

Example: $y'' - y = 0$

a) What is the order? Is the equation linear? What are $p(x)$, $q(x)$, $f(x)$?

Defn: We call such an equation **homogeneous** if $f(x)=0$.

b) Verify that $y_1(x)=e^x$ is a solution.

Is this the only solution? *Try multiplying by a constant*

c) Verify that $y_2(x)=e^{-x}$ is a solution

NOTE: We can combine these two solutions to get the most general solution

d) Verify that $y = c_1 e^x + c_2 e^{-x}$ is a solution

Discuss: How many constants? How many initial values will we need, to solve?
NOTE: $y = c_1 e^x + c_2 e^{-x}$ is the **general solution** to the differential equation

d) Solve the initial value problem: $y'' - y = 0$, $y(0)=1$, $y'(0)=3$

NOTE: What's the "hard part" of the above problem, which you were *NOT* asked to do? *Find the two basic solutions!*

ALTERNATE EXAMPLE - NONCONSTANT COEs - SKIP FOR NOW
Example: Given the differential equation: $x^2 y'' + x y' -4y = 0$

a) What it the order? Is the equation linear? What are $p(x)$, $q(x)$, $f(x)$?

*Defn: We call such an equation **homogeneous** if $f(x)=0$.*

b) Verify that $y_1(x)=x^2$ is a solution

c) Verify that $y_2(x)=1/x^2$ is a solution

NOTE: We can combine these two solutions to get new solutions by: adding, multiplying by constant.
FACT: the general solution to this differential equation is $y = c_1 x^2 + c_2/x^2$

d) Verify that $y = c_1 x^2 + c_2/x^2$ is a solution

Discuss: How many constants? How many initial values will we need, to solve?

d) Solve the initial value problem: $x^2 y'' + x y' -4y = 0$, $y(1)=2$, $y'(1)=0$

DISCUSS: In general, homogeneous linear second order equations have **two basic solutions y_1 and y_2** - and the **general solution is given by linear combination: $y = c_1y_1+c_2y_2$**
NOTE: Even linear second-order equations are often hard - so we will start simple.

Defn. The equation has **constant coefficients** if p(x) and q(x) are constant. In general, we allow a constant in front of the y" term as well:
Linear second-order homogeneous w/ constant coefficients: ay"+by'+cy=0

QUES: Did our example above fit this pattern?

SOLVING LINEAR SECOND-ORDER HOMOGENEOUS EQUATIONS WITH CONSTANT COEFFICIENTS:

Given: ay"+by'+cy=0
IDEA: guess a solution of the form y=e^rx. How do we find the correct constant r?
Take derivatives and plug in.
(ar^2+br+c)e^rx = 0

Since e^rx is never zero, this equation has a solution only when ar^2+br+c=0
The **characteristic equation**: ar^2+br+c=0
The **characteristic polynomial** ar^2+br+c

How do we solve for r? What kind of equation is it? How many solutions?

FACT: If this equation has two real roots, r1 and r2, then the basic solutions are y1=e^r1x and y2=e^r2x, and the general solution to the differential equation is: y=c1e^r1x + c2e^r2x

EXAMPLE: Find the general solution: y"+6y'+5y=0. Now find the particular solution that satisfies y(0)=3, y'(0)=-1

GENERAL SOLUTION: y=c1e^-x+c2e^-5x
PARTICULAR SOLUTION: y=7/2 e^-x -½ e^-5x

Day 10:
Section 5.2
Constant Coefficient Homogeneous Equations (p.210–217) - p.217: 1–17 odd, 18–21
Today I'll cover repeated roots & complex roots

SECOND ORDER LINEAR HOMOGENEOUS EQUATIONS WITH CONSTANT COEFFICIENTS
Given ay"+by'+cy=0:
STEP 1: GUESS a solution of the form y=e^rx
STEP 2: To find r, substitute and solve for r
SHORTCUT: This always leads to the characteristic equation: ar^2+br+c=0
REMINDER: the left side is called the characteristic polynomial
SOLVE WITH QUADRATIC FORMULA: r=[-b+-sqrt(b^2-4ac)]/2a, $r = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$
STEP 3: **If r1 and r2 are distinct real roots,** then the general solution is y=c1 e^r1x + c2 e^r2x
If r1=r2 is a repeated root, then the general solution is: _______?
If r1 and r2 are complex roots lambda+-omega*i $r_1, r_2 = \lambda \pm \omega i$, then the general solution is: _____?

Our goal for today is to fill in these two blanks!

QUESTION: Does every quadratic equation have two real roots?
NOPE! MUST CONSIDER a) repeated roots, b) complex roots

REPEATED ROOTS
Example. Find the most general solution: y"+6y'+9y=0

The characteristic equation has solutions r1=-3, r2=-3. BUT this would give a general solution y=c1e^-3x+c2e^-3x, and we can factor out e^-3x and combine constants to get: y=c3e^-3x -- *this is not the most general solution, since there are always TWO basic solutions. What is the other solution?*

STRATEGY: We have one solution, y1=e^-3x. GUESS the other solution has the form
y=uy1 = u e^-3x, where u is some unknown function.
Substitute into the original equation to find u (find y', y" first)
We get u"=0. Integrate with respect to x twice:
u'=c2
u=c2x+c1
y=e^-3x(c2x+c1) = c2xe^-3x + c1e^-3x
NOTE that if c2=0, we get our original solution. If c1=0, we get our second solution c2xe^-3x. Thus we have:

GENERAL SOLUTION: y=c1e^-3x + c2xe^-3x

RULE FOR REPEATED ROOTS:
RULE: If the characteristic equation has a repeated roots r1=r2, then the general solution is
y=c1e^r1x + c2xe^r1x

COMPLEX ROOTS

EXAMPLE. Find the general solution: $y''+4y'+13y=0$
The roots of the characteristic polynomial are: $r_1=-2+3i$, $r_2=-2-3i$.
GUESS BASIC SOLUTIONS: $y_1 = e^{(-2+3i)x}$, $y_2=e^{(-2-3i)x}$

Simplify: $y_1=e^{-2x} e^{3ix}$

How do we make sense of e raised to a complex power?

