Hiding God

The Unbelievable Logic of Arithmetic

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Introduction

1.i The Lie of the Land

12 = 3 × 4, 56 = 7 × 8. That curious pattern—which you may have noticed when you learned your multiplication tables—does not carry on through 9 and 10. But a similar pattern does use all ten digits:

> 0 + 12 = 3 × 4
>
> 5 + 67 = 8 × 9

Finding patterns, and finding out what—if anything—they mean is what mathematicians do. If mathematicians were ignoring one of the most meaningful facts about whole numbers—not the patterns above (they are meaningless coincidences)—that would be remarkable.

This book is about an incredibly meaningful mathematical fact. Its meaning is literally out of this world. You would not believe me if I told you what that meaning is without building up to telling you from something like the following puzzle—an ancient puzzle called *the liar paradox*—which is much simpler to state and to solve.

> "This description, of these nine words, is not true," said Ludwig Humdinger (a fictional performance artist), who was describing those very words.
>
> If Ludwig's description was a true description of itself, then it was as he described it. And he described it as "not true," so it was not true. But of course, if his description had been a true description, then it would have been true.
>
> So if Ludwig's description was true, then something that was not true would have been true. Since that would have been like his description was true and pigs were flying, his description was presumably not true.
>
> Except that if what Ludwig said was not true, then when he described what he said as "not true," that description would have been true.
>
> Again, something that was not true would have been true—like Ludwig's words were not true and pigs were flying—which should mean that his nine words were true.

But of course, his description being true is precisely what his description not being true rules out.

That reasoning about that description was therefore paradoxical. The word *paradoxical* comes from the ancient Greek for *contrary to common sense*, and it is only common sense that by reasoning logically we should obtain a clearer view of things, not a logical impossibility. Things are logically impossible when logic rules them out. For example, round squares are logically impossible:

> A round square would have no corners because it was round, and four corners because it was square—it would have contradictory properties, had that contradiction not shown that round squares do not exist.

But what does this contradiction—Ludwig's description being and not being true—show? Could it show that that description does not exist? But those words clearly do exist.

Perhaps they did not have the meaning that they seemed to have. Perhaps they were not a description. Descriptions are, as a rule, descriptions of things other than themselves. But "these are four words" is obviously true, so there do seem to be exceptions to that rule. And it was pretty obvious what the meaning of Ludwig's nine words was.

So presumably the contradiction shows that the apparently logical reasoning that led to it was not actually as logical as it looks.

The problem with that is that that reasoning does look perfectly logical. If the conclusion of that reasoning had not been a contradiction, its truth would hardly have been in question after such a logical argument. Indeed, some philosophers take such reasoning to be showing us that there are true contradictions.

Philosophers have entertained a lot of possibilities over the years. With the logical impossibility of true contradictions, they were scraping the bottom of the barrel. Could there be something wrong with our logic? Could it be that our evolution did not result in us having the ability to think in perfectly consistent ways about any possible description?

That explanation of this paradox—and every other logical paradox, similarly (more of them are in chapter 5)—must have seemed realistic enough when astronomers, observing Mercury in 1919, found that our intuitions about space had only ever been approximately true: Our intuitions about space had always seemed as unquestionable as our intuitions about logic.

And yet, how could failing to solve a logical puzzle mean that there was something wrong with logic? Should we not assume that a logical solution might be found one day? That question is put into stark relief by the explanation of the liar paradox that is outlined in the final section of this chapter (and fleshed out in chapter 5).

But what had already put the cat among the pigeons was some paradoxical mathematics—outlined in the next chapter—which the British philosopher Bertrand Russell (1872–1970) had become aware of in 1900. As you will see in chapter 3, there is a logical explanation of that mathematical paradox if—and arguably, only if—there is a certain very logical kind of God. But Russell would definitely have thought that our not having evolved to be capable of being perfectly logical was much more realistic. And there would have been no worries about a logical solution turning up one day if the logical solution had already been rejected. So this mathematical paradox may well have seemed like good scientific evidence that we are inherently incapable of being perfectly logical.

Although that paradox is actually good scientific evidence for there being such a God.

1.ii Expectation

> It was a sunny day at the seaside, and down on the beach a boy was looking for shells. A few yards ahead of him something was glistening. It was round and whitish, and as the boy approached it, he could see that it was rippled with ridges and furrows.
>
> The closer he got to it, the more it looked like a large shell, worn down and polished up by the sea. But when he reached it, he found that its furrows were the imprints of fingers.
>
> He knew that he had found something handmade—in all likelihood (the odds against a shell eroding like that being astronomical)—and in light of that, he saw at once that it was a bowl in the shape of a shell. He decided to keep his shell collection in it. And as he bent down to pick it up, he wondered if it had been left behind by tipsy picnickers.

If he had found something stranger than a bowl, he might still have been wondering what it was that he had found. A piece of modern art? A fragment from a sunken galleon, encrusted with oddly eroded barnacles? An alien artefact with a chameleonic surface that broke down while it was in the sea? Probably not that—although the stranger the object, the stranger the possibilities that might reasonably be entertained, and the more it would be a matter of opinion what was reasonable.

And this book is about something much stranger than a bowl on a beach. One thing, two things, three things and so on and so forth: That pattern has a property—described in the next chapter—that makes such numbers look as though they must be forever coming into existence. Since we normally think of such numbers as having always existed, it is a paradoxical property.

The paradoxical logic is laid out clearly in the first section of chapter 3.

But perhaps you are finding it hard to imagine how there could be paradoxical mathematics. The following equations are a 17th-century "proof" that 0 = 1. The first equation is 0 = 0, which is clearly true. Adding zeroes to that equation, and spreading it out:

$$0 = 0 + 0 + \dots$$

Those final three dots just mean that the addition of zero is continued endlessly. Adding zeroes should make no difference, so that equation is presumably also true. And because 1 – 1 = 0, each zero on the right-hand side can be replaced with (1 – 1):

$$0 = (1 - 1) + (1 - 1) + \dots$$

And presumably those brackets can be removed:

$$0 = 1 - 1 + 1 - 1 + \dots$$

Replacing those subtractions of units with additions of negative units gives us this:

$$0 = 1 + -1 + 1 + -1 + \dots$$

And in the next equation, some brackets have been added.

$$0 = 1 + (-1 + 1) + (-1 + \dots$$

In the next equation, each (–1 + 1) has been replaced with a zero.

$$0 = 1 + 0 + 0 + \dots$$

The endless addition of zeroes at the end of that equation can presumably be ignored, leaving us with 0 = 1, which is clearly false. That 17th-century "proof" is explained and used in the first section of chapter 4.

The paradoxical property of whole numbers that this book is about was discovered at the end of the 19th century. Some of the best mathematicians of the day thought that mathematics being the creation of mathematicians might explain how numbers could possibly come into existence. But most people—most mathematicians included—think of arithmetic as discovered, not invented. We can create fictional worlds, but we cannot create actual worlds, let alone whole numbers. Only a God could create actual worlds. And whole numbers? Could such a paradoxical property of whole numbers be a telling impression left on them by the creator of all things?

That is a logical possibility, as you will see in the third section of chapter 3.

And if that was the only logical explanation for the existence of this property, then it would follow logically that such a creator exists; and there is unlikely to be more than one logical explanation (the experts have yet to find one that many of them would call “logical”). But perhaps you think that a proof that God exists would have to look more like the words “God exists” suddenly appearing in the sky in giant letters of unearthly fire.

So suppose that such fiery words did appear. We would probably take them to be a practical joke, or an advert (something like that). If the fire was tested, and found to be nothing that we could produce, it would still be more rational to suppose that it was the work of aliens. Aliens might want us to think that they were angels, in order to control us more easily. But why would a God want to look like such aliens? Whereas, leaving a telling impression on whole numbers is not something that aliens could possibly have done.

So, chapter 3 is going to show that there is, in all likelihood, a creator of all things.

That is an extraordinary claim, but it is not a hopelessly unrealistic claim because the property in question has already proved to be extraordinarily puzzling.

This book solves a mathematical puzzle that has had mathematicians stumped for over a hundred years. And it was not the difficulty of the mathematics that had them stumped. The property in question is less complicated than most school mathematics, as you will see in the next chapter. No, what had them stumped was the property appearing to be logically impossible.

There is a logical solution to the puzzle of how such a simple pattern could have such a paradoxical property, as you will see at the end of the next chapter. But that solution only makes sense if all things were created out of nothing but the creative power of a creator who is able to change. Taking that solution seriously in the 20th century would have meant flying in the face of the prevailing atheism of an academia that had only just emerged from under the shadow of theology (and in the face of that theology, which described an unchanging God).

Academics therefore investigated other approaches to solving this puzzle, approaches that seemed more realistic. A wide range of approaches was thoroughly explored, throughout the 20th century, because none of them led to a logical solution, and those academics were trying to be logical. Academics want to be objective. But the literature they produced in pursuit of the truth of this matter—in every conceivable direction except the direction in which the truth happened to be—grew and grew until the original puzzle was buried under a mass of increasingly complicated and convoluted formalisms.

That literature amounts to a lot of evidence that there is not another logical solution to this very paradoxical puzzle; but it is not an easy read, to say the least. This book is much simpler, and far more straightforward. Although it is facing an uphill struggle. For instance, other arguments have aimed to show that God exists, and they all failed to be perfectly logical. So any new argument that God exists is bound to look like an argument that is probably illogical.

The best known of those other arguments is *the ontological argument*:

> It is presumably better for beings to be very good, than for those beings not to be very good. Does it follow that the best possible being would be very good? That does seem to follow.
>
> And it would clearly be better for a very good being to exist, than for that being not to exist. Does it follow that the best possible being would exist? If so, then we have a logical argument for the existence of the best possible being.

And religious believers do think of God as the best possible being. But what if you thought that nothing could be better than, say, eating cake? The best possible thing might then be your favourite cake. Would eating one of them now be best? You might think so. But does it follow that you are now eating one? Of course not: Making a possibility better—such as imagining cake now, rather than later—is one thing; making that possibility actual—replacing "imagining" with "having"—is another thing entirely.

That version of the ontological argument flew in the face of that rather obvious distinction; and while there have been other versions of the ontological argument, over the years, they also failed to be completely logical. As did all those other arguments.

The argument of this book is different, though. Following the discovery of the paradoxical mathematical fact that this book is about, scientific philosophers debated how best to redefine the word *logical*—that is, how best to replace logic with something else, which would then be called "logic"—and scientific philosophers would hardly have entertained such a rejection of logic had there not been some logical argument for something unbelievable. The sciences are essentially logical pursuits, of truths of various natural kinds.

Still, another reason why you might not be expecting very much from this book is that there is a lot of evidence that this world is not the creation of a God. Would any logical evidence that this book contains not be outweighed by all the biological, paleontological and other physical evidence for our evolution by natural selection? Or more simply, by just some of the evils of this world?

But all that physical evidence is primarily evidence about the development of our bodies. So all that evidence, plus the proof of chapter 3, adds up to evidence that there is probably a creator who had some reason to give us bodies that probably evolved by natural selection. And there are lots of possibilities for why a God might have wanted to create such a world. One that is also a rather logical answer to that evidential problem of evil is in the third section of chapter 3. In short, my God hypothesis is at least as consistent with the scientific facts as atheism is.

I say "at least" because of the paradoxical mathematical fact that this book is about. It was discovered by a German mathematician, Georg Cantor (1845–1918), at the end of the 19th century; although at first, it did not appear to be a fact. Cantor seemed to have proved something impossible, so he seemed to have made a mistake. But mathematicians interrogated his mathematics, increasingly thoroughly because they could not find any mistakes, and found that he had discovered something. But what? As you will see in the first section of the next chapter, there seemed to be too many whole numbers. But how could there be? There are all the whole numbers that there are.

Mathematicians tried to make sense of Cantor's paradoxical discovery, but they needed a way of working around Cantor's paradox while they carried on trying. So a hundred years ago, everything in mathematics was given a new definition, as outlined in the second section of the next chapter.

> As the mathematicians explored their new mathematics, it was mostly left to philosophers to make what sense they could of Cantor's paradox. The paradoxical nature of infinity had been the business of philosophers since Pythagoras and Aristotle. And in the 20th century, various ways of modernising logic were investigated. With a suitable redefinition, philosophers would have been able to call "logical" any explanation that cohered well enough with the best scientific theories.
>
> Since those redefinitions usually replaced logic with a mathematical model of logic, they did look scientific. But of course, rewriting logic so that it gave those philosophers what they wanted was never going to be scientific. Some philosophers developed mathematical models of truth—called *theories of truth* by scientific philosophers (scientific theories being mathematical models)—but to pursue the truth by redefining the word *truth* was never going to be scientific.

That sea change to academic mathematics trickled down to school mathematics in the form of *the new math*, which you may have heard of. It was quite controversial fifty years ago. Doing mathematics with the new definitions is very abstract—it is like doing geometry without pictures—and such abstraction has always put people off.

For such reasons, not many people heard of Cantor's paradox. And those who did hear of it tended to assume either that the new definitions had solved the problem—although they were only a way of avoiding the need for a mathematical explanation—or that the paradoxical fact was inexplicable (much as the patterns at the start of this book were meaningless).

However, Cantor's paradox seems—much as the liar paradox seems—to be a logical argument for a contradiction. So it seems to show that there is something wrong with logical thinking. It could even justify doubting advanced mathematical (and hence a lot of scientific) reasoning—much like this:

> The first part of the liar paradox—up to Ludwig's description not being true—appeared to be logical enough. But then it was contradicted by some very similar reasoning. Which raises the question, should we believe the conclusions of reasoning that looks logical? We should, of course; but if we do not know where the reasoning of the liar paradox fails to be logical, how can we be justifiably confident that none of these conclusions will be similarly contradicted, by reasoning that looks just as logical?

In any logical paradox, something that looks like logic takes us from some ordinarily unquestionable assumptions to something that does not seem to be true. But thinking logically takes us from truths only ever to truths (that is basically what the word *logical* means). To an untruth, logic would have had to have been taking something that was not a truth.

So, if we can identify all the assumptions that a logical paradox began with,

> then the surer we can be in all but one of those assumptions, and the less sure we are about that one,
>
> and the surer we are in the logic of our thinking, and in the untruth of the conclusion of our thinking,

the clearer it will be that that particular assumption is not true.

An ordinarily unquestionable assumption might, in that way, be shown to be highly unlikely. Which is how a very paradoxical logical puzzle can amount to a proof of a surprisingly high probability. And in particular, chapter 3 describes a proof that there is (very) probably a creator of all things who is able to change. I say "describes," and that word *very* is in parentheses, because reading chapter 3 will only result in such a proof insofar as all reasonable doubts are raised and answered. And it is a matter of opinion what is reasonable.

