AP PHYSICS (2011 – 2012)
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Notes for Weeks 9 & 10 Covering Ch. 9 & 10

Week 9

Chapter 9: Rotational Dynamics

After having considered Rotational Kinematics in Chapter 8, students advanced to Rotational Dynamics in Chapter 9, where forces and torques are involved.

Torques (Ch. 9.1, 9.2)

The basic relationship between torque and force is …

Torque: $\boldsymbol{\tau} = \mathbf{F} (\sin \theta)\, \mathbf{d}$,

where d is the distance from the axis along a solid body at which the force is exerted. The $\sin \theta$ is there to ensure that only the perpendicular component of the force is considered in the calculation. In vector calculus, this would look like …

$\boldsymbol{\tau} = \mathbf{F} \times \mathbf{d}$, where the **x** denotes the "cross-product" between the two vectors.

The students went over several illustrative examples, including

Example 3 (p. 252) on ***A Diving Board***, where the 3.9 m board is fixed by a bolt at one end and rests atop a fulcrum 1.4 m from the bolt. Because the board is in static equilibrium, the sum of the forces is zero, and the sum of the torques is zero. These two relations allow one to determine the two forces exerted at the bolt and the fulcrum.

Example 4 (p. 253) concerning a ***Fireman on a Ladder*** leaning against a wall which is again a static equilibrium problem, but in two dimensions. That means separating the various forces into x and y components and setting their respective totals to zero. The y component is easily solved, but the x component needs more help which comes from the torque equation. The torques are also set to zero, where one must pay attention to the appropriate lever-arm lengths for the weights of ladder (half-way up ladder), the fireman (6.3 m up ladder), and the opposing normal force exerted by the wall (completely up ladder). Here, the axis is placed at the bottom of the ladder.

Example 5 (p. 254) on ***Bodybuilding*** where the bodybuilder holds a dumbbell weight by virtue of his deltoid muscle exerting a force on his arm bone 0.15 m from the shoulder joint at an angle of 13 degrees. The torque is then the force times $\sin 13^{\circ}$. This is enough to oppose the torques exerted by the arm's weight (half-way along arm) and the dumbbell weight (at end of arm).

Centers of Gravity (Ch. 9.3)

A massive body's center of gravity is introduced, invoking the concept that the total torque on an extended body (with respect to some axis) is that equal to its total weight exerted at a distance from its center of gravity …

$\sum_i \tau = \sum_i wgt_i\, x_i = wgt_1\, x_1 + wgt_2\, x_2 + wgt_3\, x_3$ … ad nauseum.

So that $\sum_i \tau = (\sum_i wgt_i)\, x_{cg}$,

And so $x_{cg} = \sum_i wgt_i\, x_i / \sum_i wgt_i$

Example 6 (p. 257) on ***The Center of Gravity of an Arm*** illustrates this, where the arm is broken-up into 3 sections – upper arm, lower arm, and hand – each with its respective weight and mean distance from the shoulder joint. The center of gravity ends up being very close to the elbow.

Example 7 (p. 257) on ***Overloading a Cargo Plane*** is pretty amazing, where the photos show an entire jet airplane on the tarmac tilted into the air, because it had too much weight aft of the axis (the rear wheels). The excess weight shifted the center of gravity aft of the wheels, causing the plane to tilt upward.

The students then considered Conceptual Questions at the end of the chapter.

Conceptual Question #5 (p. 277) on ***Regular vs. Easy-off Toothpaste Caps***, where the large effective lever arm of the easy-off cap enables less force exerted for the same torque.

Conceptual Question #24 (p. 278) showing an incredible photo of a ***Workman and dolly full of boxes***. Students are asked to determine which box on the dolly creates the greatest torque with respect to the axis. This is a two-dimensional situation which was confusing to the students at first. Recalling that the dominant force is one of gravity, the greatest torque is created at the point where that component of the gravitational force ***perpendicular*** to the line connecting the box with the axis times the distance of that line is greatest, ie. $(F_g \sin \theta)\, d = \max$, where θ is the angle between the gravitational force vector and the line to the axis.