EULER'S FORMULA: $e^{bi} = \cos b + i \sin b$

So $y_1 = e^{-2x} (\cos 3x + i \sin 3x)$
And $y_2 = e^{-2x} (\cos -3x + i \sin -3x)$

TRIG IDENTITIES: $\cos -b = \cos b$, $\sin -b = - \sin b$

$Y_2 = e^{-2x} (\cos 3x - i \sin 3x)$

GENERAL SOLUTION: $y= c_1y_1 + c_2y_2$

$y = c_1e^{-2x} (\cos 3x + i \sin 3x) + c_2e^{-2x} (\cos 3x-i\sin 3x)$

$y = e^{-2x} [(c_1+c_2)\cos 3x + (c_1 i - c_2 i)\sin 3x]$

GENERAL SOLUTION: $y = e^{-2x} (d_1 \cos 3x + d_2 \sin 3x)$, where d_1 and d_2 are constants.

RULE: If the characteristic equation has complex roots $r_1=\lambda+\omega i$, $r_2=\lambda-\omega i$, then the general solution is $y=e^{\lambda x} (c_1 \cos (\omega x) + c_2 \sin (\omega x))$

Example: $y''+6y'+10y=0$, $y(0)=1$, $y'(0)=0$

GENERAL: $y=e^{-3x}(c_1 \cos x + c_2 \sin x)$

PARTICULAR: $y=e^{-3x} (3\sin x + \cos x)$

Day 11: Section 5.3 - Nonhomogeneous Linear Equations (p.221–227) - p.227: 1–5 odd, 9–13 odd, 16–20 even, 25–29 odd, 33–37 odd
Section 5.4 - The Method of Undetermined Coefficients I (p.229–235) - p.235: 1–29 odd
NOTE: These sections give a more theory-based presentation of this material (as opposed to Boyce & DiPrima) - I have not entirely followed Trench's exposition, choosing to focus more on working out examples.

NONHOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS
Theorem. Given a nonhomogeneous linear differential equation with constant coefficients:
$y''+p(x)y'+q(x)y = f(x)$
The general solution will be of the form:
$y(x) = c_1y_1(x) + c_2y_2(x) + y_p(x)$
Where:
- $c_1y_1(x) + c_2y_2(x)$ (sometimes called y_h) is the general solution to the the complementary (homogeneous) equation $y''+p(x)y'+q(x)y = 0$
- and $y_p(x)$ (sometimes called $Y(x)$) is any particular solution to the original (nonhomogeneous) equation.

Finding the solution involves two steps:
1. Solve the complementary equation to obtain $c_1y_1(x) + c_2y_2(x)$ *(using the techniques of the previous section)*
2. Find a single solution y_p to the original equation

It is step 2 that we will be focussing on this week. The basic idea is to guess a solution using the function f(x) on the right side as our guide - this method is called ***the method of undetermined coefficients***.

Example 1: $y''+9y=36$
Step 1: Solve the complementary/homogeneous equation $y''+9y=0$
SOLUTION: $y_h=c1 \sin(3x) + c2 \cos(3x)$
Step 2: Guess a solution to the original equation.

What kind of function is f(x)? Is it a constant function, an exponential function, a trig function, a polynomial, etc?

Since f(x) is a constant function ($f(x)=36$), let's guess our solution will also be constant:
GUESS $y_p = A$
Now find y_p' and y_p'', plug into original nonhomogeneous equation to find A.
PARTICULAR SOLUTION TO ORIGINAL EQ: $y_p = 4$

Step 3: Combine the solutions found in Steps 1 and 2 to obtain the general solution:
GENERAL SOLUTION: $y=c1 \sin(3x) + c2 \cos(3x) + 4$

Example 2: $y''-3y'-4y=3e^{2x}$, $y(0)=17/2$, $y'(0)=10$
Step 1: SOLUTION - COMPLEMENTARY: $y_h=c_1 e^{4x} + c_2 e^{-x}$

Step 2: What kind of function is $f(x)$?
GUESS solution: $y_p = Ae^{(2x)}$
PARTICULAR SOLUTION: $y_p = -1/2\, e^{(2x)}$
Step 3: GENERAL SOLUTION $y= c_1 e^{(4x)} + c_2 e^{(-x)} -1/2\, e^{(2x)}$

Now substitute the initial conditions into y, y' to find c_1, c_2
PARTICULAR SOLUTION: $y= 4e^{(4x)} + 5e^{(-x)} -1/2\, e^{(2x)}$

Save this example for later? IVP, f(x) is a polynomial
Example 2: *a) Find the general solution of* $y'' -2y' + y = -3 - x + x^2$
b) Find the particular solution satisfying $y(0)=-2$, $y'(0)=1$

a) Part 1: $y_h = c_1e^x+c_2xe^x$
Part 2: What kind of function is f(x)? What degree?
GUESS a solution that is a polynomial of degree 2.
$y_p = A + Bx + Cx^2$
Take derivatives and substitute.
PARTICULAR SOLUTION: $y=1+3x+x^2$
GENERAL SOLUTION: $y = 1+3x+x^2 + c_1e^x + c_2xe^x$

b) Find y'. Substitute initial conditions, solve for c1 and c2.
PARTICULAR SOLUTION: $y=1+3x+x^2-3e^x+xe^x$

Day 12: Section 5.4 - The Method of Undetermined Coefficients I (p.229–235) - p.235: 1–29 odd
Section 5.5 - The Method of Undetermined Coefficients II (p238-244)

WARM UP
Example 1: y" - 9y' + 14y = 212 sin(2x)

STEP 1: General solution to complementary eq: y_h = c_1 e^(2x) + c_2 e^(7x)
STEP 2: What kind of function is f(x)? What should we guess for a solution y_p?
Whatever we guess for y_p, we will have to take two derivatives and substitute on the left side - after simplifying, we should get exactly 212 sin(2x). But for the sin(2x) will have derivatives involving both sin(2x) and cos(2x). How do we accommodate this? We guess that y_p is a combination of sines and cosines:
GUESS: y_p = A sin(2x) + B cos(2x)
Take derivatives, substitute, solve for A and B
PARTICULAR SOLUTION y_p = 5 sin(2x) + 9 cos(2x)
STEP 3: GENERAL SOLUTION: y = c_1 e^(2x) + c_2 e^(7x) + 5 sin(2x) + 9 cos(2x)

GREAT! What else can happen in these problems?