Perhaps, for example, you think that the phrase "a creator of all things" has no place in a scientific hypothesis. That phrase does seem to be a religious, rather than a scientific description. But thinking logically, about any evidence there might be, for the world having been deliberately created, would help to answer, in a rather scientific way, the question of whether the world was deliberately created, or not. And note that nothing that religious people have ever said or done should count against the likelihood of a God hypothesis whose justification is a very logical bit of mathematics—just as none of the things said and done by social Darwinists in the 20th century should count against the likelihood of our evolution by natural selection.

> Because of that latter likelihood, a lot of people believe that our minds are produced by our brains, and that our brains evolved in a purely physical world. The paradoxical contradiction at the end of (the logical version of Cantor's paradox that is) the first section of chapter 3 could therefore seem to be evidence for there being, if not some mistake in the reasoning of that section, then some natural limit to our ability to reason logically about such mathematically complicated and abstract matters. So note that if there was such a limit, then there would be much less evidence for our having evolved by natural selection, because all the evidence that we have is the result of scientists thinking logically about a huge amount of scientific data.

The description "creator of all things" can be given sufficient sense, for the logical purpose of this book, via analogies. Analogies are surprisingly useful when we are trying to think logically about the fundamental nature of reality, because our languages developed as they did in order to help us to communicate about mundane matters. Even physics began with analogies: We pick things up, and we push them about, and sometimes we have to use more force to make them move, and so natural philosophers thought about the motion of physical objects in terms of forces. Physicists have devised increasingly accurate mathematical models of forcefields, over the years, but it all began with a simple analogy.

Would the creation of the world out of nothing have been like the creation of a bowl out of clay, but without the clay? Well, a magician might make a bowl like that in a fairy tale; but that bowl, and everything else in that fairy tale, would have been created by that story's author (and to some extent, by its readers). So a better analogy for the creation of the world out of nothing but a creator's creative power is the creation of a fictional world out of nothing but its author's writing (and its readers' readings).

That analogy suggests that God is, in some sense, more real than anything else. And you might wonder how anything could be more real than the world around us. But analogies should not be taken too literally. And the major alternatives to there being a God have similar implications. Materialism, for example, says that only what is physical is real—or really real (that reality is *fundamentally* physical). Many materialists do not want to say that our perceptions, thoughts and feelings are not real: That they exist is one of the most obvious facts of all. But materialism certainly seems to imply that our brains are, in some sense, more real than our perceptions, thoughts and feelings.

1.iii Solution

"This description, of these nine words, is not true," said Ludwig; and presumably those nine words were either true or, if that was not the case, not true. Ludwig's mother—who is a fictional philosopher (a professor of mathemagical logicology)—has this argument that every description is either true or else not true:

> If that was not the case, then some description would be neither true nor untrue.
>
> Such a description would not be true (from "neither true"). And yet, it would not be the case that it was not true (from "nor untrue").
>
> That being logically impossible, there is no such description.

Which does seem to mean that every description is either true or else not true. That type of argument is called *reductio ad absurdum* (Latin for *reduction to absurdity*). Despite the fancy name, such reasoning is just common sense:

> You assume, for the sake of such an argument, the opposite of what you want to prove, its contrary (such as that there is a round square).
>
> Then you take that assumption, in some logical way, to a contradiction (such as 0 = 4).
>
> To such an untruth, logic could only have taken an untruth, and so your assumption was not true (there are no round squares).

Suppose (for a non-mathematical example) that Ludwig and his mother are talking about keeping pigs as pets. His mother—professor Humdinger—thinks it a bad idea, on the grounds that pigs are dirty. But as Ludwig says, "pigs are not pigs." We normally assume that repeated words have the same meaning—that is how languages work—but here, that assumption would make Ludwig's observation absurd. He is of course not telling his mother such an obvious untruth and so she knows that here, the word *pigs* had two different meanings.

Professor Humdinger has a pet cat, and it is currently lying half in and half out of an overturned box, licking itself clean. Because it is only half in the box, the description “Humdinger’s cat is in the box” is not a very good description. But nor is it a very bad description, because the cat is half in the box.

Could that description be about as true as not?

Is that a logical possibility?

> Well, if a description was about as true as not, then it would not so much *not be the case* that it was true; rather, it would *be only about as much the case as not* that it was true.
>
> And it would not so much *not be the case* that it was not true; rather, it would *be only about as much the case as not* that it was not true.

Such a description would not be well described as “not true and not the case that it was not true,” so the first line of the professor’s reasoning (“some description would be neither true nor untrue”) was probably not true.

And it is certainly conceivable that some descriptions are about as true as not. Imagine, for instance, a blue object slowly turning red:

> Was there a moment before which the object was blue and after which it was not blue?
>
> If so, and if its colour was changing slowly enough, then its colour just before such a moment would look the same as—and would therefore be the same as—its colour just after that moment.
>
> It being contradictory for blue not to be blue, it makes sense to suppose that there is no such moment.

It makes sense—so it is conceivable—that there would instead be times when this object was about as blue as not. And it makes sense that at such times, the description “it is blue” would be about as true as not of that object. You will see how much sense it makes in chapter 5.

Although the professor thinks that God would know, of every object, whether it was blue or not. And she thinks that her cat is neither in the box, nor out of it. In particular, she thinks that the description “Humdinger’s cat is in the box” is not true. Now, there is a lot to be said for that view. If I dropped a box of something and it spilled out, for instance, I would not think that it was still in the box, even if some of it was.

However, Ludwig thinks that the professor's cat looks a bit like a cartoon dog sitting in its doghouse. So he takes "Humdinger's cat is in the box" to be true. And since some people take that description to be true, while others think that it is not true, it would make sense for it to be about as true as not.

> What about Ludwig's "this description, of these nine words, is not true"?

Well, if that was a true description of itself, then it was not true. So it was certainly not a very good description of itself.

But nor was it a very bad description: Insofar as it was false, Ludwig's description of it as "not true" was correct.

So it might make sense for it to be about as true as not. The assumption that all descriptions are either true or else not true is certainly paradoxical for this description, because either way, it would seem to be both true and not true, which is not possible.

In any logical paradox, reasoning that seems logical takes us from assumptions that would ordinarily be unquestionable to some obvious untruth, such as a contradiction.

Those assumptions might be so unremarkable, there do not seem to be any—there did not seem to be any in this liar paradox—but there must have been something that was not true (or only as true as not) being assumed. Logical thinking is essentially the kind of thinking that will, when we are thinking about truths, only ever result in truths.

And there would be consistency, instead of that contradiction, if Ludwig's "this description, of these nine words, is not true" was about as true as not:

> If it is about as true as not that Ludwig's nine words are not true, then they are about as untrue as not (this is like it being about as true as not that an object is blue, which would be because that object was about as blue as not).
>
> And when they are about as untrue as not, then because that "not" means *not untrue*—because it means *true*—they are about as true as not.

Although the professor regards Ludwig's six words as nonsensical, and hence not true. Ludwig disagrees:

> When he said those nine words, he knew that they were paradoxical, so he knew that they were not describing themselves very well. Indeed, when he said them, he was trying to say that they were not describing themselves well enough for them to count as true—and as he said them, they seemed to him to be expressing that truth rather well. They could not have

> been describing themselves well enough for them to have been true. But he thinks that on balance, they must have been about as true as not.

The professor points out that if his words were not nonsensical—if he had indeed said that what he was saying was not true—then he would have been saying that *that* was not true: He would have been saying that it was not true that *what he was saying was not true*. That is, he would have been saying that what he was saying was true. As well as explicitly asserting that his assertion was not true, he would have implicitly asserted—by making that explicit assertion—that it was true. He would have said something whose meaning was the opposite of itself. An odd way of not being nonsensical, she observes. But Ludwig is very imaginative:

> "Could the two contradictory meanings of what I said have been two equally poor descriptions of something that was such a poor description of itself, it was effectively its own contrary, but not so poor it was simply not true?" he wonders.

Does that sound like a logical possibility? An analogous case might be the descriptions "quite a good description" and "a description that is no good," because they cannot both be true (they are contraries), and they are both poor descriptions of "it is not true," which would be a poor description of a claim that was about as true as not. So Ludwig's speculation does have the right shape for a logical possibility, and I will return to it in chapter 5.

The next three chapters are about the logical explanation of Cantor's paradox. And using a God hypothesis to explain a mathematical puzzle is a bit like using a sledgehammer to crack a nut. So note that the main mathematical and philosophical responses to Cantor's paradox were to redefine words like *mathematical* and *logical*—which is a bit like redefining the word *cracked* so that the nut is already "cracked." And the reason why those were the main academic responses was that Cantor's paradox had been proving to be to be a very tough nut to crack: It is a coconut of a puzzle.

What makes a logical puzzle paradoxical is its solution not appearing to be logically possible. In order to solve the puzzle, we need to find more logical possibilities, and the widest range of logical possibilities for the nature of whole numbers includes such possibilities as their being created by a God who is able to change.

2

Cantor's Paradox

2.i Numberless Numbers

Whole numbers are very simple: Each thing is one thing, and whole numbers bigger than one are how many things there are in collections of that many things, where a collection of things is just those things (being referred to collectively).

Zero is very similarly the number of things (of some sort or other) in something that that no things (of that sort) in it. And repeatedly adding 1 to the previous number, starting with 0, gives us the counting numbers:

> 0 + 1 = 1, 1 + 1 = 2, 2 + 1 = 3, 3 + 1 = 4, 4 + 1 = 5, …

There are infinitely many counting numbers (the word *infinite* derives from the Latin for *unending*). Are there an infinite number of counting numbers? That is the usual assumption nowadays.

Although because that sequence of counting numbers gets longer and longer without end, it might seem to be getting ever longer without there ever being all of them—without there being a number of them all. Could counting numbers be popping into existence? Presumably they do not, but is there a proof that they do not? Well, what does it mean, to say that a whole number exists? Does it mean that that many things are possible? If so, then:

> If some whole number is ever going to exist, that many things will be possible. So it will always have been logically possible for there to be that many things. Which seems to mean that that whole number would always have existed.

Which would mean that every whole number that will ever exist always did exist. Although there is a flaw in that argument (as you will see at the end of this chapter). And for thousands of years, mathematicians have been puzzled by the infinity of the counting numbers. Consider, for example, how the first ten counting numbers go all the way up to ten, the first hundred all the way to one hundred, the first thousand to one thousand, and so on. What about the counting numbers as a whole?

> There are infinitely many of them, so you might expect them to go all the way to infinity, if they do all exist.

> But each and every one of them is finite, so they are all infinitely far from infinity.

And counting numbers are presumably not endlessly popping into existence.

Could it be that none of the counting numbers exist? Numbers certainly do not exist in the way that ordinary objects do. Numbers are more like colours: Much as colours are properties of coloured things, whole numbers greater than one are properties of collections; and much as the existence of a colour amounts to the logical possibility of something having that colour, the existence of a whole number greater than one amounts to the logical possibility of a collection of that size. Do logical possibilities exist? Well, we should assume that we can think logically (if we try hard enough), so we should assume that there are logical possibilities, for us to think logically about. And it makes no sense to demand more of logical possibilities than logic not ruling them out, because there is no more to them than that.

To return to the puzzling infinity of the counting numbers, the following paradox was written about by the Italian astronomer Galileo Galilei (1564–1642).

> It is only every other counting number that is an even number. So there are clearly fewer even numbers than counting numbers.
>
> But we can get the even numbers by doubling the counting numbers. So the even numbers can all be paired up with all of the counting numbers. Which means that there are as many even numbers as counting numbers.

From that apparent inconsistency, Galileo concluded that relationships like "fewer than," "as many as" and "more than" do not make sense when applied to infinitely big collections. The problem is that they clearly do make sense (that is how we could see that apparent inconsistency).

But I say "apparent" because Ludwig's "pigs are not pigs" was not contradictory. And there are, similarly, two meanings each of "fewer than," "as many as" and "more than," in Galileo's paradox.

The sense in which there are fewer even numbers than counting numbers is obvious enough: To get the even numbers, we take some of the counting numbers away.

The other sense is more important, because it applies much more widely. If we have, say, five oranges and three lemons, then although we have fewer lemons than oranges, we do not get the lemons by taking some of the oranges away. Rather, we have two oranges left over after we have paired up as many of the oranges and lemons as possible.

And it is a similar pairing up that means that there are as many even numbers as counting numbers. In general:

> There are *as many* things in one collection as there are in another if the things in one collection can all be paired up with all the things in the other collection.

Mathematicians say that such collections are *equinumerous* (which just means that they have the *same number* of things in them). Equinumerosity is essentially a logical concept, because the sense of the word *can* in "can all be paired up" is that of logical possibility.

> If no such pairing is possible, there will be some pairing up of just some of the things in one of the collections with all of the things in the other. The former collection has more things in it, the other fewer.

Thinking logically about equinumerosity results in arithmetic, both the ordinary arithmetic of the counting numbers and the arithmetic of such infinite numbers as the number of all the counting numbers (if they do all exist).

In Galileo's time, the Church regarded infinity as an attribute of an ineffable God, and Galileo had problems enough disagreeing with the Church's geocentric view of the solar system, so it is understandable that he did not try to explain this paradox away. But the ineffability of infinity became less plausible once the calculus had been invented; and it was as Cantor was working on the calculus that he found a proof that for any collection, the collection of all of its subcollections—where a *subcollection*, of a collection of things, is just some of those things—is a bigger collection. That proof is outlined in the final section of this chapter, and it means that if there is an infinite number of all the counting numbers, then there are also a lot of bigger infinities, and their arithmetic. But that proof also gave Cantor inconsistencies, such as this:

> The collection of all the other mathematical collections should be the biggest collection.
>
> But the subcollections of that collection would comprise an even bigger collection.

Cantor took that inconsistency to mean that the collection of all the other mathematical collections was an inconsistent collection. Cantor's explanation for the existence of a collection so big, it was not a logical possibility seems to have been that God is unimaginably big (Cantor was a Lutheran). But most mathematicians wanted a different explanation of Cantor's paradox.

You might think that if a mathematical innovation produced inconsistencies, mathematicians would reject that innovation. But Cantor's innovations were so clearly logical, most mathematicians decided to work around his inconsistencies. That was possible because those inconsistencies were all at something like an outer edge to Cantor's new arithmetic of infinity. Mathematics was replaced with a very accurate mathematical model of itself (as sketched in the next section); and in that model, Cantor's paradox became a proof that any collection—or rather, class (see below)—that was too big to consistently be a collection was not part of that model, not part of the new, redefined mathematics. Mathematicians found that they could manage perfectly well without an explanation of Cantor's paradox.

Before set theory, philosophers who were classifying things used *classes* to refer collectively to all the things of some kind. Classes are a lot like collections; but a class might not be a collection—a class might have no members, for example, and the membership of a class might be variable—and even the non-variable classes are more like sets than collections (a set might have no members).

2.ii Set Theory

Russell heard about Cantor's paradox in 1900. A decade later, he and another British philosopher, Alfred North Whitehead (1861–1947), had an axiomatic model of classes, as part of their redefinition of mathematics—their *Principia Mathematica* (published by Cambridge University Press in three volumes, from 1910 to 1913). And a decade after that, mathematicians had settled on the axioms of a standard set theory.