Students in class finished off centers of gravity by considering

Example 8 (p. 258) on ***Rollover of an SUV taking a turn towards the right.*** The frictional force between the road and tires provides the centripetal force keeping the SUV in the turn. Because, $F_f = F_c = m v^2 / r$, solving for velocity yields $v = (r F_f / m)^{1/2}$. The SUV's center of gravity is at a height above the road (in the middle of the SUV), and so the rightward frictional force on the tires exerts a (counterclockwise) torque, where $\tau = F_f h = (m v^2 / r) h$, and the axis is set at the left-hand tires. Once the right-hand tires just barely lose contact, the forces and torques are all with respect to the axis at the left-hand tires. The sum of the torques is just equal to zero, so that $F_f h = F_N w/2$, where F_N is the normal force of the road on the SUV which exerts an opposing (clockwise) torque at a distance from the axis corresponding to half the width of the car. All this distills to

$m v^2 h / r = m g w / 2$, or $v^2 = r g w / 2 h$, so that

$v = (r g w / 2 h)^{1/2}$.

In retrospect, this was a pretty ugly example, with several uncertainties about where to draw the forces and corresponding torques.

Problems #15, and #23 (p. 280) revisit the torques exerted by arm muscles, when the hand is pulling against a spring (#15) and lifting a ball (#23).

Problem #24 (p. 281) revisits the forces and torques involved in Example #4 (p. 253), this time with a woman leaning against a wall, such that the x and y force components again must be worked separately.

Moments of Inertia and Rotational Dynamics (Ch. 9.4)

Rotational motion of matter about a designated axis was introduced using the rotational analogy to the linear situation of $\sum_i F_i = m a$, which is

$\sum \boldsymbol{\tau} = I \boldsymbol{\alpha}$**,**

Where $\boldsymbol{\alpha}$ is the angular acceleration, and I is the moment of inertia for a particular distribution of matter with respect to a designated axis. In general, $I = \text{const } m r^2$. The constant ranges from nearly 0 to 2, depending on the matter's distribution wrt the axis. Table 9.1 on p. 262 lists some common shapes and their moments of inertia. Moments of inertia and the corresponding rotational dynamics were explored by working through the following examples.

Example 9 (p. 263) on the ***Moments of Inertia vs. Axes.*** The placement of the axis determines the distribution of matter with respect to that axis and hence the moment of inertia.

Example 10 (p. 264) on the ***Torque of an Electric Saw.*** The saw starts from rest and achieves an angular velocity after so many revolutions. Once the angular velocity is

converted from rev/s to rad/s, one can determine the angular acceleration in rad/s^2. This is done by using the kinematic equation: $\omega_f^2 - \omega_i^2 = 2\ \alpha\ \Delta\theta$
And solving as $\alpha = (\omega_f^2 - \omega_i^2) / (2\ \Delta\theta)$, where the angular displacement $\Delta\theta$ is determined from the number of revolutions. The net torque is then: $\tau = I\ \alpha$.

Conceptual Example 11 (p. 265) on ***Archery and Bow Stabilizers… do they really work?*** The long thin rod jutting forward of the bow increases the bow's moment of inertia with respect to the archer's shoulder. According to $\alpha = \sum\tau / I$, the larger the moment of inertia, the smaller the angular acceleration that would disturb the archer's aim.

Rotational Work and Energy (Ch. 9.5)

By analogy to the linear case of work: $W = \mathbf{F}\ \mathbf{\Delta d} \cos\theta$,
The work done in a rotational case is: $W = \tau\ \Delta\theta$.

This work can produce a change in rotational kinetic energy, whereby

$W = \tau\ \Delta\theta = ½\ I\ (\omega_f^2 - \omega_i^2)$.

The total mechanical energy remains conserved, such that

$E_{mech} = KE_{translational} + KE_{rotational} + PE =$ constant,
or
$E_{mech} = ½\ m\ v^2 + ½\ I\ \omega^2 + m\ g\ h =$ constant.