1. What if the right side f(x) contains a solution to the complementary equation?
2. What if the right side f(x) is a combination of functions?

Example 2: y"-7y'+12y = 5 e^(4x)

STEP 1: y_h = c_1 e^(4x) + c_2 e^(3x)
STEP 2: What should we guess for y_p?
NOTE: The obvious guess, y_p = Ae^(4x), won't work - because this is already a solution to the complementary equation. We need a function that is not equal to Ae^(4x), but has Ae^(4x) among the first and second derivatives.
GUESS: y_p = xe^(4x)
PARTICULAR SOLUTION: y_p = 5xe^(4x)
STEP 3: GENERAL SOLUTION: y = c_1 e^(4x) + c_2 e^(3x) + 5xe^(4x)

Example 3: WHAT GUESS SHOULD WE MAKE FOR y_p, BASED ON DIFFERENT f(x)?
a) ay" + by' + cy = 4e^(-5x)
GUESS: y_p = Ae^(-5x)
WHAT IF e^(-5x) is a solution to the complementary equation?
GUESS: y_p = Axe^(-5x)
WHAT IF xe^(-5x) is also a solution to the complementary equation?
GUESS: y_p = Ax^2e^(-5x)

b) ay" + by' + cy = 4x+3x^3
What kind of function is f(x)? What is the degree?
GUESS: y_p = Ax^3 + Bx^2 + Cx + D ("a polynomial of degree 3")

NOTE: For equations with constant coefficients, the complementary equation will never have a solution that is a polynomial - so this guess will work!

c) ay" + by' + cy = 4e^(2x)sin(6x)

What kind of function is f(x)?

GUESS: y_p = Ae^(2x)sin(6x)+Be^(2x)cos(6x)

Day 13: Section 5.6 Reduction of Order (p.248–252) p.253 1–3, 5, 9, 13, 17, 19, 25, 31

Given a single solution to the complementary equation, find the general solution using reduction of order

Today:
$P_0(x)y'' + P_1(x)y'+P_2(x)y=F(x)$
And
$P_0(x)y'' + P_1(x)y'+P_2(x)y=0$

What kind of equation is it? Second order, linear, (nonconstant coefficients), homogeneous or nonhomogeneous

How do we solve these? WHO KNOWS?
But today we'll learn a technique that works IF we have a hint.

Reduction of order
A method to find the general solution to

$P_0(x)y'' + P_1(x)y'+P_2(x)y=F(x)$
And
$P_0(x)y'' + P_1(x)y'+P_2(x)y=0$

PROVIDED you know ONE solution y_1 to the complementary equation.

STEP 1: Guess a solution to the original equation of the form $y=u\,y_1$. Take derivatives and plug in to the original equation, simplify.
STEP 2: The result should have only u'', u' (not u). Substitute $w=u'$, $w'=u$. Solve this first-order linear equation for w.
STEP 3: Now replace w with u'. Integrate to find u.
STEP 4: Combine u and y_1 to obtain the general solution $y=u\,y_1$.

Reduction of order
A method to find the general solution to

$P_0(x)y'' + P_1(x)y'+P_2(x)y=F(x)$
And
$P_0(x)y'' + P_1(x)y'+P_2(x)y=0$

PROVIDED you know ONE solution y_1 to the complementary equation.

STEP 1: Guess a solution to the original equation of the form $y=u\,y_1$. Take derivatives and plug in to the original equation, simplify.
STEP 2: The result should have only u'', u' (not u). Substitute $w=u'$, $w'=u$. Solve this first-order linear equation for w.
STEP 3: Now replace w with u'. Integrate to find u.
STEP 4: Combine u and y_1 to obtain the general solution $y=u\,y_1$.

Example 1: Solve $x^2y''-3xy'+3y=0$ given that $y_1=x$ is a solution.

HINT: What is the complementary equation? Same as original!
Step 1: x^3u”-x^2u’=0 -> x^3w’-x^2w = 0
Step 2: w=C1x, u=C1x^2/2 + C2
Step 3: y=ux=C1x^3/2+C2x or y = c1x^3 + c2x.

Example 2: Find the general solution of xy”-(2x+1)y’+(x+1)y=x^2,
Given that y1=e^x is a solution of the complementary equation xy”-(2x+1)y’+(x+1)y=0.
Step 1: u”-u’/x=xe^-x
Step 2: w = -xe^-x + C1x, u=(x+1)e^-x + C1x^22 + C2
Step 3: y = ux = (x+1) + c1x^2e^x + c2 e^x

5.7 Variation of Parameters (p. 255-262)
p.262: 1-5, 7, 11, 13, 31, 33, 34

Today we look at a method called variation of parameters:

- Used for finding a *particular solution* of $P_0(x)y'' + P_1(x)y' + P_2(x)y = F(x)$. We call the particular solution y_p.
- To use the method, we must already know the general solution to the complementary equation, $P_0(x)y'' + P_1(x)y' + P_2(x)y = 0$, which we call $y_h = c_1y_1 + c_2y_2$
- Once we find the particular solution to the original equation, the general solution of the original equation will be: $y = y_p + c_1y_1 + c_2y_2$.
- NOTE: We must assume that the leading coefficient, $P_0(x)$is nonzero on any interval we consider.

QUESTION: This method is similar to 'reduction of order', but in reduction of order we only need ONE solution to the complementary equation. Why do we need this method, too?
ANS:

- Usually simpler than reduction of order (provided we know two solutions to comp eq)
- This method is more general - the idea can used to solve more complicated differential equations (higher order equations, linear systems of equations).
- This method is a powerful tool used by researchers in differential equations! *If you study more diffy q's, you will a most certainly see it!*

BEWARE: Much of the method will seem familiar, but there will be one or two extra things to remember!
BEWARE: Things will get messy for a while, but they will get better!