> See Ivor Grattan-Guinness's comprehensive history, *The Search for Mathematical Roots, 1870–1940: Logics, Set Theories and the Foundations of Mathematics from Cantor through Russell to Gödel* (published by Princeton University Press in 2000), for the details.

Redefining their entire subject was unusual, but not unprecedented. Mathematics had been rigorously redefined by the ancient Greeks—ostensibly by Euclid—and that had worked out well. Euclidean space is a mathematical model of our natural conception of space. It is defined by the Euclidean axioms, which describe the simplest and most obvious properties of space. Euclid's first axiom, for example, says that between any two points in space there is a straight line. Thinking logically about Euclid's axioms gave the ancient Greeks all of their mathematics (not just geometry, because magnitudes like fractions and whole numbers were defined to be ratios of lengths). And in the 19th century, non-Euclidean axioms were postulated and studied, facilitating the discovery by astronomers in 1919 that spacetime is non-Euclidean.

Axiomatic sets are mathematical models of collections. The axioms of a set theory are mathematically precise descriptions of the most basic properties of collections. The first axiom of the standard set theory says—in a very abstract way—that sets containing the same things are the same set. Now, as collections have subcollections, so sets have subsets. The *subsets* of a given set are sets that contain nothing that is not in the given set. For example, the set {apple, banana} is a subset of the set {apple, banana, coconut}, while the set {apple, cherry} is not. The curly brackets indicate a set. They are like a picture of a bag containing the set's members, which are in no particular order in that bag (unlike words in parentheses). That is, {banana, apple} is the same set as {apple, banana}.

> Incidentally, {apple, banana} is called a *proper subset* of {apple, banana, coconut}.
>
> The whole set {apple, banana, coconut} is an improper subset. It is a subset because it is not the case that there is something in it that is not in it, but it is not a subcollection because the corresponding collection is not just some of those things.
>
> And a set with no members at all is an improper subset of every set, because it is not the case that there is something in it, and there is no collection that is not a collection of some things.

The replacement of mathematical collections with sets, and the redefinition of everything else in terms of sets, was designed to isolate the bulk of mathematics from the contradictions that followed from taking classes as big as the totality of all the sets to be collections. This redefinition was able to do that because once everything in mathematics was a set, Cantor's paradox became the following proof that the class of all those sets is not itself a set.

> If that class was a set (of all the other sets), then the class of all of its subsets would be a bigger class of sets—via the proof outlined in the next section.
>
> Whereas, the class of all the sets should be the biggest class of sets.

That contradiction means that that class is not a set. Mathematicians call it a *proper class*. And because there are no sets like that, the inconsistencies that Cantor found do not exist within set theory. Replacing mathematics with a mathematical model of itself was about protecting almost all of it from some very peripheral inconsistencies, so most mathematicians continued to do mathematics much as they had been doing it, with all the new infinities to explore, but without having to worry about the meaning of Cantor's paradox. Consistency could not be guaranteed, as the Austrian logician Kurt Gödel (1906–1978) proved in 1931. But because everything was redefined within the same artificial "language," consistency was expected.

Mathematicians have since designed other artificial "languages," which are to some extent analogous. When a computer, for example, calculates the solution to a mathematical problem, it begins by translating that problem into its machine "language." The resulting computer code is more complicated than the original problem, but it facilitates a faster and more reliable calculation. And the set-theoretical translation of such a problem is just another very complicated bit of mathematics that most mathematicians do not need to know anything about.

So, the inconsistencies that Cantor had found in his extension of arithmetic do not exist within set theory. But those inconsistencies are still there in that arithmetic, in the logic of possible collections of things. Not too dissimilarly, fictional characters are exactly as their canonical stories describe them, but historical figures were not necessarily as history describes them. Embarrassing facts about historical figures are sometimes discovered, and this book is about the rather embarrassing meaning of the fact that there is no number—because there is no collection—of all the whole numbers. This section is about a figurative fig leaf, which you will not need to know anything about in order to understand the proof of the next chapter. But to see why that proof is a proof of a high probability, it may help you to see a little of what the now standard meaning of "whole number" is.

The "whole number" 0 is defined to be the set with no members, the empty set, whose symbol is Ø. The meaning of that symbol is given by the standard axioms. It has a very precise and abstract meaning. But informally, it is like an empty bag, in a story about bags.

The existence of a set with no members is asserted by an axiom called *the axiom of infinity*. That axiom has that name because it also says that there are all those sets that are defined to be the "counting numbers," as follows. The counting numbers can be thought of as the products of repeatedly adding one, starting with nothing; and in the standard set theory, the "adding" of "one" to a "whole number" is defined to be the putting of the set that is that "whole number" into that same set. The following examples should make that description clearer.

0 + 1 = 1, and if you put the empty set, Ø, into a set with no members you get a set containing Ø. So, the standard 1 is {Ø}.

> The standard 1 is like a bag that contains an empty bag, in a story about bags.

1 + 1 = 2, and if you put {Ø} into {Ø}—just before the last curly bracket—you get {Ø, {Ø}}.

> The standard 2 is like a bag that contains both an empty bag and a bag containing an empty bag—in a story in which those are the same empty bag (via axiom 1), which is almost too odd.
>
> Counting numbers themselves are no odder than the patterns with which this book began. They are this simple: If there are as many things in a collection (such as you and I) as there

are units in a sum of units (such as 1 + 1) then there are that many things in it (we are two people).

But the standard 2 is not too odd. Fictional bags are as their stories say they are—if a story says that there is a round square bag, then there are round squares *in that story*—and similarly, axiomatic sets have whatever properties their axioms say they have.

Whitehead and Russell are infamous for spending hundreds of pages of their *Principia Mathematica* proving that 1 + 1 = 2. Why did they not take that equation to be true by definition? Because their 2 was no more our 2 than any model is the thing being modelled.

2 + 1 = 3, and if you put {Ø, {Ø}} into {Ø, {Ø}} you get {Ø, {Ø}, {Ø, {Ø}}}.

If the empty set had been {}, that would have been {{}, {{}}, {{}, {{}}}}, which is much harder to read. Mathematical symbols are there to make the mathematics easier to read. Unfamiliar symbols may look like hieroglyphs or graffiti, but familiar symbols—like those for the counting numbers—are more like nicknames.

2.iii Proofs

Cantor's proof that every set has more subsets than members is called Cantor's generalised diagonal argument. It is a generalisation of his original argument, which worked by way of the main diagonal of an infinitely large matrix. A matrix is a rectangular array of symbols. When a matrix has as many rows as it has columns, it is a square matrix. Below, for example, is a list of four sequences. The first is 0, 1, 0, 1. Each sequence has four elements, each of which is 1 or 0.

0	1	0	1
1	0	1	1
1	1	1	0
0	1	1	1

Main diagonals of square matrices start at their top left-hand corners and go down and to the right, ending at the bottom right-hand corners. The matrix above has a main diagonal of 0, 0, 1, 1 (its first element is the first element of the first sequence, its second element is the second element of the second sequence, its third is the third of the third and its fourth the fourth of the fourth).

A quick way to get a fifth sequence of four such digits is to replace each 1 in the main diagonal of that square matrix with 0 and each 0 with 1. That sequence—1, 1, 0, 0—differs from the first sequence in its first digit (and in its fourth digit), from the second sequence in its second digit (and elsewhere), from the third in its third digit, and from the fourth in its fourth (and elsewhere). So, it differs from every sequence in that list and is therefore a fifth sequence of that kind.

The beauty of that method of finding another sequence—of the same kind as those in a list that is as long as those sequences—is that it is just as straightforward however big the matrix is. In particular, it works just as well when the sequences and their list are infinite and equinumerous. Cantor's original diagonal argument concerned the set of all the counting numbers. It showed that that set had more subsets than members—or that the corresponding collection has more subcollections than things in it (axiomatic sets were only introduced following Cantor's paradox). The diagonal argument that I will shortly describe concerns the collection of all the counting numbers, instead of a set of all the counting numbers, and logically possible selections from that collection, instead of the subsets of such a set, but it is essentially the same as Cantor's original argument.

Let me give the collection of all the counting numbers the nickname N.

If the sequence of all the counting numbers gets longer and longer forever without there ever being a collection of all of them—if N does not exist—then my God hypothesis (in the third section of chapter 3) can explain how that sequence could possibly get longer and longer forever. The alternative—N exists—gives us a version of Cantor's paradox (in the first section of chapter 3), which that God hypothesis will be explicitly explaining.

So, I will be assuming that N exists. And given that N exists, each subcollection of N also exists—those numbers being just some of the numbers that are collectively N—and so each subcollection could, just possibly, be selected by a logically possible selector, such as a God. That is, each subcollection of N is a logically possible selection from N.

The selection of none of the numbers in N could also be classed as a logically possible selection.

> It is more natural for us to think of that as not making a selection, but it does not really matter—the following diagonal argument is all about equinumerosity, and a few selections here or there makes no difference to whether two collections are equinumerous or not (as Galileo's paradox showed). Counting the selection of no counting numbers as a logically possible selection does make my diagonal argument more like Cantor's:
>
> The subsets of a given set are all those sets whose members are some, or none, or all, or one of the things in that given set.

And a logically possible selection from a collection of things is some, or none, or all, or one of them. The last two are because given that N exists, N is there to be possibly selected, and each counting number is there to be possibly selected.

Now, for each logically possible selection from N, each counting number will either be in that selection or else it will not. A neat way of representing that possible selection is therefore to give each counting number a label, such as *In* when it is in the selection and *Out* when it is not; each possible selection can then be represented as an endless sequence of such labels (the first label saying whether 1 is in the selection or not, the second label saying whether 2 is, the third whether 3 is, and so forth). For example, the endless sequence *In*, *Out*, *In*, *In*, … could represent the possible selection of 1, 3, 4, ….

> Any pair of labels—such as I and O, or 1 and 0—would do just as well. Each of the four sequences in the square matrix above could—if we read each 1 as “in the collection” and each 0 as “not in it”—be a description of a possible selection from, say, the first four counting numbers, with the fifth sequence, 1, 1, 0, 0, representing the selection of 1 and 2.

The following *reductio ad absurdum* shows that the collection of all the logically possible selections from N is not the same size as N.

If those two infinite collections were the same size—if they were equinumerous—then all those possible selections could be paired up with different counting numbers. Such a pairing could be pictured as an endless list of sequences—each as long as that list and composed of some combination of the labels *In* and *Out*—the first paired with 1, the second with 2, and so on. Its top left-hand corner might look like this:

1 is paired with	*In*,	*Out*,	*In*,	*In*,	…
2 is paired with	*Out*,	*Out*,	*In*,	*In*,	…
3 is paired with	*Out*,	*Out*,	*Out*,	*In*,	…
4 is paired with	*In*,	*Out*,	*Out*,	*In*,	…
And so on	…	…	…	…	…

Every logically possible way of giving each counting number one of those two labels should appear somewhere in that list. But to get a representation of a selection that is not in that list we need only take the main diagonal of that list and replace every *In* with an *Out* and every *Out* with an *In*. Which means that for any putative pairing up of all the counting numbers with all of the logically possible selections from them there is going to be at least one possible selection not in that pairing. So there is no such pairing, and so the collection of all the logically possible selections from N is not the same size as N.

That was the *reductio ad absurdum*, and the rest of the diagonal argument is trivial: The collection of all the logically possible selections from N is not smaller than N—because for each counting number, it includes the selection of that number—so it is bigger than N.

In the next section (the first section of chapter 3), the logic of that argument is spread out, to make it even easier to see; and that argument is then generalised, from N to any collection of things, for the following purpose:

That section begins by describing a collection of possible selections that is as big as the collection of all the counting numbers. Presumably those logical possibilities always existed—as did all of the logically possible selections from that collection (which comprise a bigger collection), and all possible selections from that bigger collection (which comprise an even bigger collection, via the generalised diagonal argument), and all possible selections from that even bigger collection, and so on, endlessly. And presumably those logical possibilities always existed, so there are presumably all of the logically possible selections from them, and all of the logically possible selections from those logically possible selections, and so on and so forth:

> For every such collection, there are—and presumably there always were—all the logically possible selections from it.
>
> And for every endless sequence of collections, there is yet another totality of all the logically possible selections in all of those collections.

The problem is that, logical possibilities having always been logically possible, there is presumably some huge collection of all the logically possible selections that could, in that way, be found to have always existed. Each logically possible selection from that huge collection is just some, or none, or all, or one of them; and in particular, all the subcollections (the "some" above) are clearly there, to be possibly selected. They mean that there are even more of these logically possible selections. Whereas, they should already be part of that huge totality: They have just been found to have always existed in exactly the same way as all the others were found to have always existed.

That contradiction—which is reached more rigorously at the end of the next section—is a version of Cantor's paradox because Cantor also built his collections up out of previously collected things, in much the same kind of way. The rest of chapter 3 describes how, although all logical possibilities were always logically possible, logically possible selections could, just possibly, become more and more numerous.

Let me start by outlining a simpler way in which some other logical possibilities could conceivably become more and more numerous. Imagine a logically possible creator, creating a person. She might begin with a perfectly precise picture of that person. That person is then created, in some sort of universe. This creator did not have to create that person. And She does not have to actualise that possibility again, making an identical duplicate of that person, in an identical universe. But let us suppose that She did just that. Presumably She could actualise that possibility again, and again, if She wanted to.

There would always have been the logical possibility of the original person, and the logical possibility of that first identical duplicate; and for each extra duplicate that She creates, there will always have been the possibility of that person. But what about the duplicates She does not create? Would there be more than one possibility of them? All of them would share the same description. And what distinguishes one duplicate from another is their being different instances of that description. Similarly, when we refer to something, it makes a difference whether it exists or not: Only when something exists can it be pointed to; when something is only a possibility, it can only be described.

There was always the logical possibility of the original person, and the logical possibility of the identical person; but perhaps they were originally indistinct parts of the original possibility of someone satisfying that description. Once those two people exist, that distinctness exists, but was that distinctness always there, or do we read it back into the original possibility when we know that it exists? Well, suppose that this creator wants to create an apple.

> She begins with a comprehensive conception of a particular apple, and then She actualises that possibility—She creates that apple, out of nothing but the creative power that She happens to have. She might then create another, identical apple.

There was always the possibility of the first apple, and the possibility of the second apple, but try to imagine that She does not create the first apple, only the second. She begins with the same conception of an apple, and then She actualises that possibility. But that is exactly the same as the creation of the first apple. So it is just as though these possibilities do only become distinct from each other when there are distinct duplicates.

For another scenario, suppose that this universe had never existed. You would still have been a logical possibility. So it seems as though that particular possibility would still have been distinct from the more general possibility, even if this universe had never existed. Which would conflict with it being the case that if this universe had never existed, there would not have been that particular possibility. However, that particular possibility—of you in particular—was being read into the possibilities that would exist if this universe had never existed only because it was us supposing that this universe had never existed. Had this universe never existed, we would not have been supposing anything.