When a body is rolling on a surface without slipping (ie. a tire), this reduces to
$E_{mech} = ½\ m\ v^2 + ½$ const $m\ r^2\ (v/r)^2 + m\ g\ h =$ constant
or
$E_{mech} = m\ (v^2/2 +$ const $v^2/2 + g\ h) =$ constant. For a thin bicycle tire, const = 1, and so
$E_{mech} = m\ (v^2 + g\ h)$, with the kinetic part being twice that of pure translational motion or pure rotational motion.

This rolling without slipping was further explored with

Example 13 (p. 271) on ***Rolling Cylinders,*** where solid and hollow cylinders of the same mass are rolled down an inclined plane. The solid cylinder has $I = ½\ M\ R^2$, while the hollow cylinder has a greater moment of $I = M\ R^2$. Solution of the energy equation (see textbook) yields velocities that are greater for lower moments of inertia... ie. for the solid cylinder.

Angular Momentum (Ch. 9.6)

Besides having kinetic energy, matter in rotational motion also has momentum – known as angular momentum: **L** = I **ω.**
Consider the simplest case of a ball or planet following a circular trajectory. Then
L = I **ω =** m r^2 v / r = m v r. That's worth remembering.

Big Idea: *Even more important is the principal of angular momentum conservation. Just like momentum conservation in the linear case, angular momentum conservation can be used along with energy conservation to solve some pretty hairy problems in rotational dynamics. But let's continue with the more simple stuff.*

Conceptual Example 14 (p. 273) considers a ***Spinning Ice Skater***. The velocities will change according to **L** = I **ω =** constant. That means $I_f\ v_f/r_f = I_i\ v_i/r_i$, or

$v_f = v_i\ (I_i/I_f)\ r_f/r_i$.

When the skater's arms contract, they reduce I, so that (ignoring her body!!)…

$v_f = v_i\ (r_i^2/r_f^2)\ r_f/r_i = v_i\ (r_i/r_f)$. This is obviously an upper limit to how much the velocity can increase.

Example 15 (p. 273) on ***A Satellite in Elliptical Orbit*** is actually easier, because the moment of inertia of the satellite is simply: $I = m\ r^2$.
This can be plugged into the momentum conservation equation so that
$L = m\ v_f\ r_f = m\ v_i\ r_i$,
and so
$v_f = v_i\ (r_i/r_f)$ … this time exactly!
In other words, as a satellite (or planet) gets farther away from the focus, its velocity decreases – and vice versa. Kepler's 2nd law of planetary motion stating that "equal areas are swept out in equal times" is a direct consequence of this momentum-conserving decrease and increase in velocity as a function of distance from the focus. Kepler didn't know this, but we do ;-)

Reflections: *Laboratory activities were noticeably lacking during this week. We need to think about developing some good activities that reveal rotational dynamics as they relate to both energy and momentum conservation.*

<u>Week 10</u>

Chapter 10: Simple Harmonic Motion and Elasticity (Yea!)

Tracing Harmonic Motion (Ch. 10.1, 10.2)

This week began with the simple spring and Hooke's Law which relates the force exerted by the spring to the displacement of the spring:

$F = - k \Delta x$, where the negative sign indicates that the force is always a "restoring" force pulling or pushing the spring back into its equilibrium state. Once an attached mass is displaced from equilibrium, the spring's restoring force will cause the mass to move back toward the equilibrium position, overshoot, and then get forced back again in the opposite direction, overshoot, etc. The resulting "harmonic motion" can be traced on a moving chart recorder via a projection technique shown in the textbook. The upshot is a cosine curve, whereby

$x = A \cos \omega t$, where $\omega = 2 \pi / T = 2 \pi f$

The textbook uses the same projection technique to develop relations for the velocity and acceleration. However, differential calculus does the job much quicker!

$v = dx / dt = -A \omega \sin \omega t$, where $v_{max} = A \omega$ … which occurs at the equilibrium point.

$a = dv / dt = -A \omega^2 \cos \omega t$, where $a_{max} = A \omega^2$ … which occurs at the extremities.