THE IDEA BEHIND VARIATION OF PARAMETERS:
GUESS a solution of form $y_p = u_1y_1 + u_2y_2$
NOTE: We have TWO unknown functions, but only ONE equation to satisfy. This means we have some freedom to restrict u1 and u2.

Take derivative: $y_p' = u_1y_1' + u_1'y_1 + u_2y_2' + u_2'y_2$

Here is the "trick": Let's impose a condition on u1, u2 that will make this easier to solve.

LET US REQUIRE THAT: $u_1'y_1 + u_2'y_2 = 0$**, **IMPORTANT EQUATION 1.**

So $y_p' = u_1y_1' + u_2y_2'$

Take derivative again:

$y_p'' = u_1y_1'' + u_1'y_1' + u_2y_2'' + u_2'y_2'$

Plug y_p, y_p', y_p'' into the original equation, and group together terms with u1, u2:

- $u_1(P_0y_1'' + P_1y_1' + P_2y_1) + u_2(P_0y_2'' + P_1y_2' + P_2y_2) + P_0(u_1'y_1' + u_2'y_2') = F(x)$

NOTE: what are y1, y2? *Solutions to complementary equation.* When we plug them into LHS of original equation we get 0. This means the coefficients of u1 and u2 are 0!
Thus we have: $P_0(u_1'y_1' + u_2'y_2') = F(x)$, or

$u_1'y_1' + u_2'y_2' = \frac{F}{P_0}$ ** **IMPORTANT EQUATION II**

By combining the two "important equations", we can solve for u1' and u2', then integrate to find u1, u2 (since we are looking for a particular solution yp, take the constants of integration to be 0 for simplicity).

VARIATION OF PARAMETERS:
TO SOLVE: $P_0(x)y'' + P_1(x)y' + P_2(x)y = F(x)$, given y_1, y_2 (independent) solutions to the complementary equation:

1. GUESS A PARTICULAR SOLUTION: $y_p = u_1y_1 + u_2y_2$
2. WRITE DOWN THE SYSTEM OF EQUATIONS:

$$u_1'y_1 + u_2'y_2 = 0$$
$$u_1'y_1' + u_2'y_2' = \frac{F}{P_0}$$

3. Solve the system of equations for u_1', u_2'
4. Integrate to find u_1, u_2 (let constants of integration =0)
5. Substitute into y_p
6. The general solution is $y = y_p + c_1y_1 + c_2y_2$

Example 1: Find the general solution $x^2y'' - 2xy' + 2y = x^{9/2}$, given that $y_1 = x$ and $y_2 = x^2$ are solutions of the complementary equation $x^2y'' - 2xy' + 2y = 0$.

ANS: $y_p = \frac{4}{35}x^{9/2}$

Example 2: Find the general solution: $y'' + 3y' + 2y = \frac{1}{1+e^x}$

HINT: First find y_h, the general solution to the complementary equation.

ANS: $y_h = c_1e^{-x} + c_2e^{-2x}$, $y_p = (e^{-x} + e^{-2x})\ln(1 + e^x)$

NOTE: the term $-e^{-x}$ *in yp is a solution to the complementary equation, so it disappears.*

General solution: $y = (e^{-x} + e^{-2x})\ln(1 + e^x) + c_1e^{-x} + c_2e^{-2x}$

NOTE: NEED TO Add notes & WW assignments for the following two lectures, Spring 2019

6.1 Spring Problems I 6.2 Spring Problems II	268-277 279-284	p.277: 1, 3, 7-13 odd, 19, 21 p.288: 3, 4, 7-11 odd, 14-16
6.2 Spring Problems II (continued) 6.3 The RLC Circuit	284-287 290-295	p.288: 13, 17-20 p.295: 1-10

Day 14: Section 7.1 Review of Power Series (p.307–316) p.317: 1, 11, 13, 15–17
Section 7.2 Series Solutions Near an Ordinary Point I (p.320–328) p.329: 1, 3, 8, 11–13, 19–25 odd
7.1 Find radius of convergence of a power series. Simplify expressions involving series.
7.2 Find power series solutions to second-order linear ODEs with polynomial coefficients (either find first 7 terms, or give formula)

DISCUSSION: Follow-up to "flipped class" assignment.
- what is a power series? taylor series/maclaurin series?

RECALL: a **power series** is like a polynomial - but with infinitely many terms: a0+a1x+a2x^2+...

NOTE: We write this in sigma notation $\sum_{n=0}^{\infty} a_n x^n$

QUESTION: What is cos(0.7)? Can you find it without using any trig functions on your calculator, only +/-//?*

RECALL: Taylor and MacLaurin Series are just a particular kind of power series - used to solve a basic problem

BASIC IDEA: (Desmos demonstration)
Start with any function, y=cosx.
Pick a point on the graph, say x=0 -- this gives the point (0,1).
I am going to try to make a polynomial that matches my function at that point.
Here goes:
First try: y=1
Second try: y=1-1/2 x^2
Third try: y=1 - 1/2 x^2 + 1/24 x^4-1/720 x^6

QUESTION: What is cos(0.7)?
Find it without using any trig functions on your calculator! Plug into our third approximation.
NOTE: cos(0.7) == 0.764842187
Third approx gives y(0.7)=0.764841

NOTE: is our polynomial up to x^6 **exactly** equal to cos(x)? How could we make it better?

If we use infinitely many terms, the result is a power series that is:
- exactly equal to cosx
- equal to cosx not just at x=0 but for *any* x in the real numbers.

QUESTION: What makes one power series different from another? *The coefficients.*
QUESTION: How do we find the "right coefficients" so that our power series matches our function, in this case cosx? *That's the theory of Taylor Series:*

DEFN: If a function f(x) has derivatives of all orders at x=0, then the MacLaurin Series of f(x) is a0+a1x+a2x^2+a3x^3+a4x^4+...= $\sum_{n=0}^{\infty} a_n x^n$. If we consider a different point x=c, then we get the Taylor Series of f(x) at x=c, a0 + a1(x-c) + a2(x-c)^2+a3(x-c)^3+... $\sum_{n=0}^{\infty} a_n (x-c)^n$.