To sum up, logical possibilities could conceivably become more numerous as the original possibilities become increasingly individuated, through being increasingly actualised. Now, the sort of logical possibility whose existence is what the existence of a whole number amounts to—the logical possibility of that number of things—is not obviously of that kind. But in the third section of the next chapter you will see how that sort of logical possibility could conceivably be of that kind. As for how that would explain Cantor's paradox, let me close this chapter by considering the much simpler possibility that N does not exist. What would possible selections of counting numbers be like if there were always more and more counting numbers, with no collection of all of them?

> Some of the possible selections would have a describable pattern to them—for example, the selection of a finite number of counting numbers, or the selection of all of the even numbers—and these possible selections would become more and more numerous as the counting numbers got bigger and bigger.
>
> But most of the possible selections would have no such pattern. There would be no shorter way of specifying these possible selections, other than listing all the numbers in them. So, if there were always more and more counting numbers, then none of these possible selections would ever be distinct from all of the others of this kind.
>
> Possible selections from those possible selections would be very similar—there would be more and more of the distinct possibilities, plus similarly indistinct possibilities.
>
> So, if there were always more and more counting numbers, with no timeless collection of all of them, then there would be no number of possible selections of counting numbers, and no number of possible selections from those possible selections. There would just be more and more distinct possibilities of both kinds. No diagonal argument could show that they were not equinumerous, because they would not be two collections.

The counting numbers are probably timeless—N probably does exist—but if the next chapter's possible selections were somehow growing in number forever, then there would similarly be no such collection as the paradoxical collection of all of them. No diagonal argument would be able to show that it had paradoxically inconsistent properties, because it would not exist.

3

The Way of Things

3.i Paradox

The apple in my fruit bowl suddenly looks attractive (I must be feeling peckish after all that writing). That bowl, which is in the shape of a leaf, also contains a banana and a piece of coconut. If I call the apple A, the banana B, and the coconut C, the possibilities are:

> Have none of them (write this list instead).
>
> Have one of them, A or B or C.
>
> Have two of them, A&B or A&C or B&C.
>
> Have all three of them.

Those 1 + 3 + 3 + 1 = 8 possibilities are not things in the way that apples are things, but they are things in the sense that each counts as one thing.

> Because those possible selections are just different combinations of those three pieces of fruit, they differ only by having different pieces of fruit in them, so they are just as distinct from each other as those pieces of fruit are.

I was implicitly selecting the possible selection of A, from those eight possibilities, when I thought about having the apple; these possibilities are themselves things that might themselves be selected. And because possible selections from those eight things are just different combinations of those eight things, they are also just as distinct from each other as the pieces of fruit are. There are, in total, 256 different possible selections that might be made from those eight possibilities. The numbers are not important here—what matters will be whether various infinite collections are equinumerous or not—but incidentally, 256 is eight twos multiplied together, which mathematicians call two to the power of eight, which they write as 2^8.

> The possible selections from the bowl are like that too, because $2^3 = 8$.
>
> Incidentally, subsets of sets are also like that, which is why the set of all the subsets of a given set is called the *powerset* of that set. For example, the powerset of {a, b, c} contains 2^3 sets, {Ø, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}, {a, b, c}}.

The powerset of {a, b, c, d} contains twice as many sets: There are those eight sets, plus the eight that you get by adding the letter *d* to each of them, which is 2^4 sets.

There are, similarly, twice as many possible selections from four things as there are from three; and twice as many again from five things, 2^5 of them.

And there are a huge number of possible selections from those 256 things: Two to the power of 256 is bigger than ten to the power of 77, which is a one with seventy-seven zeroes after it. And there are many more logically possible selections from them; and so on, endlessly.

So, there are infinitely many of these logically possible selections, all of them as distinct from each other as the pieces of fruit are.

These collections of possible selections get bigger and bigger in increasingly huge strides; but this is the same infinity as the infinity of the counting numbers, even though the counting numbers get bigger and bigger in steps of 1. Whether you take short steps endlessly, or you take longer and longer strides endlessly, you would be walking slowly or leaping and bounding along exactly the same endless path. Still, infinitely many possibilities is still an awful lot of things for us to have got, given only the three pieces of fruit in my fruit bowl. You may well be wondering if there really are so many things. If logical possibilities do exist, then where are they?

Well, the physical possibility of me selecting the apple resides in my physical ability to pick such things up and in the apple being in the bowl. But if someone else had picked it out of the bowl, the result would have been the same selection. And if no one had, its selection would still have been possible. So, the logical possibility of selecting the apple resides in the apple being in the bowl, and there being some logically possible selector who would therefore be able to select it. Given that the God described in the third section of this chapter is logically possible, it is therefore the presence of the things that could be selected, in the collection that they could be selected from, that makes their selection a logical possibility.

To make the following reasoning more readable, I will give these possible selections shorter names. The possible selection of none of A, B and C is called [1], and [2] names the possible selection of A, [3] that of B, and so on. The list at the start of this section is now, more neatly:

[1]

[2], [3], [4]

[5], [6], [7]

[8]

The possible selection of none of those eight possibilities is called [9], with [10] naming the possible selection of [1] from those eight, [11] that of [2], and so on, endlessly.

To refer collectively to all those infinitely many possibilities, I will use the letter M (short for *many*). That is, [1] is in the collection M, [2] is in M, [3] is in M, and so forth—for every counting number *n*, [*n*] is in the collection M—and there is nothing else in M.

Now, it is logically possible for there to be a being who could spend half a second deciding whether or not to select [1], a quarter of a second deciding about [2], an eighth of a second deciding about [3], and so on—each decision being made in half the time of the previous decision—so that in one second, such a being could have made a decision about every [*n*] in M, and could therefore have made a selection from M.

I call the collection of all the logically possible selections from M the *power-collection* of M, and write that more neatly as P(M).

The following diagonal argument is based on Cantor's original diagonal argument, and it shows that P(M) is bigger than M.

Most of it shows—with a *reductio ad absurdum*—that P(M) is not the same size as M. So, the next bit of this section's (paradoxically logical) argument for a contradiction is itself a logical derivation of a contradiction, from the supposition that:

(a) There are as many things in P(M) as there are in M

The following contradiction will show that (a) is false.

But to begin the *reductio ad absurdum*, we suppose that (a) is true—that the possible selections from M can all be paired up with all of the things in M:

> One possible selection can be paired up with [1], another with [2], yet another with [3], and so forth, for every [*n*] in M, in such a way that every possible selection from M—everything in P(M)—is somewhere in that infinite sequence of pairs.

And as described in the last section of the previous chapter, each possible selection from M can be represented as an endless sequence of such labels as I and O. For example, (O, I, O, O, …) might represent the possible selection of [2].

So, given that (a) is true, those sequences could all be listed; first the one paired with [1], then the one paired with [2], then the one paired with [3] and so on, for every [*n*] in M. The top left-hand corner of such a list might look something like this:

(I,	O,	I,	I,	…
(O,	O,	I,	I,	…
(O,	O,	O,	I,	…
(I,	O,	O,	I,	…
…	…	…	…	…

The main diagonal of that particular list of sequences is (I, O, O, I, …), and the sequence (O, I, I, O, …) has a O where each of that diagonal's I's is, and a I where each of its O's is. It makes sense to call it the *negative diagonal* of that list.

For any such list, there will be some such negative diagonal, with the following contradictory properties:

> Because it is an endless sequence of the symbols I and O, it represents some logically possible selection from M. Which means—by (a)—that it is somewhere in that list.
>
> But it differs from the sequence corresponding to [*n*] in its n^{th} place, for every [*n*] in M, so it is not anywhere in that list.

That contradiction means that there is no such negative diagonal, and hence no such list, which means that (a) is not true: P(M) and M do not contain the same number of things.

And P(M) is not of finite size, so it is not smaller than M. So, P(M) is bigger than M.

That proof will be easier to generalise, from M to any collection of things, if the reasoning from (a) to here is rewritten more neatly. That reasoning was, firstly, that any pairing whose existence follows logically from (a) could, in theory, be represented as a list of sequences of I's and O's, and secondly, that from any such list we could, in theory, get a sequence of I's and O's that was not in that list. So, the next two steps after (a) are:

(b) List is a pairing of each [*n*] with some sequence of I's and O's, in such a way that all logically possible endless sequences containing nothing but I's and O's are paired up.

(c) ND is a sequence of symbols with the following property: For each [*n*] in M, if the sequence paired with [*n*] in List has I in its n^{th} place, then O is in the n^{th} place of ND, and if not, then I is in that place; and there is nothing else in ND.

Since ND is an endless sequence containing nothing but the symbols I and O,

(d) ND represents a possible selection from M

But ND was defined so that it differed from everything in P(M), so

(e) ND does not represent a possible selection from M

The contradiction between (d) and (e) shows that there is no such thing as ND.

If (a) was true, there would be some such list of pairings as (b) describes, and hence, via (c), such a sequence as ND. Since there is no such thing as ND, (a) is not true: P(M) and M do not contain the same number of things.

And P(M) is not smaller than M because for each [*n*] in M, P(M) contains the possible selection of just [*n*]. So, P(M) is bigger than M.

> Generalising that proof, to show that P(T) is bigger than T, for any collection of things T, is relatively straightforward.

Given any collection of things, it is logically possible for some logically possible being to select some arbitrary number of those things. So if T exists, then P(T) exists.

To show that P(T) is bigger than T, with a *reductio ad absurdum*, we begin by supposing that:

(1) There are as many things in T as there are in P(T)

If (1) was true, the things in T could all be paired up with all the possible selections from T. Let some such pairing be called PAIR. So, it follows from (1) that:

(2) PAIR exists

PAIR can be used to define a collection that I shall call NEDI:

(3) For each thing in T, if the possible selection that PAIR pairs that thing with includes that thing, then that thing is not in NEDI, and if that possible selection does not include it then it is in NEDI, and there is nothing else in NEDI.

It follows from (2) and (3) that NEDI exists. And it follows from (3) that the only things in NEDI are things in T. Which means that NEDI is a possible selection from T:

(4) NEDI is in P(T)

But it also follows from (3) that NEDI differs from every possible selection that PAIR pairs the things in T with, which is every possible selection from T according to the definition of PAIR. So NEDI is not a possible selection from T:

(5) NEDI is not in P(T)

Because (5) contradicts (4), NEDI does not exist, which means that PAIR does not exist. And since PAIR could have been any of the pairings whose existence was implied by (1), it follows that (1) is not true: P(T) and T do not contain the same number of things.

And P(T) is not smaller than T—for example, P(T) includes, for each thing in T, the possible selection of just that thing—so P(T) is bigger than T.

> When T is M, we get the proof that P(M) is bigger than M.
>
> And presumably some logically possible being could select any of the things in P(M). In particular, a God is a logically possible being (or so the proof of this chapter assumes). Taking T to be P(M) gives us a proof that P(P(M)) is even bigger.
>
> And presumably some logically possible being could select any of the things in P(P(M)). And taking T to be P(P(M)) shows that P(P(P(M))) is bigger still. And so on.

There are, in short, infinitely many increasingly large infinite collections. All the things in all of those collections are logically possible selections. They are logical possibilities, so presumably all of those things have always existed. Let us refer to them all collectively as U (for *union*).

Presumably there are all of the logically possible selections from U (a being who could select any of them is presumably a logical possibility). And taking T to be U proves that there are more things in P(U) than there are in U. Similarly, there are more things in P(P(U)) than there are in P(U), and so on. Again, there are infinitely many increasingly large infinite collections. And we can again refer collectively to all the things in all those collections, say as V.

And presumably P(V), P(P(V)) and so on exist (for the same reason). We can again refer collectively to all the things in all those collections, say as W. And so on and so forth. In short, there is always a larger collection to be found to be already there:

> Either there is another collection of all the possible selections from the previous collection (another power-collection).
>
> Or there is another collection of all the things in all the collections that we have, in this way, found to be there (another union).

Presumably there are all the logically possible selections that such steps—power-collection and union—could possibly get to, because if they could possibly be got to then they are already logical possibilities. So, their collection—let us call it X—presumably exists. And given that X exists, there is presumably a logically possible being who would be able to select one, or some, or all, or none of the things in X.

> Taking T to be X gives us a diagonal argument that there are more of these logically possible selections than there are things in X.
>
> Whereas, X was the result of considering all such steps to have been taken. So there should not be more of these logically possible selections than there are things in X.

That contradiction means that something that I have been assuming in this section is not true. And I have not been assuming much. I assumed that there are things like apples, for example, and that a God is a logical possibility.

The rest of this chapter considers how the logically possible selections of this section could conceivably become ever more numerous.

3.ii The Shape of Time

The apple on my desk was always a logically possible object. And there was always the possibility of something exactly like that apple—which is not necessarily the same thing. To see why not, consider how there might, just possibly, be an extra spatial dimension that we have not noticed (because nothing in this universe moves in that dimension). Off in that extra dimension, there might be other universes, and one of them might, just possibly, be exactly the same as this universe. The apple just like A in that universe would not be A, of course, so the logical possibility of that apple is not the same as the logical possibility of A.

There might, just possibly, be a lot of identical universes, containing a lot of apples just like A (whether their separation is spatial or not). Does that mean that there are already that many logical possibilities? There is no logical limit to how many identical copies of this universe there could be, so that would be a lot of possibilities.

And there is nothing to tell one of those possibilities from another—unless they are not just possibilities, but actualities—so that would be a lot of identical possibilities. If there actually is another apple just like A, then that apple's possibility is distinguished from all those others by it being the possibility of that particular apple. But the others would still be a lot of identical possibilities. Maybe too many.

It may be more realistic to suppose that if more and more copies of this universe were coming into being, then as the originally indistinct possibilities became increasingly individuated, more and more individual logical possibilities would be coming into being.

Imagine a simple Euclidean space that contains nothing but three completely identical spheres at the corners of a perfectly equilateral triangle. That seems to be a logically possible space; as does a space that contains nothing but four identical spheres at the corners of a square, or five at the corners of a pentagon, and so on.

Consider some such space being created by some logically possible creator. Because those objects would all be exactly the same, nothing in their creator's conception of them could tell them apart before they are created.

Once they have been created, each might be referred to individually. For each of them, the possibility of that particular sphere would always have existed—in the sense that that description, "the possibility of that particular sphere," could then be read into the general possibility of that many identical spheres—but that possibility could originally have been an indistinct part of that general possibility.

If possibilities could become more numerous over time. But perhaps you think that if we exist within an unchanging spacetime (as contemporary physics does seem to say we do) then possibilities could not become more numerous over time. So consider a calendar and a clock.

> As the current date moves along the calendar, and the current time moves around the clock, they remain that calendar and that clock.