Big Idea: *Note that in harmonic motion, the acceleration is always changing. This is very different from the situation of constant acceleration under the influence of gravity.*

Natural Frequencies of Oscillation

On p. 293, the textbook guides the reader to derive the natural frequency of oscillation. We can begin by equating the forces in Newton's 2nd Law and Hooke's Law:

$F = m a = - k x$, so that

$a = -k x / m$
or
$-A \omega^2 \cos \omega t = - k A \cos \omega t / m$
so that
$\omega^2 = k / m$
yielding the natural frequency of oscillation:
$\omega = (k / m)^{1/2}$.

How cool is that!?
This expression can be interpreted as a battling between the restoring force (represented by k) and the inertia of the system (represented by m). The stiffer the spring, the stronger the restoring force and the faster the oscillation – and vice versa.

Laboratory Activity – at last!
Determining the spring constant k

and testing the inverse-root relationship between natural frequency and the mass.

Students attached a spring to the top of a pole and suspended increasing amounts of mass from it. They measured the total masses, converted them to weights (forces in Newtons), determined the corresponding displacements (in meters), and plotted the results.

The Force vs. Displacement plots were fairly linear (how good were they really?) yielding slopes that equal the spring constant. The absence of a minus sign is because of using the weights for the force rather than the spring's equal but opposite restoring force. On another day, the students timed the oscillation period (T) as a function of attached mass. I had displayed on a computer an online stopwatch, but they ended up using their phones!! They then derived the "natural" angular frequency $\omega = 2\pi / T$.

I asked them to plot the angular frequency vs. $(1/m)^{1/2}$. This led to fairly linear behavior, confirming the theoretical relationship, where the slope could be identified with $k^{1/2}$. The resulting spring constant was acceptably close to that derived using the displacement method. ➔ The theory works!

Example 6 (p. 293) on ***A Body-Mass Measurement Device*** is way cool. On the Space Shuttle – and now the International Space Station – the felt weight is essentially zero, so how can you measure your mass? The trick is using a spring and the oscillation equation derived from Hooke's Law: $\omega = (k / m)^{1/2}$, and thus obtaining the mass m as a function of the observed natural frequency ω and previously calibrated spring constant k, yielding $m = k / \omega^2$. This mass will be the total mass, including that of the chair and of the astronaut. It is then a simple "matter" of subtracting the chair's mass to get the astronaut's mass.

Energy and Harmonic Motion (Ch. 10.3)

Using Hooke's Law for the spring's restoring force, one can derive the corresponding potential energy in terms of work: $PE = W = F \Delta x$, assuming aligned forces and displacements. In integral calculus, this would be expressed as

$PE = W = \int F\, dx = \int k\, x\, dx = \frac{1}{2} k x^2$, where the force here is that exerted by the agent stretching or compressing the spring (hence the positive sign). You can also obtain this result graphically by plotting $(F = k x)$ vs. (x). The sloping straight line has an area under it equal to ½ the height $(k x)$ times the base (x), yielding $W = \frac{1}{2} k x^2$.

We can now consider the very rich energy equation:
$E_{mech} = KE_{trans} + KE_{rot} + PE_{grav} + PE_{spring} = \text{constant}$

And use this to solve problems involving springs and the resulting motions. These are illustrated in Example 7 (p. 297) involving ***A Horizontal Spring*** and Example 9 (p. 298) involving ***A Vertical Spring***, where the gravitational potential energy must be included.

The Pendulum (Ch. 10.4)

The derivation of a pendulum's motion is fairly straightforward, but I always need to check my notes to get things started. A weight at the end of a massless but rigid rod of length (L) that is pivoted at the top, is displaced by an angle (θ). The resulting gravitational force vector has a component that is perpendicular to the rod…

$F_{perp} = m\ g \sin\theta$.

For small angles (less than 10 degrees), θ(radians) ~ $\sin\theta$. The students confirmed this with their calculators. This approximation makes the derivation easier.

If $F_{perp} = m\ g\ \theta$, then the corresponding torque is
$\tau = F_{perp}\ L = -L\ m\ g\ \theta$,
or
$\tau = -k'\ \theta$, where the minus sign indicates a restoring torque,
and where we have let $k' = L\ m\ g$.