1. The coefficients a_n are given by: $a_n = \frac{f^{(n)}(c)}{n!}$
2. Amazingly, for most functions, the Taylor Series will be equal to the original function $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x-c)^n$

(TALK ABOUT INTERVAL OF CONVERGENCE?)

HOW DOES THIS FIT INTO DIFFERENTIAL EQUATIONS?

Sometimes we cannot solve a differential explicitly by finding a formula for y.
However, if we think of a power series for y, we might be able to find the coefficients - this will give us an approximation for y (it won't match y exactly unless we use infinitely many terms, but it will be very close to correct - especially near the point x=c that we started with).

NOTE: Instead of the example below, I started the example from the following lecture:
Example 1. Find the series solution of the initial value problem **y''-xy=0, y(0)=3, y'(0)=1.** Estimate the value of y(2) using the first eight terms of the power series solution.

I only got the setup done - then, the following day, I finished it out and had a little more discussion about recurrence relations, etc. I never came back to the example below.

~~Example 1. Find the general solution of the differential equation y''+y=0 as a power series. *Give the coefficients in terms of constants a_0 and a_1.*~~

a. Write y as a power series about about x=0 $y = \sum_{n=0}^{\infty} a_n x^n$

b. Take derivatives of y and substitute into the differential equation.
c. equate coefficients on both sides to obtain a series of equations involving the an.
d. Take a0, a1 as given. our goal is to use these equations to EXPRESS LATER TERMS IN TERMS OF EARLIER ONES!
e. Make a list of a0, a1, a2, ... up to a7, all IN TERMS OF a0 and a1
f. put these back into the original power series to obtain an expression for y in terms of a0, a1

QUESTION: Normally we have two basic solutions, combined with two constants. How does that show up here? What are the basic solutions?

TWO DIRECTIONS WE CAN GO FROM HERE:

1. Solve IVP by finding values for an up to some point - get an approximate solution!
2. Simplify this expression, try to write it in sigma notation, relate it to other (known) series

ON BOARD: Write out y=a0+a1x+a2x^2+... (up to a5).
QUES: How many constants? (infinite)!
- Find y’, y”
- substitute y”, y into the differential equation
- collect like terms (terms with same power of x). Ask them to generate the next couple of terms (up to x^8) by pattern matching.
- equate left and right sides -- set each coefficient expression equal to 0.

Day 15: Section 7.3 Series Solutions Near an Ordinary Point II (p.335–338)

p.338: 1–5 odd, 19–23 odd, 33–37 odd, 41–45 odd
7.3 Find power series solutions to second-order linear ODEs with polynomial coefficients (examples with no explicit formula possible - calculate first few terms)

I had started this example the previous day, finished it today.

Example 1. Find the series solution of the initial value problem y''-xy=0, y(0)=3, y'(0)=1. Estimate the value of y(2) using the first eight terms of the power series solution.

PROCEED AS BEFORE
Do we know any of the an? (y=f(x), initial conditions: a0=y(0)/0!=3, a1=y'(0)/1!=-3)
Use these to find later terms (first eight). Use the resulting partial power series to estimate y(2).

RECURR. REL: a_(n+3)=1/(n+3)(n+2) a_n, with a_2=0.
TAYLOR: y=3+x+1/2 x^3+1/12 x^4+1/60 x^6+1/504 x^7+...
APPROX: y(2)= 11.654
ACTUAL ANSWER: y(2)=11.8037

*Did *not* cover this one:*

Example 2. Find the general solution to (1+8x^2)y"+2y=0 (give the first 6 terms in terms of constants a0 and a1)

RECURR. REL: a_(n+1)=-(8n(n-1)+2)/(n+2)(n+1) *a_n
a2=-2/2 a0
a3=-2/6 a1
a4=-18/12 a2
a5=-50/20 a3
a6=-98/30 a4
a7=-162/42 a5

Example 2 (GROUP WORK VERSION). Find the first 6 terms (up to a5) of the power series solution to (1+8x^2)y"+2y=0, y(0)=30, y'(0)=6, and use them to estimate the value of y(1.6)

a0=30
a1=6
a2=-2/2 a0 = -a0 = -30
a3=-2/6 a1 = -1/2 a1 = -3
a4=-18/12 a2 = -3/2 a2 = -3/2 (-30) = 45
a5=-50/20 a3 = -5/2 a3 = -5/2 (-3) = 15/2

$y(x) \approx 30 + 6x - 30x^2 - 3x^3 + 45x^4 + \frac{15}{2}x^5$

$y(1.6) \approx 324.0672$

Day 16: Section 7.4 Regular Singular Points Euler Equations (p.344–346) p.347: 1–12

*NOTE: This section does *not* use series to find solutions. Instead, the techniques are similar to those used second-order linear constant coefficient equations (solutions depend on roots of characteristic equation)*

Solve Euler equations for which the characteristic equation has 2 real roots, repeated root, or complex roots.

Are there any differential equations that cannot be solved by the series methods we've discussed so far? YES. For equations of the form:

$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0$

Our method will have problems if we try to create a series around a point $x=c$ where the leading coefficient is 0 - that is, if $P_0(c)=0$.

NOTE: Today, we will just look at *one* kind of equation of this form. BUT this type of equation can actually be solved without series. Lucky!

NOTE: There are extensions of our series methods that will work in these cases.

Defn: Euler Equations: Can be written in the form

$ax^2y''+bxy'+cy=0$

Where a,b,c are constants and a is not=0. Assume $x>0$.

TO SOLVE THE EULER EQUATION $ax^2y''+bxy'+cy=0$ (with $x>0$)

GUESS solution of form $y=x^r$ (r is a constant)
Indicial equation: $ar(r-1)+br+c=0$
Suppose indicial equation has roots r_1, r_2.
TWO REAL ROOTS: BASIC SOLUTIONS $y_1=x^{r_1}$, $y_2=x^{r_2}$
REPEATED ROOT: $r_1=r_2$, BASIC SOLUTIONS: $y_1=x^{r_1}$, $y_2=\ln(x)x^{r_1}$
COMPLEX ROOTS: r_1=lamba+omega*i, $r_2=\lambda-\omega i$
BASIC SOLUTIONS: $y_1=x^{\lambda}\cos(\omega \ln(x))$, $y_2=x^{\lambda}\sin(\omega \ln(x))$

NOTE: we will still have problems when $x=0$. Because of this, we will consider only solutions for $x>0$.