Now, if we imagine those dates and times laid out horizontally, in a timeline, with the names of significant events above that line—in between the start and end times of those events—then we have a picture of reality that includes time as a dimension.

However, the weight on a beam, for instance, might be one of the dimensions of a graph that showed how that load varied along that beam; and weight is not much like a spatial dimension. So there are two basic possibilities for the nature of time:

> *Four-dimensionalism*—named after the three dimensions of space plus a temporal dimension that is assumed to be very like them—is the hypothesis that reality is all of the past, the present and the future.
>
> *Presentism* is the hypothesis that reality is the changing present, with the past being the way reality was, and the future the way reality will be.

Presentism is not about living in the moment. The future is going to happen, whether presentism is true or not. No, the basic difference between four-dimensionalism and presentism is that spacetime is full of events while the presentist present is full of things—ordinary objects like clocks, and the particles of which they are composed; and people and so forth. Some of the things that will be around in the future already exist, but most of them are not as they will be. And there might be some new things, things that are not already part of reality if presentism is true. Although scientific philosophers tend to assume that four-dimensionalism is true. Still, most think that presentism is a logical possibility.

Some argue that presentism is logically impossible, though; for example, some have argued that the question of *how long the present moment lasts* would be a dilemma if presentism was true:

> Either the present moment has a duration, or else it does not.
>
> If it had a duration, it would have parts that were earlier and later. So it would be some mixture of past, present and future times, not just the present moment.
>
> But if it had no duration, there would not even be enough time in the present moment for us to perceive anything. So how could it be the only real moment?

However, if presentism is true then it is only *the changing present* that is real. Since those changes are what time is, if presentism is true, the presentist present lasts for all time. There is plenty of time for things to happen in the presentist *present*. As for the present *moment*, a moment is just a very short period of time, whether presentism is true or not. The duration of the present moment is something that neuroscientists might measure, whether presentism is true or not.

Another argument against the logical possibility of presentism is that if presentism is true, there would be nothing to *make* various truths about the past *true*: What, for example, makes it true that you are now reading this question? Presumably it was your reading of that question that made it true that you were reading it. So, the existence—in the past—of your reading of that question makes it true—now—that you read it. But if presentism is true, the event of you reading that question no longer exists.

But given that God is a logical possibility, God's memory could make true all the truths about the past. Scientific philosophers trying to refute presentism usually want a different kind of answer to that argument. But the logical possibility of presentism only needs there to be one logical answer, and this one suits the proof of this chapter. It is a proof because although there are other arguments against the logical possibility of presentism, none are any better than those two fairly typical examples. And arguments of a similar kind can be made against four-dimensionalism. For example, what is it that makes you the same thing—the same person—as you were yesterday? For such reasons, the main criticism of presentism is not that it is logically impossible, but that modern physics shows four-dimensionalism to be (very) probably true.

And the spacetime of Einsteinian dynamics does suit four-dimensionalism. But in another bit of modern physics—quantum mechanics (the bit that explains chemistry, electricity, light and so on)—there are experimental results (connected to Bell's inequality) that make little sense if the future state of every particle is already part of spacetime. So although Einsteinian dynamics currently has only inelegant presentist descriptions, Einsteinian dynamics plus quantum mechanics currently add up to a simple lack of empirical evidence either way. And it is anyone's guess how that will change in the future (especially if the sciences become less materialist).

And even if four-dimensionalist models do prove to be the most elegant mathematical models in physics, that would not necessarily be much evidence against presentism. The elegance and utility of Newtonian physics was not, after all, much evidence against space being non-Euclidean. And if there is a God, then the most basic bit of reality would not be fundamental particles, it would be a person. The nature of time is therefore as much about the natures of consciousness and responsibility, as it is about physics.

Furthermore, the proof of this chapter only needs presentism to be logically possible. That is because even if four-dimensionalism is true, spacetime could conceivably have been created by a creator who is able to change in a second temporal dimension. If we call that logically possible dimension *hypertime*, then the question of whether presentism is true or not is the question of whether the creator whose existence explains Cantor's paradox is temporal or hypertemporal.

3.iii Plain Talking

> The Plain is a two-dimensional place inhabited by round people and square people.
>
> The Plain is so two-dimensional, round and square people argue over whether pentagons are spikey circles or squares with one corner squashed flat and the sides on either side of that corner pushed out slightly by that squashing.
>
> In the three-dimensional space around the Plain, a cylindrical person called Cyril has been watching them. He decides to give them something to talk about.
>
> As he passes through the Plain, he can look like a round person or a square person because his height is the same length as the diameter of his circular cross-section. He pauses at various places in the Plain—sometimes looking like a round person, sometimes a square person—and says, "I am Cyril."

The round people take the Cyrils to be a race of round and square people who are able to flip over to this side of the Plain from the other side, somehow.

The square people assume that Cyril is one person. They deduce that he is a round square person who is somewhere impossible when he is not visiting them. And they think that he must be visiting them in order to show them that they were all—round people as well as square people—made by him in his image.

Should I let them know that it was I who made them up? A square person called Martin appears and says, “I made the Plain and everyone in it.”

The square people take him to be Cyril, telling them that they are right. They set out to correct the round people. I blame myself.

I could have described the Plain axiomatically, but that would have been silly. Fictions are naturally sketchy.

Reality is the opposite of sketchy. Although quantum mechanics does describe physical reality as being a bit sketchy. And some philosophers think that we become more real through the commitments and creations that we choose to make.

And if the world was created, then it might be analogous to a fiction. Our knowledge of such transcendent aspects of reality as our world’s creator is bound to be sketchy, at best; but the question here is: Is such a creator a logical possibility?

An answer of “it almost certainly is” is indicated by the fact that most of the most scientific atheists claim only that there is (very) probably no God. Certainly, such arguments as there have been against the logical possibility of God have only ever shown that certain conceptions and definitions of God are not perfectly logical.

Consider, for example, the paradox of the stone: Can God make a stone that is too heavy for anyone—including God—to lift?

If not, there is something that God cannot do: Make such a stone.

But even if God could make such a stone, there would be something that God could not do: Lift that stone.

Either way, there is something that God cannot do.

Whereas, religious believers usually claim that there is nothing that God cannot do.

However, that just shows that, instead of saying "there is nothing that God cannot do," a better way to describe the creative power of a creator of all things out of nothing is to say that such creation would be, to some extent, like the creation of a fictional world:

> Fictional worlds are exactly as their authors describe, and what their readers thereby imagine, when those are competent authors and readers.
>
> And authors can write anything, any combination of words.

A God whose existence could explain Cantor's paradox would certainly be able to make an entire universe like this one. And presumably, such creations could obey any physical laws desired. No stone in such a universe could be so heavy, such a God could not make it rise up if desired. Not being able to create a stone too heavy for anyone to lift is more a reflection of the power of such a creator than a limitation of that power—which could conceivably have limits—let alone a reason why such a God could not exist.

A more natural argument against the logical possibility of such a God is the problem of evil: Is the existence of such a God compatible with the hundreds of millions of years of senseless suffering that seems to have occurred on this one planet alone?

> Why would such a God not have created good and only good people in a world in which good and only good things happened?

I think that if there is a God, then that is what God would (probably) have done. God may well, I think, have originally created a more spiritual world (a less physical world) in which a wide range of good people were all much closer to their creator than we are. Imagine them as wiser and much better informed about creation than any of us are:

> Might some of them have wanted to spend some of their limitless time in a less heavenly world (a world that could keep itself ticking over, to a greater extent)?

You might not think so, but the relationships that those people would have been having with each other could conceivably have benefited from them spending a short amount of time in a world like ours. Note that I am not saying that this is (probably) why you were born; this is just one answer to that argument against the *logical* possibility of a good God.

From the heavenly perspective of those hypothetical people, being born here might have seemed like going camping. It may not have looked like that once they were here, of course, but a God would presumably have been able to guarantee that they would end up at least as well off as they started.

They might reincarnate, for example, with some of their later incarnations being more therapeutic. Indeed, this might be one of those more therapeutic worlds.

Furthermore, the relationships that those people could possibly have been having with their creator would have been intrinsically limited, even in their heavenly home: Much as any story's author is above and beyond that story, their creator would have been above and beyond even that world. Could those people have gained a valuable perspective on such limits by spending some time in a world even further from their creator? One reason why that is likely is that a God would always have known about the most horrible logically possible creations, as well as all sorts of other things; whereas, good creatures would never have thought of such things in their heavenly home.

Less obscurely, all the generally accepted facts appear to be consistent with that story—*the Odyssey theodicy*, as I call it—and with many other ways in which the creation of a God might be like this world (ways called *theodicies* by philosophers). The fact that most of us do not recall choosing to be born, for example, does not count against the logical possibility of that story, because most of us cannot even recall being babies.

This chapter's proof (of a high probability) does not depend upon whether the creator who explains Cantor's paradox is called "good" or not, though: This is a logical book, not a theological book.

What about the mysteriousness of how anyone could possibly create anything out of nothing? Would it be too much like taking the possibility of magic seriously, to take the possibility of such a creator seriously? Well, consider Galileo's rejection of the hypothesis that the movement of the moon causes the tides:

> Although the timing and strength of the tides clearly correlate with the position and phase of the moon, Galileo could see that there was nothing but clouds and seagulls in between the moon and the seas. There was clearly no way in which the moon could pull all that water about, Galileo thought; or none that was not some "magical" action at a distance.
>
> The British scientist Isaac Newton (1643–1727) is said to have wondered, upon seeing an apple fall from a tree, what had pulled it down; there was clearly nothing under the tree, except the earth. But something had caused the apple to move, and everything on earth fell in the same kind of way; and the same mathematical pattern was found in the orbits of the planets around the sun.

The difficulty of imagining how the moon could pull the seas about was less important, to Newton, than that pattern. And it was easy enough to see how there did not have to be anything "magical" about the action at a distance that Newton described. You could, for example, picture gravity as the bending of a rubber sheet: Masses rest on the surface of the sheet, depressing it slightly so that smaller masses fall down the sloping sides of those depressions towards the larger masses (like apples falling) or roll around them (like asteroids around the sun).

> In the first section of this chapter, a mathematical pattern of logical possibilities becoming increasingly numerous was described. The rest of this section is a mere sketch of one way in which such possibilities could conceivably be endlessly created.

Our knowledge of logically possible Gods is bound to be sketchy. But certainly, in order to create things, of any kind, a God would presumably have had to have had the concept of *a thing*, in the sense of a collection or number of things.

That is the simplest concept in logic and mathematics, but if there was originally nothing but God, then God's uniqueness would plausibly have been simpler—simpler from God's perspective, and hence objectively simpler. It is therefore conceivable that God is above and beyond the concept of a thing. And if that concept had not already been instantiated by God, then it could have been chosen by God from a range of similar concepts.

A God would presumably know an unimaginably wide range of possibilities for creation. So a God might know a wide range of concepts similar in various ways to the concept of a thing, and so the concept of a thing might, just possibly, be—from God's perspective—a mass of very closely related concepts. If each of those narrower concepts of a thing would have facilitated the creation of exactly the same things, with exactly the same mundane properties, then why not? As for why, that would depend upon the purpose of creation.

If there is a God then creation presumably has a purpose; and the nature of created things would presumably have been chosen to suit that purpose. Perhaps, for example, the point of creation was to have other people to interact with, people who might contribute to their own creation by making responsible choices. If so, then God might have wanted to facilitate their discovery that they were deliberately created, and that there was therefore more to their lives than they might otherwise have thought plausible.

Could the reason why the concept of *a thing* has the property described in the first section of this chapter be that the only logical explanation for the existence of that property is the endless creation of possible selections? Maybe, if the creation of possible selections does make sense. There is certainly no way in which whole numbers could be individuating themselves (a fiction in which they were individuating themselves would be as realistic as a fiction in which the number zero was Julius

Caesar). So, does the creation of possible selections make sense? Not even a God could create possible selections that were not already possible.

And how could they be individuated, from broader possibilities, when each collection of them has a predetermined form? Well, you could say that they are possible because God can create them. That is because if there was originally nothing but God, then such possibilities would originally have been part of God's power. And although each collection of them does have a predetermined form—that of a power-collection—the totality of all those collections has the form of an endlessly growing class of things.

And if the concept of a thing is objectively *a mass of very closely related concepts*, then those narrower concepts could conceivably give rise to various inconsistent properties. As those properties came to God's mind, God would choose which of the narrower concepts to keep, in order to have created things with the Cantor-paradoxical property in question. By deciding which of those narrower concepts to keep, God might be said to be creating those possible selections.

Those properties would presumably be in God's mind instantaneously, as the things with those properties were thought of. But even someone with the power to create things from nothing but their own creative power could conceivably have some intrinsic limits. Even a God might have to take some time—or hypertime—thinking about collections much bigger and far more complicated than those described in the first section (although it would presumably be no more effort for a God to do that than it is for you to keep your heart beating while you sleep).

> And because there is always another of those steps to be taken, even a God might be creating such possibilities forever (out of some more general possibility).

That particular way in which a God could conceivably be creating the possible selections of the first section of this chapter is unlikely to be how a God would actually create them, of course; but at least it illustrates the conceivability of such a God creating such possibilities forever. Naturally, it is extremely mysterious how possible selections could be created. But it is even more mysterious how a God could possibly make anything out of nothing but the possibility of stuff of all kinds. But creation out of nothing is presumably a logical possibility (or at least, the proof of this chapter has assumed that it is).

4

Reasonable Doubts

4.i Bad Math

Mathematicians are usually working—implicitly or (less commonly) explicitly—within a standard set theory; and within that axiomatic model of collections, the paradoxical contradiction of the first section of chapter 3 cannot be derived. You could be forgiven for thinking that that means that the proof of the previous chapter was based on bad mathematics.

But while Cantor's paradox does not exist within the standard set theory, it does exist in the mathematics that that theory models (that is, after all, why set theories exist). If mathematics was translated into any other artificial "language," and if that translation was as good as the standard redefinition, then mathematical models and methods would work just as well. Indeed, those models and methods could use actual numbers (as most scientists assume they do). A model of a model is just another model.

As for pure mathematics, it is all about proofs. Given some abstract structure—such as arithmetic—mathematicians prove all sorts of facts about it by starting with the definition of that structure and then deducing those facts logically from that definition. Proofs have to start somewhere, and the standard axioms are very good descriptions of the most basic properties of collections—when those are not paradoxically large collections—so those axioms are a pretty good place for any mathematical proof to begin.

And even when the collections of interest to a mathematician are paradoxically large, set theorists need only add another axiom to the standard axioms—such as an axiom asserting that there is a set of all the standard sets—in order to create a new set theory with which to study such large collections. To study very large collections of the new sets, another axiom could be added; and so on, indefinitely. Set theory is therefore a pretty good model of the endless creation of unimaginably large whole numbers.