This equation for the torque has the same form as $F = -k\ x$, and so has the same kind of solution…

$\omega = (k' / I)^{1/2}$, where the moment of inertia has been substituted for the mass in this semi-rotating system, yielding
$\omega = (L\ m\ g / m\ L^2)^{1/2}$
and
$\omega = (g / L)^{1/2}$

Big Idea: *Lo and behold, the mass once again disappears as a key player in this gravitationally-driven system, leaving only the source of restoring force (g) vs. the remaining contributor to the inertia (L) as determinants of the natural angular frequency.*

The corresponding period (T) giving the "tick" of the grandfather clock is then

$T = 1/f = 2\pi / \omega = 2\pi\ (L / g)^{1/2}$.

Example 10 (p. 300) on ***Keeping Time*** uses this relation to derive the length of a pendulum necessary to create a ticking period of 1 second. The solution is
$L = T^2\ g / 4\ \pi^2 = 0.248$ m.
This length is actually much shorter than most grandfather clock pendulums, suggesting that they "tick-off" greater time intervals.

Elastic Deformation (Ch. 10.7)

Other applications of Hooke's Law concern the stretching, compressing, shearing, and volume deformation of materials. Common to all these phenomena is some sort of restoring force that combats the initial deviation.

Stretching/Compressing: $P = F/A = Y (\Delta L / L_o)$,
where Y is Young's Modulus
(in units of N/m^2, Pascals, or J/m^3).
Table 10.1 on p. 304 shows that most values of Y range from 10^8 N/m^2 (Teflon) to 10^{11} N/m^2 (Tungsten). That means typical deformations ($\Delta L/L$) are extremely small.

This was shown in Example 13 on ***Lifting a Jeep*** with a steel cable suspended from a helicopter. A 16 m length of cable gets stretched by only 0.024 m = 2.4 cm.

Shear Deformation: $P = F/A = S (\Delta X / L_o)$,
where S is the Shear Modulus (in units of N/m^2, Pascals, or J/m^3), ΔX is the horizontal shearing, and L_o is the thickness of the material being sheared. The force is parallel to the area, so it's not really producing a pressure, even though the units are N/m^2.
Table 10.2 on p. 307 gives values ranging from 10^9 N/m^2 (lead) to 10^{11} N/m^2 (tungsten) – once again showing that most solid materials don't shear much.

The big exception is a colloid as in Example 14 on ***JELLO,*** where enough information was provided to derive the Shear Modulus which was a much lower 460 N/m^2, indicative of the greater shearing.

Bulk Distortion: This involves a change in pressure… $\Delta P = -B (\Delta V / V_o)$,
where B is the Bulk Modulus (in units of N/m^2, Pascals, or J/m^3).

Here, one can formulate $\Delta P / P_o = \Delta V / V_o$, and so see that B is just like a pressure… resisting the deformation. ***Big Idea:*** *In many ways, B represents the substance's electrostatic energy density – that which keeps the material bound together and resistant to distortion.*

Table 10.3 on p. 308 gives values ranging from 10^9 N/m^2 (oil and ethanol) to 10^{11} N/m^2 (steel and diamond). Once again, such high values indicate very high electrostatic binding and hence very little relative distortion. It is fun to compare these energy densities with the thermal energy densities of certain fuels (see http://en.wikipedia.org/wiki/File:Energy_density.svg). For example, diesel fuel has a listed energy density of 37 MJ/Liter x 10^6J/MJ x 1 Liter/1000cm^3 x $10^6 cm^3/1m^3$ = 3.7 x 10^{10} J/m^3 – pretty close to its Bulk Modulus! Perhaps this "coincidence" is worth pondering some more(?)

All of these distortions can be summarized in terms of **Stress vs. Strain:**

Stress is the input: F/A or ΔP

Strain is the output: $\Delta L/L_o$, $\Delta X/L_o$, $-\Delta V/V_o$,

such that they follow a generalized Hooke's Law – ***stress is proportional to strain*** – mediated by their respective "spring constants" (Y, S, and B).