Example: Find the general solution $x^2y''-xy'-8y=0$

IDEA: Assume a solution of the form $y=x^r$ (for some constant r).

Take derivatives and substitute. Factor out x^r.

$r(r-1)-r-8=0$

The result is like the characteristic equation, but here we call it the *indicial equation*.

Two real roots.

Two basic solutions:

$y_1=x^4$

$y_2=x^{-2}$

General solution: $y = c_1x^5 + c_2x^{-2}$

EXAMPLE 2: x^2y”-5xy’+9y=0, y(1)=3, y’(1)=5
ANS: y=3x^3-4x^3ln(x)

EXAMPLE 3: x^2y”+3xy’+2y=0
ANS: y=c1x^-1 cos(lnx)+c2x^-1 sin(lnx)

Day 25: Section 8.1 - Introduction to the Laplace Transform (p.394–402)
[NOTE: use table on p.463 of textbook for homework]
p.403: 1(a,b,d,e), 2(b,c,f,g,h,i), 4, 5, 18

ASIDE: The Laplace Transform is basically a continuous version of a Power Series -- *Where the Laplace Transform comes from (Arthur Mattuck, MIT). video: part 1, part 2*

Overview: What is the Laplace transform all about? How do we use it/why do we study it?

- The Laplace transform of a function f(t) is another function called ℒ{f(t)}, or F(s).
  - NOTE: This is similar to the way that the derivative of a function f(t) is another function f'(t) or dfdt.
  - NOTE: The variable changes when we compute the Laplace transform - if the original f(t) is a function of t, then the Laplace transform is a function of another variable s.
- We use it to make solving differential equations easier, following this outline:
  1. Start with a differential equation.
  2. Take the Laplace transform of both sides.
     This replaces the differential equation with a much simpler (algebraic) equation.
  3. Solve the algebraic equation.
  4. Simplify the solution.*
     ** requires partial fraction decomposition*
  5. Take the inverse Laplace transform of the solution.
  6. This gives the solution to the original differential equation.
- Yes, but WHAT is the Laplace transform? Ask me about it sometime (go on...)

TRICKY PARTS: #2, #4, #5.
Today: #2 - how to take the Laplace transform.
- Finding the Laplace transform using the definition
- some basic Laplace transforms
- linearity of the Laplace transform
- Finding the Laplace transform using a table

Defn. If f(t) is a function** defined for t>0, then the Laplace transform of f(t) is:

$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)dt.$

** NOTE: In fact, we also need the function f(t) to satisfy two additional conditions:

1. f(t) must be piecewise continuous on any interval [0,inf]
2. f(t) must be of exponential order: $|f(t)| \leq Ke^{at}$ *when* $t \geq M$ *(for some constants K, a, and M, with K and M positive).* This says that it cannot grow too quickly.

Example 1: Find the Laplace Transform of the constant function f(t)=1.
ANS: F(s)=1/s, s>0
NOTE: Interval of convergence!!

Example 2: Find the Laplace Transform of f(t)=e^at where a is constant
ANS: F(s)=1/(s-a), s>a

HANDOUT: Laplace Transforms for common functions (p463-464)

Example 3: Find the Laplace Transform of:

A. $f(t) = t^3 + e^{6t}$

B. $f(t) = \sin(5t) + e^{2t}\cos(4t)$

Day 25: 8.2 The Inverse Laplace Transform (p.405–412)
[NOTE: use table on p.463 of textbook for homework]
p.412: 1(a,b,d,e), 2(a–e), 3(a–d), 4(a,d,e), 6(a), 7(a), 8(a,d)

TODAY: taking Laplace transform, inverse Laplace transform using table
RECALL: Laplace transform of: $\mathcal{L}\{1\}$, $\mathcal{L}\{e^{at}\}$, $\mathcal{L}\{t^n\}$
LINEARITY RULE: For function *f*,*g* and constants *a*,*b*, $\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$
Ex: $\mathcal{L}\{5 + 6t^5 + 2e^{3t}\}$

GROUP WORKSHEET
(Submit answers on a separate sheet)

Part I. Find the Laplace Transform of each function, and determine the interval on which it is defined.

1. $t^2 + 4t^3 - 7t^6$
2. $\sin t + \cos 3t$
3. $5e^{2t} - 4\sin 3t$
4. $5t^2 + 3\sin 5t - 2e^{6t}\cos 2t$

Part II. Find the inverse Laplace Transform of each function.

1. $\frac{3}{s} + \frac{4!}{s^5} + \frac{1}{s-7}$, $s > 7$
2. $\frac{1}{s^2+25} + \frac{s}{s^2+25}$, $s > 0$
3. $\frac{1}{(s-4)^2+9}$, $s>2$
4. $\frac{1}{(s-6)^7} + \frac{5}{2s-7}$, $s > 6$
5. $\frac{5}{s^2-8s+41}$, $s > 4$

Day 27: Section 8.3 Solution of Initial Value Problems (p.414–419)
[NOTE: use table on p.463 of textbook for homework] p.419: 1–31 odd

Ex: Find the Inverse Laplace Transform:
a) $\frac{5}{(s-4)^2+25}$, s>4
b) $\frac{5}{s^2-8s+41}$, s>4
c) $\frac{1}{s-1}+\frac{4}{s+4}$, s>1
d) $\frac{5s}{s^2+3s-4}$, s>1

(HINT: a =b, c=d)
Starting from b), how do we get to a)? Sim. from d to c?

PARTIAL FRACTION DECOMPOSITION (discuss, do the example above).

Resources on Inverse Laplace and Partial Fractions:
Paul's Notes on Partial Fractions:
http://tutorial.math.lamar.edu/Classes/Alg/PartialFractions.aspx
OpenLab assignment from 2014 with videos on Inverse Laplace and Partial Fractions:
https://openlab.citytech.cuny.edu/2014-spring-mat-2680-reitz/?p=383
More details if the numerator has higher degree than the denominator:
https://openlab.citytech.cuny.edu/2015-spring-mat-2680-reitz/?tag=partial-fractions

SOLVING IVPs USING LAPLACE TRANSFORM
We will need to be able to take the Laplace transform of the derivative of a function.