And because set theory is not in conflict with, but is rather compatible with my explanation of Cantor's paradox, actual mathematicians are unlikely to think of chapter 3's proof as bad mathematics. To see this more clearly, it may help to consider a simpler but similar scenario. And the paradoxical equations in the second section of chapter 1, which took us from 0 = 0 to 0 = 1, are not a bad mathematical model of something that may well be a physically possibility. To begin with, here are those equations:

$$0 = 0$$

$$0 = 0 + 0 + \dots$$

$$0 = (1 - 1) + (1 - 1) + \dots$$

Removing the brackets:

$$0 = 1 - 1 + 1 - 1 + \dots$$

$$0 = 1 + -1 + 1 + -1 + \dots$$

Adding more brackets:

$$0 = 1 + (-1 + 1) + (-1 + \dots$$

$$0 = 1 + 0 + 0 + \dots$$

$$0 = 1$$

Those steps from equation to equation cannot all have been correct, but which was incorrect? Was it the first? Adding zeroes to any amount will not change it, as a rule; but infinitely many zeroes can add up to a nonzero amount:

> A straight line might be crossed by another line anywhere, and where they cross there is a point. So there are points everywhere in that line. The length of each point is 0, but that line—which is composed of nothing but points—has some nonzero length.

Still, the zeroes in the equations above are simply added together, they are not arranged in a previously existing line. And you will soon see a reason why the zeroes in the equations above do not amount to more than adding zero.

What about the other steps? Well, the replacement of each 0 with (1 – 1) was clearly alright, because one minus one does equal zero.

And removing any of those pairs of brackets would clearly be unproblematic, so why not remove each and every one of them?

What about the introduction of negative units? To replace subtractions of units with additions of negative units is basically to introduce directions. Adding to a debt, for example, might be thought of as moving in the opposite direction to increasing wealth. So, there were directions as well as magnitudes, following that introduction, and so the nature of the equation was changed by that introduction. When the subtraction of a unit became the addition of a negative unit, the addition of a unit became the addition of a positive unit. But what happened to zero? Directed magnitudes are called vectors, so a mathematician might say that zero became the zero vector. But it is strange to think of something with no length having any direction. And if it had no direction, how could it be a directed magnitude?

Still, a mathematician might say that the zero vector has both directions potentially. And a mathematician would certainly say that that argument went wrong when the brackets were removed. The top three equations are clearly true, and the bottom three are clearly false, so there was clearly something odd about the middle two.

The removal of the brackets left us with an equation whose right-hand side was this:

$$1 - 1 + 1 - 1 + 1 - 1 + \dots$$

That is called the Grandi series, after the Italian mathematician Guido Grandi (1671–1742). Mathematicians generally do such infinite sums by working through such series step by step and seeing what happens. And summing the first two terms of the Grandi series gives us $1 - 1 = 0$. Summing the first three terms gives us $1 - 1 + 1 = 1$. The first four terms gives us $1 - 1 + 1 - 1 = 0$. And so on: The sums are clearly going to alternate between 0 and 1, endlessly.

In general, if such initial finite sums were getting closer and closer to some number as the length of those sums increased, then that number would be called the sum of the infinite series. (Adding zeroes endlessly does therefore amount to adding zero.) But with the Grandi series, the initial finite sums are alternating endlessly, so mathematicians say that the Grandi series has no sum.

The removal of the brackets took us from a series that has a sum—in an equation that was correct—to a series that has no sum. The removal of the brackets was not good mathematics.

Nevertheless, some of those equations do give us quite a good mathematical model of creation out of “nothing,” as follows.

In physics, a particle and an antiparticle might appear, as a pair, out of the background energy of the vacuum. Once formed, the particle and the antiparticle repel each other.

> A picture in which their separation is represented horizontally, with the vertical axis representing time, might be: \/ The particle is on the left, the anti-particle on the right. The particle and antiparticle begin at the bottom of that V, moving away from each other.

An infinite line of such particle/antiparticle pairs could—hypothetically—appear from the vacuum of an infinite space; and they might, just possibly, move in such a way that each antiparticle collided with the particle immediately to the right of it, like this:

\/…

The bottom of that endless zig-zag represents an infinite line of pairs of particles and antiparticles spontaneously appearing in empty space.

If each of those particles was modelled mathematically by +1, and each antiparticle by –1, that appearance could be modelled by this infinite sum:

$$(+1 \quad + \quad -1) \quad + \quad (+1 \quad + \quad -1) \quad + \quad (+1 \quad + \quad \dots$$

That is the same as an infinite sum of zero-vectors, so it sums to the zero-vector, which would not be a bad mathematical model of the empty space that these hypothetical particles and antiparticles are spontaneously appearing within (especially if we think of it as having all the directions potentially).

The middle of the zig-zag represents an infinite line of particles and antiparticles. A mathematical model for that could be the Grandi series with directed units:

$$+1 \quad + \quad -1 \quad + \quad +1 \quad + \quad -1 \quad + \quad +1 \quad + \quad \dots$$

The collision of all the antiparticles with particles leaves just one particle moving away, which could be modelled mathematically by this series:

$$+1 \quad + \quad (-1 \quad + \quad +1) \quad + \quad (-1 \quad + \quad +1) \quad + \quad \dots$$

Would a mathematician object to those mathematical models on the grounds that the middle addition has no sum? The Grandi series has no sum in the calculus, but that does not make it wrong to say that it has no particular sum; indeed, the Grandi series having no particular sum coheres with it being the case that adding brackets in two different ways gives rise to two different sums.

> Quite generally, mathematical objects that were originally thought to be non-existent—negative numbers, for instance—tend to be thought by later mathematicians to have had, more precisely, no definition in the earlier mathematics.

For such reasons, mathematicians are unlikely to object to those simple models. Especially if it is emphasised that they do not exist within the standard calculus. And similarly, while the logical puzzle described in the first section of chapter 3 does not exist within the standard set theory, it does exist in any world in which there are things as we know them.

4.ii A Logical Joke

> All humans are animals,
> and you are a human,
> so you are an animal.

From the fact that all humans are animals, it follows that you in particular are an animal.

Quite generally, logic just gives back to us something that we already had—that is why logic takes truths only ever to truths, why logical thinking is reliable thinking. Nevertheless, logic can be surprising because what it gives back to us might not be something that we knew we had.

You will have read this section's first words, "all humans are animals," as saying that we are all members of the animal kingdom because with that reading, those four words are clearly true. When we read words, we subconsciously give them meanings in such a way as to make the most sense we can of them. We do the same kind of thing when we hear words—which is how we were able to learn our first languages—and we do something similar when we recognise faces.

It can, as you probably know, be harder to recognise someone when you see them out of their usual contexts. When it comes to recognition, context is important. And similarly, you will have read the second sentence, "and you are a human," as another biological fact.

From those two facts it follows that you are—in the same biological sense—an animal.

Nevertheless, the normal meaning of "you are an animal" is that the person being described is being beastly. And that rude assertion might even have looked like it was being proved, because the context is a proof. So as well as that proof—of a biological truth—there was also a "proof," of something surprisingly rude (a logical joke).

Logical paradoxes are also "proofs" of untruths. But they are less funny-ha-ha. And in particular, the following paradox is so funny-peculiar, it could make anyone wonder about the reliability of logic. It concerns the following claim:

> If this claim is true, there is a round square.

For any claim to be true, what it claims has to be the case. So:

> If that claim is true, then it is the case that if that claim is true, there is a round square.

In other words:

> If that claim is true, then if that claim is true—which it would then be—there is a round square.

That repetition of “if that claim is true” can presumably go, because if the claim is true then it is true. So the line above presumably means the same as:

> If that claim is true, there is a round square.

But that is just what that claim claimed. So:

> That claim is true.

And it follows logically, from “if that claim is true, there is a round square” and “that claim is true,” that there is a round square. Since there are no round squares, one of the steps of that argument was not as logical as it looked. But there were only four steps in that argument, and each of them looked perfectly logical.

The first step was just what it means for a claim to be true. The second step was the removal of a repetition—and it is hard to see how that step could have been wrong (if the claim is true then it is true)—the third step was again what it means for a claim to be true, and the final step was a logical deduction. And the more you try to discover where the illogical step was, in that “proof,” the clearer it becomes that you do not have to be crazy to wonder about the reliability of logic.

There is a clue to where the illogical step is in the fact that “if this claim is true, there is a round square” is basically claiming not to be true, just like Ludwig’s nine words. But it is hard to see how the logical explanation of the liar paradox could help to explain this paradox, because none of the four steps above appears to have assumed that the claim is either true or else not true.

> The logical explanation of this paradox is described in the third section of the next chapter.

Incidentally, this paradox was originally a mathematical puzzle, devised by the American logician Haskell Curry (1900–1982) to test mathematical models of logic.

> For an overview, see “Curry’s Paradox,” by Lionel Shapiro and Jc Beall, in *The Stanford Encyclopedia of Philosophy* (Winter 2021 Edition), edited by Edward N. Zalta, URL = <https://plato.stanford.edu/archives/win2021/entries/curry-paradox/>.

The mathematical modelling of logic is usually called *formal logic* (Russell more sensibly called it *mathematical logic*). The word *formal* suggests that arguments in actual languages are informal, that they are less rigorous than formal-logical proofs. So note that to make an argument formal in this sense is to just model it mathematically (algebraically, as a rule). Which means capturing just some of its meaning. Now, if there was something irredeemably inconsistent about that argument—if it had too much meaning (if it proved something and its contrary) because of a flaw in logic itself, for example—then capturing just some of its meaning would be ideal. Nevertheless, a formal-logical proof is a mathematical proof. To get an actual argument from such mathematics, meaning would have to be put back into those symbols, using words.

Logic is the study of what logical arguments have in common. So logic is about the form—the shape or structure—of logical arguments. And the science of patterns is mathematics. But still, the only way to get a *rigorously logical* argument is to think logically, in ways that suit the subject matter, and then express those thoughts clearly.

Chapter 3 was a rigorously logical argument that there is (very) probably a God. It did not fail to be logical by not being formal-logical.

Although to a lot of atheists, that description of chapter 3 is bound to look like "a rigorously logical argument that there is (very) probably a round square." You have just seen an apparently logical argument that there is a round square, and you won't have concluded that there is probably a round square. You probably assumed (correctly) that that version of Curry's paradox was not as logical as it looked. Atheists reading chapter 3 are similarly likely to assume that chapter 3 was not as logical as it looked. Most of them will not want to waste their time looking for the mistake in that chapter. And even if some atheist logicians do look—and look quite hard, because they can find no mistake—they could always conclude that evolutionary pressures must have left us without the ability to reason in perfectly reliable ways about such abstract matters. And yet:

> If the logical pursuit of truth led actual scientists to a truth that they had thought false, would they change logic until it gave them what they originally expected? Or would they revise their beliefs in light of what logic was showing them?

As the 20th century began, Russell was finding paradoxes that were similar to Cantor's but more obviously logical—see the second section of the next chapter for an example—and so Whitehead and Russell presented a mathematical model of logic in their *Principia Mathematica*, with the aim of building from it a consistent model of mathematics.

Most mathematicians chose to work within a standard set theory; but for a hundred years, scientific philosophy has been applying a mathematical model of logic—classical logic—that developed out of the model in the *Principia Mathematica*. Classical logic models the logic of deductions; and it is a very good model, when those deductions are not paradoxical. The axioms of classical logic encode the steps of arguments that were so obviously logical, they were widely regarded as proofs. If some such proof had included a step that did not correspond to any of the classical axioms, logicians would have had that reason to add another axiom to classical logic. They would not have had any reason to reject that proof. And when the use of classical logic results in something implausible, logicians have that reason to investigate non-classical logics, or to wonder if they have reached some limit to the utility of such mathematical modelling; but of course, they have no reason to reject logic.

4.iii Doubting Reason

What makes a good explanation is common sense. For an example to focus on, picture yourself in a derelict warehouse, looking at half a brick. It is surrounded by shattered glass, underneath a broken window. The hypothesis that it broke the window would explain your observations—up to a point (you have no idea why it was thrown)—so, because you have no reason to doubt that that is what happened, you naturally assume that the half brick broke the window. Consequently you have no reason to examine the warehouse.

If you did look around, you might find other halves of bricks surrounded by shattered glass and none of them underneath broken windows (perhaps this half brick is part of a work of modern art). Or, if you had examined the glass, you might have found that it was shattered by a bullet (perhaps the half brick was put there later to conceal a shooting). There are lots of possibilities. But if you did look around, you would probably find little more than dirty rubble and a few beer cans. The window was probably broken by the half brick. If you had looked around and found nothing, you might have said that the window was very probably broken by the half brick. But for you to have wanted to look around, you would have had to have been investigating something more serious than a broken window. So you continue on your way.

Straight away you find that the next window along is also broken. Underneath it, surrounded by shattered glass, is the other half of the brick. When you get outside, you find that an oversized spectacles frame has been fixed over those two windows. One of the window frames is painted red, the other blue, and a sign underneath says, “The Spectacle of the End of Society.” Googling that phrase, you find it to be the title of a performance by Ludwig Humdinger, who had used the half bricks to break the windows. Your hypothesis was true, although not in the way you imagined. Unless what you read online was false. Perhaps, for example, the windows were broken when an alien fired a ray-gun at another alien.

For all we know for sure, aliens might cover up such evidence of their presence with artistic installations (crop circles might cover up their usual comings and goings). And there are lots of similarly unlikely possibilities—although, how do we know that such aliens are unlikely? Evidence for that unlikelihood might have been produced by them, as part of their cover-ups. Conversely, even if an alien ray-gun had broken the windows, and you made a thorough examination of the glass and saw how strangely it was broken, you would not have any real evidence for aliens (scientists might have been using the deserted warehouse to test a secret device).

In the absence of unambiguous evidence for aliens—such as capturing one of them—it would obviously be reasonable to ignore this logical possibility, and plenty of others. The unquantifiable possibility that some very secretive and very capable power has been interfering with evidence for their own unfathomable reasons, without leaving any sign of their presence behind, is just not a reasonable doubt for, say, a jury to entertain, whether that power is alien, angelic, demonic or foreign. And when scientists are looking for a scientific explanation of their observations, they similarly ignore the logical possibility of God.

> For thousands of years, God was invoked to explain all sorts of natural events; but as successful scientific explanations of nature mounted up, God was seen to be an unnecessary hypothesis. Scientific philosophers usually dismiss arguments that invoke the existence of a God in order to explain a current gap in our scientific knowledge—a gap that could eventually be filled properly by scientists—as "God of the gaps" arguments.

Which raises the question, is the proof of this book a "God of the gaps" argument?