Laplace transform of a derivative
Theorem. Suppose *y(t)* is continuous and of exponential order, and y', y"*, are* piecewise continuous.
Then the Laplace transform of y(t) is Y(s)
... of y'(t) is sY(s)-y(0)
... of y"(t) is s^2 Y(s)-sy(0)-y'(0)

Example 1. Use the Laplace transform to solve the differential equation $y'' - y' - 2y = 0$ with initial conditions $y(0) = 1,\ y'(0) = 0$.

ANS: $Y(s) = \frac{1/3}{s-2} + \frac{2/3}{s+1}$, $y = 1/3\, e^{2t} + 2/3\, e^{-t}$

Example 2. Find the solution of the differential equation $y'' + y = \sin 2t$ satisfying initial conditions $y(0) = 2,\ y'(0) = 1$.

ANS: $Y(s) = \frac{2s}{s^2+1} + \frac{5/3}{s^2+1} - \frac{2/3}{s^2+4}$, $y = 2\cos t + 5/3\sin t - 1/3\ \sin 2t$

More complicated example worked out in some detail:
Example: Solve using Laplace Transform $y'' + 2y' + y = 6\sin t - 4\cos t,\ y(0) = -1,\ y'(0) = 1$
Take the Laplace transform of both sides:

$$s^2F(s) - sf(0) - f'(0) + 2(sF(s) - f(0)) + F(s) = 6 * \frac{1}{s^2+1} - 4\frac{s}{s^2+1}$$

$$s^2F(s) + s - 1 + 2(sF(s) + 1) + F(s) = \frac{6-4s}{s^2+1}$$

$$s^2F(s) + s - 1 + 2sF(s) + 2 + F(s) = \frac{6-4s}{s^2+1}$$

$$F(s)(s^2 + 2s + 1) + s + 1 = \frac{6-4s}{s^2+1}$$

$$F(s)(s^2 + 2s + 1) = \frac{6-4s}{s^2+1} - s - 1$$

$$F(s) = \left(\frac{6-4s}{s^2+1} - (s + 1)\frac{s^2+1}{s^2+1}\right)\frac{1}{s^2+2s+1}$$

$$F(s) = \left(\frac{6-4s-(s+1)(s^2+1)}{s^2+1}\right)\frac{1}{s^2+2s+1}$$

$$F(s) = \left(\frac{6-4s-(s^3+s^2+s+1)}{s^2+1}\right)\frac{1}{s^2+2s+1}$$

$$F(s) = \left(\frac{-s^3-s^2-5s+5}{s^2+1}\right)\frac{1}{(s+1)^2}$$

$$F(s) = \frac{-s^3-s^2-5s+5}{(s^2+1)(s+1)^2}$$

Find the partial fractions expansion:

$$\frac{-s^3-s^2-5s+5}{(s^2+1)(s+1)^2} = \frac{As+B}{s^2+1} + \frac{C}{s+1} + \frac{D}{(s+1)^2}$$

$$-s^3 - s^2 - 5s + 5 = (As + B)(s + 1)^2 + C(s^2 + 1)(s + 1) + D(s^2 + 1)$$

$$-s^3 - s^2 - 5s + 5 = (As + B)(s^2 + 2s + 1) + C(s^3 + s^2 + s + 1) + Ds^2 + D$$

$$-s^3 - s^2 - 5s + 5 = As^3 + 2As^2 + As + Bs^2 + 2Bs + B + Cs^3 + Cs^2 + Cs + C + Ds^2 + D$$

$$-s^3 - s^2 - 5s + 5 = s^3(A + C) + s^2(2A + B + C + D) + s(A + 2B + C) + (B + C + D)$$

Set coefficients of like terms equal:

$$-1 = A + C$$
$$-1 = 2A + B + C + D$$
$$-5 = A + 2B + C$$
$$5 = B + C + D$$

Solve the system of equations to obtain:

$$A = -3, B = -2, C = 2, D = 5$$

Thus $F(s) = \frac{-s^3-s^2-5s+5}{(s^2+1)(s+1)^2} = \frac{-3s-2}{s^2+1} + \frac{2}{s+1} + \frac{5}{(s+1)^2}$

$$F(s) = \frac{-3s}{s^2+1} + \frac{-2}{s^2+1} + \frac{2}{s+1} + \frac{5}{(s+1)^2}$$

Take Inverse Laplace Transform of both sides:

$$y = -3\cos t - 2\sin t + 2e^{-t} + 5t^{1}e^{-t}$$

ANS: $y = -3\cos t - 2\sin t + e^{-t}(2 + 5t)$

1	$\frac{1}{s}$, $s > 0$
e^{at}	$\frac{1}{s-a}$, $s > a$
t^n, n=positive integer	$\frac{n!}{s^{n+1}}$, $s > 0$
$\sin at$	$\frac{a}{s^2+a^2}$, $s > 0$
$\cos at$	$\frac{s}{s^2+a^2}$, $s > 0$
$t\sin at$	$\frac{2as}{(s^2+a^2)^2}$, $s > 0$
$t\cos at$	$\frac{s^2-a^2}{(s^2+a^2)^2}$, $s > 0$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$, $s > a$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$, $s > a$
$t^n e^{at}$, n=positive integer	$\frac{n!}{(s-a)^{n+1}}$, $s > a$
$af(t) + bg(t)$	$aF(s) + bG(s)$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$

Day 27: Revisiting first-order linear

Today, I am going go back over first-order linear equations and show how we can shortcut to the solution.

RECALL: First-order linear differential equations. TODAY: Shortcut!