The hypothesis that all things were deliberately created, out of nothing but the possibility of such things, is certainly not an ordinary scientific hypothesis. Ordinarily, matter and energy change only superficially; they can change from one form to another—a lump of clay might be turned into a bowl, for example, and engines turn the potential energy of fuel into the kinetic energy of motion (and in stars, matter is turned into energy)—but the sum total of all the energy in the universe is (if we include the energy-value of all the mass in the universe) unchanging. Nevertheless, there is a lot of evidence that the universe has been expanding out from an original concentration called the Big Bang, before which there may well have been none of that energy. That would not violate the law of the conservation of energy (including mass-energy), it would just mean that that law did not apply beyond this universe. Or perhaps there was originally some sort of potential for the Big Bang. We simply do not know much, if anything, about such matters. What we do know is that there are physical objects and people. And they certainly appear to be two different kinds of things. So there are two basic possibilities:

The hypothesis that the world is fundamentally impersonal is commonly called materialism.

The hypothesis that a person created everything else is a God hypothesis.

Materialism is certainly counter-intuitive, because we tend not to think of each other as biochemical mechanisms, but that does not stop it being a reasonable hypothesis. And similarly, we do not naturally think of the universe as being created by some extraordinary person out of nothing but that person's ability to create. (Creation myths always had gods or giants creating the world out of something that had always been there.) But again, that does not make God hypotheses unreasonable.

And Cantor's paradox makes it unreasonable to ignore the logical possibility of God—much as a spaceship arriving from outer space, in clear view of all our telescopes, would make it unreasonable to ignore the possibility of aliens—and so the proof of this book is not a "God of the gaps" argument.

It is normally reasonable to ignore the possibility of aliens, even though aliens could conceivably be here already. There is little or no evidence of their presence. Now, that is what we would find if there were powerful aliens hiding here. And while nothing in the universe can travel faster than light, and the nearest galaxies are millions of light-years away, secretive and powerful aliens could conceivably have fabricated the evidence for one or both of those facts. But of course, none of that evidence is actually thrown into question by that possibility. It is a logical enough possibility, and there is no way of knowing how unlikely it is. But it is simply not a reasonable doubt for scientists to entertain.

Why it is not—how it could be rational to treat such intrinsically unknowable probabilities as though they were extremely high improbabilities—is a hard problem in scientific philosophy known as the problem of *Cartesian scepticism*, after the French philosopher René Descartes (1596–1650).

That it is rational to ignore such unknowable probabilities, that is obvious.

One of Descartes' examples was the possibility that you are currently having a very vivid dream. If this was a dream, then the objects you see around you would not be real objects, so a lot of your common-sense beliefs about them would not be true. According to Cartesian scepticism, you do not have the knowledge that common sense says you have, because you do not know such things as that you are not dreaming.

Perhaps you think that you do know that your dreams are not at all like your current experiences? But if this was a dream, your recollection of what dreams are like would be a recollection within a dream, which would hardly be a reliable kind of recollection.

Another of Descartes' examples was demons who might be bewitching us. The possibility of secretive and powerful aliens is a modern version of that possibility. As is the possibility that you have been hypnotised by a hypnotist who has made you forget that people can be that good at hypnotism. Any of your common-sense beliefs might, just possibly, have been put into your head by such a hypnotist, so how could you possibly justify the common sense that that has not happened?

Nevertheless, this is simply not a reasonable doubt to have about any of your common-sense beliefs. Understanding why it is not, that is a problem for philosophers; assuming that it is not, that is necessary if we are to explain anything. And much the same is true of the logical possibility that you are dreaming all of this.

For a third kind of Cartesian doubt, consider how there might, for all we know, be other universes exactly the same as this one. Furthermore, objects could, just possibly, be swapped instantaneously between these identical universes. Such a swap would have no physical consequences. So there is no evidence that such swaps occur—this is not a scientific hypothesis—but nor is there any evidence that they do not occur. Nevertheless, we all assume that they never do.

To see that we do assume, if only implicitly, that such swaps never occur, consider what would happen if Ludwig was swapped for one of his duplicates. There would be no observable difference; but if his mother saw Ludwig's duplicate waiting for a bus, she would of course not know that Ludwig was waiting for a bus: It would not have been Ludwig. The question is, when it is Ludwig who is waiting for a bus, how does she know that Ludwig is waiting for a bus? She would be seeing that he was, so she would know that he was. So, she must have been assuming, implicitly, that it was not some logically possible duplicate. That would be an implicit assumption because, like most people, Ludwig's mother is not even thinking of such possibilities when she is not philosophising.

It is like she can acquire knowledge by not thinking. Such is the logical problem. It is a logical problem in the sense that it is a problem for people like the professor to ponder, when she is back in her study. It is not a logical problem in the sense that it gets in the way of our thinking logically:

> In order for us to be able to gain any knowledge, about anything, we need to be able to recognise things as the same things in different situations. So in the absence of evidence to the contrary—and there never could be evidence that such swaps are occurring—we implicitly assume that things remain the same things, that it is their properties that are changing.

How many other Cartesian-sceptical scenarios are there? No one knows. But it would certainly be very difficult to show that every logically possible Cartesian-sceptical scenario was very unlikely. And even if that could be done, philosophers would be little better off: Adding up a lot of very low probabilities can result in quite a high probability. In practice, we simply ignore each of these possibilities, which does add up to ignoring them all. Note that it is not so much that we can ignore them, as that we must ignore them in order to think logically.

> Should we simply ignore the logical possibility that we did not evolve to be able to think logically about such things as all possible collections, or just those in chapter 3, or Ludwig's nine words, and the ten of Curry's paradox, or the world as a whole, as philosophers do, or just the scientific subjects?

Insofar as such a possibility seems realistic, it does seem wrong to ignore it—but that is true of all the Cartesian-sceptical scenarios. The possibility that you are in a coma right now, for example, is not unrealistic. People do fall into comas, and dreams can be very vivid. And yet, to take that possibility seriously—say, as a reason why some very unwelcome news need not be taken very seriously—would not be rational. And much the same is true of this possibility. To see why, imagine a defence lawyer suggesting to a jury that they should not trust their logical consideration of the evidence, on the grounds that logic might, for all they know, be revealing one of its natural imperfections in this case:

> Such a lawyer might run through the basics of the theory of evolution and show the jury some simple logical paradoxes, in order to make that seem like a reasonable possibility. Character witnesses who assure the jury that the accused is not the sort to commit such a crime might then be called. Those jurors might end up feeling that the accused must have been misrepresented by that evidence. But that would of course not have been a reasonable doubt for them to entertain. What they should have doubted was those witnesses' testimonies, and that lawyer's account of those logical paradoxes.

5

The Net of Language

5.i Vagueness

Ludwig was waiting for a bus to take him to his studio, where the current performance had begun several years earlier, with a heap of sand from which individual grains of sand were taken one by one by whoever turned up. Ludwig had removed the first grain.

Whenever a single grain of sand is taken away from a heap of sand, it leaves a heap behind. So when Ludwig removed the second grain, he was again taking a grain away from a heap, and so he was again left with a heap. The removal of the third grain was therefore another taking away of a grain from a heap, and so the fourth grain was also taken away from a heap. And so on. It seemed to Ludwig that he would always have a heap of sand, so he offered free sand to anyone who picked their own, one grain at a time, and called it *The End of the Society of the Spectacle*.

People turned up, off and on. Some of them even joined in. But many years—and hundreds of thousands of grains of sand—later, Ludwig has only enough sand for a sand castle. As he waits for his bus, he decides to end the performance as soon as he gets to his studio, by making that castle. Just then, his mother—professor Humdinger—comes over, having seen him waiting at the bus stop. He decides to call the castle *Hogs Fort*.

His mother asks him about his work, and he tells her about his disappointment. She agrees that it does indeed follow, from the fact that taking a single grain of sand away from a heap of sand leaves a heap of sand behind, that no heap of sand could possibly get used up one grain at a time. But that just means, she tells him, that heaps of sand are not logical objects. She asks him to consider how different it would have been, had he been removing sand from a mill of sand, where a *mill* is her nickname for *a million or more grains*. She points out that:

> Whenever a grain of sand is taken away from a mill of sand, either that mill was exactly a million grains, or else a mill remains.

There is no mystery about how a mill of sand stops being a mill. And every heap of sand is actually some precise number of grains of sand, she adds. She prefers formal logics to logic, and made-up words like *mill* to natural words like *heap*, but it was not really very mysterious how Ludwig's heap of sand stopped being a heap:

> If you have a lot of things, and a few are taken away, you will still have a lot of them; but if a lot are taken away, then of course you might not be left with very many.

Why did professor Humdinger think it mysterious? Well, a heap is a large amount. And it is only natural for us to think of large amounts as the business of mathematics. And mathematical operations can, as a rule, be repeated indefinitely. Nevertheless, while mathematics is an exact science, a heap is a vague amount. A heap is a lot of stuff that has been piled up, according to my dictionary.

Another dictionary might say that a heap is a large pile, and that a pile is an amount of something that has been heaped up. Dictionary definitions tend to be a bit vague and circular, because they are only describing the meanings of our words. We learn those meanings by coming across various uses of the words, and subconsciously abstracting their meanings from what we know by acquaintance with those uses. The way in which our words have acquired their meanings is, in short, a rough-and-ready process. Our words are defined well enough for their ordinary uses, but their meanings are certainly not as precise as those of mathematical objects.

When we look at things like Ludwig's heap, in ways that would ordinarily be logical enough, what we see can therefore—especially if we are mathematically inclined or trained—seem contradictory. Suppose, for example, that each time a grain of sand was taken away from his heap, Ludwig had asked himself, "is this pile of sand a heap?" The answer would at first have been "yes," but it would now have been "no." None of those piles of sand stopped being a heap with the removal of a single grain, so they must have stopped being heaps gradually; and as they gradually stopped being heaps, "yes" would gradually have become incorrect. So there would have been times when "yes" was in between correct and incorrect. The puzzle is how "yes" could be neither correct nor incorrect:

> If "yes" was not the correct answer, it would have been incorrect.

Well, what do we know about this? I suppose that when Ludwig had a pile of sand that was not obviously a heap and not obviously not a heap, some people would call it a heap, some would say that it was not a heap, and a lot of people would just call it a pile of sand. Why not describe it as a pile of sand that was about as much a heap as not?

That description is rather vague, but a heap is a vague amount. And if that description was correct, then the question "is this pile of sand a heap?" would be answered correctly by "about as much as not." And that question being answered by "yes" about as correctly as not would solve the puzzle: "Yes" would be, not so much not correct, as only about as correct as not; and it would be, not so much not incorrect, as only about as incorrect as not. Those descriptions are rather clumsy, but that is only because it is not natural for us to use the word *heap* in the way that this logical puzzle requires us to. When something is about as much a heap as not, we are more likely to call it a pile.

> Would it be wrong to say that it was a heap? Well, it is not a fact that it is a heap. But nor is it a fact that it is not a heap. The fact is that it is only about as much a heap as not. To say that it was—or that it was not—a heap would be, not so much wrong, as only about as right as not. And for some piles, it might even be right to say that it was a heap (it is as much a heap as not), or that it was not a heap (it is not so much a heap as a pile).

Like the liar paradox, this heap paradox has been exposing natural limits to the ordinary rules of logical thinking since those rules were first formulated. One of the ordinary rules says that for each way that things could be—such as being a pile, or being a heap—any given thing is either that way or else it is not that way.

With Ludwig's heap of sand, that rule gradually stopped applying to "being a heap," a year into his performance, and then started to apply again; and then that rule stopped applying with "being a pile," before starting to apply again.

Now, when descriptions are not good enough for that rule to apply, we would usually find better descriptions, to which it does apply. It is a very natural and useful rule (arguments by *reductio ad absurdum*, for instance, assume that that rule applies).

However, with these paradoxes it is the descriptions *in the puzzle* that are not good enough. In order to solve the puzzle—rather than a different puzzle (such as a mathematical model of the puzzle)—we need to think clearly about those poor descriptions, not about better ones.

The meanings of almost all of our words are to some extent vague—even words like *apple* (as follows). That is because our words are defined only as well as our uses of them have made them. And even where the meaning of a word is perfectly clear, similar paradoxes might arise with the use of that word. As you saw in chapters 1 and 2, the use of the word *true* can be liar-paradoxical and Curry-paradoxical. That is because it is descriptions that are true, or not; and descriptions can have all sorts of words in them.

Regarding apples, suppose that a futuristic machine that can pluck molecules one by one from ordinary objects is set to work on an apple. Taking one molecule away from any apple cannot stop it being an apple, so after the first molecule has been taken away, an apple remains. The second molecule is therefore taken away from an apple—and so on. It could seem as though this machine would always be taking molecules away from an apple, but of course, it will eventually be taking them away from something much less than an apple.

Professor Humdinger takes that scenario to mean that ordinary objects like apples are not logical objects. Ludwig, who can juggle with three apples, is sure that apples are things. He takes that scenario to mean that in between being an apple and not being an apple, this piece of fruit will have been about as much an apple as not.

There would have been no fact of the matter of whether it was an apple or not. And when it was in between being an apple, and being about as much an apple as not, there would have been no fact of the matter of whether it was an apple, or only about as much an apple as not. If he had problems regarding it as an apple, he would think of it as being only about as much an apple as not; and if he had problems with that too, then he would think of it as being a bit more apple than not. And if he needed to describe it in even more detail, then he would take himself to have that need to improve his terminology (perhaps he would talk of myriads of molecules).

For just one more scenario, picture a man staggering through a desert and seeing a mirage that he takes to be a pool. He staggers towards it, muttering "that pool looks cool," over and over to himself. Coincidentally, there really is a pool, just where he thinks it is, but it is obscured from his view by that mirage. As he staggers towards it, the mirage is gradually replaced by the image of the pool. So, the thing that he is muttering to himself about gradually changes from the illusory pool to the real pool. And at some point, he refers to the pool as much as not.

When the water in it had evaporated sufficiently, that body of water would actually be about as much a pool as not. And that water might then be about as blue as not.

> The line that separates the blue colours from the other colours is not like a mathematical line, it is more like the line that a pencil might draw on a picture of a rainbow—a thin line, with fuzzy edges. There is no fact of the matter of whether the colours on that borderline are blue or not. And for some colours, there is no fact of the matter of whether they are on that borderline or not. The fact of this matter is that colours on that borderline are about as blue as not (for such reasons as the one in the third section of chapter 1).

5.ii Russell's Paradox

Descriptions like "that pool is blue" have two basic parts. There are words like "that pool," which tell us what is being described. And there are words like "is blue," which describe it. The latter are called *predicates*.

Descriptions tend not to be descriptions of themselves. But “this sentence contains five words” does describe itself. And predicates can also describe themselves. “Has five words in it,” for example, has five words in it.

Paradoxically, if the predicate “does not describe itself” *does not* describe itself, then it would have described itself pretty well. So it would be correct to say that it does describe itself, as well as correct to say that it *does not*.

> So, if “does not describe itself” *does not* describe itself, then it does and it does not describe itself.

Does it follow from that contradiction that “does not describe itself” *does* describe itself? Was that a *reductio ad absurdum* of it *not* describing itself?

But if “does not describe itself” *does* describe itself, then “does not describe itself” is *correctly* described by “does not describe itself,” so it would be *correct* to say that “does not describe itself” does not describe itself.