EXAMPLES:

1. $y' + 2y = x^3 e^{-2x}$ (ANS: y=e^-2x (x^4/4 + C))
2. y'+4xy=x (ANS: y=¼ + Ce^(-2x^2))
3. $y' + (2/x)\,y = 2x^{-3} + x^{-1}$
4. $y' - x^2 y = 0$

SHORTCUT TO SOLVE LINEAR EQUATIONS:

Step 1: Rewrite the differential equation in the standard form: $y' + p(x)y = f(x)$

Step 2: Find u(x) by plugging in: $\mu(x) = e^{\int p(x)dx}$

NOTE: all you need is a single antiderivative - so no "C" in P(x)

Step 3: If the original equation is:

- not homogeneous, then the solution is $y = \frac{1}{\mu(x)} \int \mu(x) f(x) dx$
- homogeneous, the solution is: $y = C/\mu(x)$

PROJECT:

Check out https://www.programiz.com/python-programming/online-compiler/ for online Python compiler

Day 28: Spring problems
Section 6.1: Spring problems
WeBWorK: Modeling - Vibrations

Setup:
Spring hanging. Has a natural length L.
Put an object on the spring. Object has mass m. What does it do?
Define equilibrium. l is the distance the spring is stretched from the natural length L at equilibrium.

A spring-mass system: an object of mass *m* suspended from a spring.

- The system is in *equilibrium* when the object is at rest, forces acting on it sum to zero - the position of the object in this case is the *equilibrium position*.
- y = displacement of object from its equilibrium position (measured positive upwards)
- The following forces act on the object:
    - Gravity: *-mg*
    - Force of the spring: F_s, which is proportional to the the distance the string is stretched from its natural length (the length of the spring with no weight on it). If the weight (without motion) stretches the spring a distance of l from its natural length, then $F_s = k(l - y)$ where ***k*** is the spring constant.
    - A damping force F_d that tends to slow down the motion of spring over time. F_d is proportional to the velocity of the object, $F_d = -cy'$
    - We can also have an external force F(t) (other than gravity) that may vary over time but is independent of displacement and velocity.
        - The motion is *free* if F=0, or *forced* if $F \neq 0$.

These are all descriptions of *forces* acting on the object. Since force equals mass times acceleration or F=ma, and acceleration is the second derivative of position y, we have:
$my'' = -mg + F_s + F_d + F(t)$

Now, when the weight is initially put on the spring, it will stretch until the force of gravity is balanced against the force of the spring, at which point it is at equilibrium. So at equilibrium we have $kl = mg$.

Substituting, we have:
$my'' = -mg + k(l - y) - cy' + F(t)$

Simplify.

Defn. The equation of motion for a spring system is: $my'' + cy' + ky = F$
(sometimes we divide through by m to isolate y")

As our first example, we will consider undamped, free systems (undamped means c=0, free means no forcing function, or F=0)

Example 6.1.1 An object stretches a spring 6 inches in equilibrium (in a free, undamped spring system).
(a) Set up the equation of motion and find its general solution.

We don't know the mass m, ,but we can find k/m (=l/g)
NOTE: gravity is g=32ft/sec.
Free: F=0
Undamped: c=0

(b) Find the formula for the displacement y(t) of the object for t>0 if it's initially displaced 18 inches above equilibrium and given a downward velocity of 3 ft/s.
(c) Use trig identities to express y(t) in terms of a single cosine function.

LINEAR COMBINATION OF SINE AND COSINE A trigonometric identity you may not have seen before: Theorem. $a\cos cx + b\sin cx = A\cos(cx - D)$, where $A = \sqrt{a^2 + b^2}$, and $D = \arctan\left(\frac{b}{a}\right)$ (adjust for quadrant: add pi to D whenever (a,b) is in quadrant II or III)

Example 2: A 32 pound object is suspended from a spring with spring coefficient 40 lb/ft. The whole system is suspended in a viscous liquid that has a damping coefficient of 4 lb*sec/ft and subjected to an external force $F(t) = 88\sin(4t) + 24\cos(4t)$.

a. Find a general formula describing the position of the object at time t.
NOTE: The "forced response" in homework refers to a single solution to the original equation.
b. Suppose the object is initially moved up a distance of 10 ft (y(0)=10), and is set in motion in the downward direction at a speed of 40 ft/sec (y'(0)=-40). Find the formula describing the position of the object at time t.
c. Express your answer to b. in terms of cosine and exponential functions.

RECALL: mass=weight/gravity (gravity = 32 ft/sec^2)

ANS Example 2: Equation of motion: y"+4y'+40y=88sin(4t)+24cos(4t)

a) y(t) = c_1 e^(-2 t) sin(6 t) + c_2 e^(-2 t) cos(6 t) + 3 sin(4 t) - cos(4 t)
b) y(t) = -5 e^(-2 t) sin(6 t) + 11 e^(-2 t) cos(6 t) + 3 sin(4 t) - cos(4 t)
c) y(t) = sqrt(146) e^(-2t) cos(6t+.4266)+sqrt(10) cos(4t-1.8925)

RESOURCES

SIMIODE SIDE NOTE: There is a great "Day 1" activity on modeling death with M&Ms here: https://www.simiode.org/resources/1798/download/1-1-S-MM-DeathImmigration-StudentVersion.pdf

SLOPE FIELD GENERATOR (DESMOS): https://www.desmos.com/calculator/p7vd3cdmei
SLOPE FIELD GENERATOR (GEOGEBRA): https://www.geogebra.org/m/W7dAdgqc
SLOPE FIELD GENERATOR (this one works well when re-adjusting the window - e.g. for applications): https://bluffton.edu/homepages/facstaff/nesterd/java/slopefields.html

APPLICATIONS OF FIRST-ORDER DIFFERENTIAL EQUATIONS IN MECHANICAL ENGINEERING ANALYSIS
https://www.engr.sjsu.edu/trhsu/Chapter%203%20First%20order%20DEs.pdf

Fluid dynamics: design of containers and funnels
Heat conduction analysis: design of heat spreaders in microelectronics
Heat conduction & convection: heating and cooling chambers
Applications of rigid-body dynamic analysis

REQUIRED IN:

MECH 4730 Finite Element Methods
MECH 4760 **Vibration** and Advanced Dynamics

APPLICATIONS OF DIFFERENTIAL EQUATIONS IN COMPUTER ENGINEERING TECH

Electrical circuits

REQUIRED IN:

CET 4705 Component and Subsystem Design I
CET 4711 Computer-Controlled Systems Design I
CET 4762 Electromechanical Devices