> So, if “does not describe itself” *does* describe itself, then it does and it does not describe itself.

And to conclude from that contradiction that “does not describe itself” *does not* describe itself would just take us back to the beginning of this paradox.

That paradox is called *the Grelling-Nelson*, after Kurt Grelling (1886–1942) and Leonard Nelson (1882–1927), two German mathematicians who wrote an academic paper on it in 1908.

The Grelling-Nelson paradox is the predicate version of a paradox that Russell discovered in 1901, as he was thinking about Cantor’s paradox. The class version of that paradox is called Russell’s paradox (described below). For a simple example of the use of classes in place of predicates, consider the deduction with which the second section of chapter 4 began:

> All humans are animals, and you are a human, so you are an animal.

Instead of using the predicate “is an animal,” to say that someone is an animal, you could say that that person *is in the class of animals*.

Now, because every human is an animal, the class of humans is a subclass of the class of animals (the subclasses of a given class are those classes whose members are all in that given class).

So, the logic of the deduction is that because you are in that subclass (because you are one of the humans), you are in the original class (you are one of the animals).

That deduction can be pictured as a circle, representing the class of humans, inside a bigger circle, representing the class of animals, with “you” written inside the smaller circle, and hence also inside the bigger circle (such pictures are called Venn diagrams).

Is there a class of animals, though? Is there a subclass of humans? There is no collection of all and only the humans that there have ever been, because there was no first human: Humans evolved gradually. There is no perfectly precise definition of what a human is. And there is no collection of all and only the humans that are alive now, similarly (people grow slowly from single cells, and they do not instantaneously stop being alive). And most of our words are like that.

Instead of the predicate “is blue,” for example, we might consider the class of blue colours, or the class of logically possible blue things—except that because some possible things are about as blue as not (as shown in the third section of chapter 1), there is no collection of all and only the logically possible blue things. If there is a class of logically possible blue things, then those possible things that are about as blue as not would have to be in that class about as much as not.

Russell was following in the footsteps of such British mathematicians as George Boole (1815–1864) and John Venn (1834–1923) and taking a mathematical approach to logic. And axiomatic entities have whatever properties their axioms say they have. So axiomatic classes can have, as much as they do not have, certain members. However, Russell was primarily interested in mathematics. And assertions whose terms are precisely defined ought to obey the ordinary logical rule that for every assertion, either it or its contrary is true.

Russell’s primary problem, as he tried to axiomatize classes, was therefore Cantor’s reason why the collection of all of his other mathematical collections was an inconsistent collection—Cantor’s paradox—which Russell had become aware of in 1900.

One difference between Russell’s classes and Cantor’s collections was that classes are naturally able to contain themselves as members. The class of classes, for example, is itself a class (much as “is a predicate” is a predicate), so the class of classes is a member of itself.

When Cantor used collections, it was like he was putting things that he had already got into an imaginary bag, and then putting those bags into bigger bags, and so on. And it makes little sense to think of anything like a bag as being one of the things that it contains.

Cantor was naturally thinking about the collection of all of his *other* collections. But as Russell thought about Cantor’s paradox, he was naturally thinking about the class of those classes that were not members of themselves. Which was a paradoxical class for a different reason:

> The class of those classes that are not members of themselves would be a member of itself if it was a class that was not a member of itself.
>
> Conversely, if it was a member of itself, then that would be because it was a class that was not a member of itself.

That is Russell's paradox. It did not arise for Russell's axiomatic classes (much as Cantor's paradox does not arise within set theory). And it does not arise if chapter 3 is correct: If there are always more and more collections, then there are always more and more collections that are not members of themselves; there is never any Russell-paradoxical collection.

The predicate version of Russell's paradox is not very puzzling either, because predicates can describe themselves as much as not (if the claim "this is blue" was printed out in a colour that was about as blue as not, for example, then those three words would describe themselves about as much as not).

Because predicates can sometimes describe themselves as much as not, I should have said that insofar as "does not describe itself" describes itself, it does not describe itself, and that insofar as it does not describe itself, it does. It follows that "does not describe itself" describes itself as much as not.

5.iii Vaguely True

Aristotle (the ancient Greek philosopher who tutored Alexander the Great) defined the truth predicate "is true" like this:

(T) To say of what is the case, that it is the case—or to say of what is not the case, that it is not the case—that is to say something that is true.

Modern philosophers like *show* what the meaning of the truth predicate is, with lists like this:

> "Roses are red" is true *if*, *and only if*, roses are red;
> "snow is white" is true *if*, *and only if*, snow is white;
> "the sky is blue" is true *if*, *and only if*, the sky is blue; and so forth.

Now, truths are what our words express when our words are describing the world accurately. So for example, insofar as snow is white, "snow is white" is true. That list might therefore be more accurately written like this:

“Roses are red” is true *insofar as* roses are red;

“snow is white” is true *insofar as* snow is white;

“the sky is blue” is true *insofar as* the sky is blue; and so forth.

And insofar as snow is blue, “snow is white” is not true. Aristotle defined untruth like this:

(U) To say of what is the case, that it is not the case—or to say of what is not the case, that it is the case—that is to say something that is not true, something untrue.

Furthermore, snow might be only vaguely bluish. “Snow is white” might be about as true as not.

> Ludwig’s pile of sand was at times a heap only as much as it was not a heap. At such times, “yes” answered the question “is Ludwig’s pile a heap?” correctly only as much as it answered it incorrectly. At such times, “Ludwig’s pile is a heap” was about as true as not.

(T) and (U) are therefore joined by

(V) To say of what is the case, or of what is not the case, that it is about as much the case as not, or to say of what is about as much the case as not, that it is the case or that it is not—that is to say something that is about as true as not, something vaguely true.

Will anything else need to be added? Well, a description that is not much truer than not will be about as true as not. And a description that is much truer than not will be true enough to count as true. So (V) does appear to be an adequate adjunction to (T) and (U).

In the rest of this final section, some increasingly paradoxical puzzles involving the truth predicate will be solved.

To begin with a puzzle that is not paradoxical, is the following description true?

> This description, of these eight words, is true.

With those eight words I was saying that what I was saying was what I was saying it was. So, what I was saying had very little, if any meaning. And if it was meaningless, then it was not true.

On the other hand, the question “what am I now, with these ten words, talking about?” can be answered quite well with “nothing but that question, which is about nothing else.” Since that question has a fairly good answer, it may seem to have been a sensible question. But it was clearly just as vacuous as those eight words. So perhaps those eight words were not nonsensical.

And we do have a natural tendency to assume, in the absence of any evidence to the contrary, that people are speaking the truth. So, some people might take those eight words to be true. And since some people may well take them to be untrue, it makes sense to think of them as being about as true as not. Although it does not really matter, because no contradiction needed to be explained.

A puzzle that does give us a paradoxical contradiction, but which is easier to solve than the liar paradox, is this question:

> Is the answer to this question "no"?

Normally, when questions of the form "is such and such the case?" are asked, we assume that the answer is "yes" or "no." But if the answer to "is the answer to this question 'no'?" was "no, the answer is not 'no'," then that would be contradictory, because the answer would be, and would not be, "no." And if the answer was "yes, the answer is 'no'," then both "yes" and "no" would be answers, which is also contradictory, because "yes and no" means that the answer is, and is not, "no."

Still, that is only a little paradoxical. When we get "yes and no" as the reply to a question of the form "is such and such the case?" we do not normally assume that a contradiction was meant. Why would anyone assert such an obvious untruth? Whereas, something like "yes, in one sense, but in another sense no" might well be true. So we tend to assume that something like that was meant. And in answer to the question "is the answer to this question 'no'?" I might say "yes and no," meaning "as much 'yes' as 'no'," or "as much as not."

> Insofar as the answer is "yes," it is "yes, the answer is 'no'," so it is "no."
>
> And insofar as the answer is "no," it is "no, the answer is not 'no'," so it is not "no."

So, insofar as it is "yes" it is "no," and insofar as it is "no" it is not "no."

It follows that good answers are "as much 'yes' as 'no'," and "as much 'no' as not."

The paradoxical reasoning about that question was a bit like the liar paradox. If Ludwig had been telling the truth, with his "this description, of these nine words, is not true," then what he said would have been the case: His description would not have been true. Conversely, if his description had not been true, then what he said would have been true.

It would be better to say that insofar as what Ludwig said was true, it was not true, and that insofar as it was not true it was true. And it follows logically that what Ludwig said was as true as not.

About as true as not is a more natural kind of thing for words to be, though. That is because the line between true and not true is more like the line that a pencil might draw than a mathematical line. So in conclusion, Ludwig's description was about as true as not.

The more puzzling problem with Ludwig's "this description, of these nine words, is not true" is that with those words, Ludwig was saying that his description was true, as well as that it was untrue.

Now, a contradiction—such as "it is blue and it is not blue"—is a pair of contraries. And a claim and its contrary cannot both be true. If it is the case that something is not blue, for example, then it is not the case that it is blue. So, contradictions describe impossibilities—as a rule.

A pool whose water is about as blue as not is a possibility, though. And "it is blue" and "it is not blue" would both be about as true as not of such a pool. Could "it is blue and it is not blue" be about as true as not too?

> That contradiction is certainly not a very good description of that possible pool; it is not "it is about as blue as not."
>
> But nor is it a very bad description; it is essentially the same as "yes and no," in answer to the question "is that pool blue?" And that would be one way of saying that the pool is about as blue as not.

Contradictions are very good descriptions of impossibilities—when the contraries conflict. But these contraries complement each other: The description "it is blue" overstates the pool's blueness, while "it is not blue" understates it.

It may well be that some contradictions are not so much false as only about as true as not.

> And Ludwig's explicit description of what he said as not true, and his implicit description of it as true, were both about as true as not. And those contraries also complement rather than conflict with each other.

Ludwig used the same nine words to express them both because they are the same self-description. It is just that it is such a poor description, of the same poor description, it is its own contrary. Now, that is not the kind of thing that we would think might exist, had we not been thinking so logically about just such a self-description. But that does seem to be how Ludwig's self-description was.

> And why would it be logically impossible for something to be its own contrary, if some contradictions are not so much false, as only about as true as not?

Now, it would be tedious for you to read through a complete list of all the known versions, with each being shown to be explicable in this way (and if I left a few out, you might wonder if they really could all be solved this way). But the following version is the sort that is generally considered to be the hardest to explain.

Consider the following claim:

> This claim is not at all true.

That claim says of itself—explicitly—that it is not even as true as not. It is also—implicitly—saying of itself that it is not at all true that it is not at all true. In other words, that it is at least as true as not.

This version of the liar paradox is that if that claim is about as true as not, then:

> It seems to be true, because it says of itself that it is at least as true as not.
>
> But it also seems not to be true, because it says of itself that it is not even as true as not.

Nevertheless, the logical solution is indeed that that claim is about as true as not.

> The key word is *about*, because even if that claim is about as true as not, it might be a little less true than untrue.
>
> You have seen how easily descriptions that are about as true as not can seem to be both true and not true. The reason why they can is that it is not false that they are true. It is only as true as not that they are. And similarly, it is not false that they are not true.
>
> The added detail in the more precise description of that claim—as about as true as not and, moreover, a little less true than untrue—positions that claim slightly further away from true. And that means that saying that *that claim is at least as true as not* is less true, and that saying that *it is not even as true as not* is less untrue.

But the most paradoxical logical puzzle involving the truth predicate is the logical version of Curry's formal-logical puzzle. It was in the second section of chapter 4; here it is again:

Consider the following claim:

> If this claim is true, there is a round square.

From the meaning of the truth predicate, it follows that:

> If that claim is true, then what that claim claims is the case.

From the wording of the claim, it follows that:

> If that claim is true, then if it is true, there is a round square.

And of course, if that claim is true then it is true, so:

> If that claim is true, there is a round square.

Which is just what the claim claimed. So from the meaning of the truth predicate:

> The claim is true.

And from those last two results—the claim is true, and if it is true then there is a round square—it follows that there is a round square. Since that is not true, there must be an illogical step somewhere in that "proof."

Now, the claim "if this claim is true, there is a round square" is, in effect, saying of itself that it is not true. So this claim is about as true as not. And so the meaning of the words "if the claim is true" would be expressed more accurately by the words "insofar as the claim is true."

That means that "if the claim is true, then what it says is the case" has this meaning:

> Insofar as the claim is true, it is the case that if it is true, there is a round square.

And the meaning of that is:

> Insofar as the claim is true, there is a round square insofar as the claim is true.

And that repetition cannot simply be eliminated. To see why not, it may help to model the phrase "about as much as not" mathematically, as 50%:

> If the claim in question was 50% true, then the meaning of "insofar as the claim is true, there is a round square insofar as the claim is true" would be that it was 50% true that it was 50% true that there is a round square.
>
> That is clearly not the same as it being 50% true that there is a round square—which is what "insofar as the claim is true, there is a round square" would mean, if the claim was 50% true—because fifty percent of fifty percent is twenty-five percent, not fifty percent.

Mathematical models of logic do have their uses.

Although like any analogy, they should not be taken too literally. Percentages are too precise to accurately model something as inherently vague as "about as much as not."

And scientific philosophers should never have thought that formal logics could be more reliable than actual logic. In order to apply a formal logic scientifically, they would have to be thinking logically. Otherwise, they would be applying it illogically. And it would of course not be very scientific to apply mathematics illogically.

As for the question of how highly evolved primates like ourselves could possibly be capable of thinking in perfectly reliable ways, one possible answer is that our creator wants us to be able to think for ourselves, and that to think properly is to think logically. And that answer is not that unlikely, in light of chapter 3.

Of course, if materialism is assumed, then our ability to think logically about such abstract matters does seem unlikely. But if we assume materialism, then our ability to have feelings also seems unlikely. How could molecular structures—even highly evolved molecular structures—possibly have feelings? And yet, we clearly do have feelings.

Just to be scientific, scientific philosophers should have been assuming that there was some good answer to the question of how highly evolved primates could possibly be capable of thinking in perfectly reliable ways about very abstract matters. To see why, consider the physicists of a hundred years ago, who discovered that space was non-Euclidean:

> Had those physicists taken Einstein's theories to be beyond the logical reach of even such highly evolved primates as themselves, they could have kept the very successful and relatively simple physics with which they were familiar.
>
> But of course, that would not have been scientific. It would not have been the most logical assessment of all the evidence.

This book has been defending logic, and such logical pursuits of truth as history, and science. Science, history and logic might seem a bit remote from the world around you, but what could be more realistic than the pursuit of the truth?

A God could also seem to be a remote and unrealistic thing. So note that if there really is a God, then that is where the foundations of everything in the real world would be. As the British poet William Blake (1757–1827) put it, at the start of his *Auguries of Innocence*,

> To see a World in a Grain of Sand
> And a Heaven in a Wild Flower
> Hold Infinity in the palm of your hand
> And Eternity in an hour