Linear Algebra
Lesson 11:
Part 1: Dimension,
Part 2: Finding a Basis (using Pivot columns)
Part 3: Finding an Orthonormal Basis (using Gram Schmidt)

Please be sure to mark down the date and time that you start this lesson. Carefully take notes on pencil and paper while watching the lesson videos. Pause the lesson to try classwork before watching the video going over that classwork. If you work with any classmates, be sure to write their names on the problems you completed together. Please wear masks when meeting with classmates even if you meet off campus.

You will cut and paste the photos of your notes and completed classwork and a selfie taken holding up the first page of your work in a googledoc entitled:

MAT313F21-lesson11-lastname-firstname

and share editing of that document with me sormanic@gmail.com and with our graders. If you have a question, type QUESTION in your googledoc next to the point in your notes that has a question and email me with the subject MAT313 QUESTION. I will answer your question by inserting a photo into your googledoc or making an extra video.

This lesson has three parts and may be done on three days. Please do the homework as classwork as you come up to each problem in the videos. Students who are behind schedule may skip Part 3.

Classwork is the notes for the lesson.

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Part I Dimension (required)

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Before starting, review Parts 3-4 of Lesson 10 and check that your homework was completed correctly.

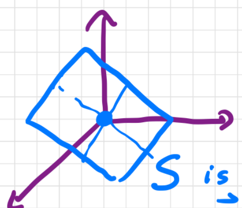
Watch Playlist 313F20-11-1to3 and complete the HW in these photos as classwork while watching.

Lesson 11

Dimension of a Subspace

Finding a basis using 2 methods:

- Pivot Columns
- Gram Schmidt



$\mathbb{R}^n$ Euclidean Space of dimension n

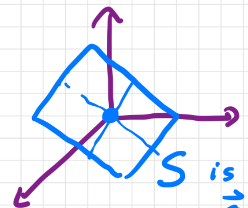
S is a subspace:

$\vec{0} \in S$

closed under + → $\forall \vec{v}, \vec{w} \in S \quad \vec{v} + \vec{w} \in S$

closed under scalar → $\forall R \in \mathbb{R} \ \forall \vec{v} \in S \quad R\vec{v} \in S$

$\vec{v}_1 \dots \vec{v}_k$ are lin. indep if $\sum_{i=1}^{k} a_i \vec{v}_i = 0 \iff$ all $a_i = 0$, $i = 1 \dots k$



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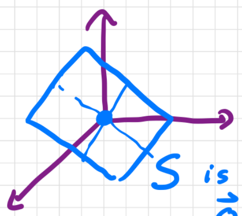
$S = \langle \vec{v}_1, \vec{v}_2 \dots \vec{v}_k \rangle = \left\{ \sum_{i=1}^{k} a_i \vec{v}_i \mid a_i \in \mathbb{R} \right\}$ span

Defn: $\vec{v}_1 \dots \vec{v}_k$ are basis for S if $S = \langle \vec{v}_1 \dots v_k \rangle$ and $\vec{v}_1 \dots \vec{v}_k$ are lin. indep.



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$\vec{v}_1 \dots \vec{v}_k$ are linearly independent if $\sum_{i=1}^{k} a_i \vec{v}_i = 0 \Leftrightarrow$ all $a_i = 0$, $i = 1 \dots k$

$S = \langle \vec{v}_1, \vec{v}_2, \dots \vec{v}_k \rangle = \left\{ \sum_{i=1}^{k} a_i \vec{v}_i \mid a_i \in \mathbb{R} \right\}$ span

Defn: $\vec{v}_1 \dots \vec{v}_k$ are basis for S if $S = \langle \vec{v}_1 \dots v_k \rangle$ and $\vec{v}_1 \dots \vec{v}_k$ are lin. indep.

Defn The dimension of $S \subset \mathbb{R}^n$ is equal to k if it has basis with k vectors.

Linear Algebra

$$\mathbb{R}^n = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \middle| \; x_i \in \mathbb{R} \text{ free} \right\}$$

$$= \left\{ x_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \dots + x_n \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \middle| \; x_i \in \mathbb{R} \right\}$$

n dim has n basis vectors

standard basis of $\mathbb{R}^n$

$$\hat{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \hat{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \dots \hat{e}_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

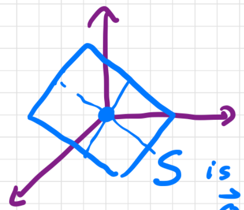
basis has 3 vectors

$\boxed{\text{HW1}}$ Verify that $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ are a basis for $S = \mathbb{R}^3$

3dim

Check: span $\mathbb{R}^3$ (by defn of $\mathbb{R}^3$)

Check: linearly indep.



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$\vec{v}_1 \dots \vec{v}_k$ are linearly independent if $\sum_{i=1}^{k} a_i \vec{v}_i = 0 \Leftrightarrow$ all $a_i = 0$ $\; i=1 \dots k$

$S = \langle \vec{v}_1, \vec{v}_2 \dots \vec{v}_k \rangle = \left\{ \sum_{i=1}^{k} a_i \vec{v}_i \mid a_i \in \mathbb{R} \right\}$ span

Defn: $\vec{v}_1 \dots \vec{v}_k$ are basis for S if $S = \langle \vec{v}_1 \dots v_k \rangle$ and $\vec{v}_1 \dots \vec{v}_k$ are lin. indep.

Defn The dimension of $S \subset \mathbb{R}^n$ is equal to k if it has basis with k vectors.

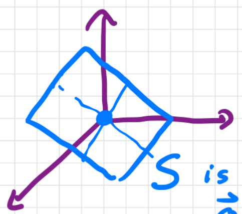
Linear Algebra

Is dimension well defined?

Could S have 2 different collections of basis vectors $\vec{v}_1, \vec{v}_2 \dots \vec{v}_k$ and $\vec{w}_1, \vec{w}_2 \dots \vec{w}_m$ where $k \neq m$?!!

If so, then which is the dimension ?!

In fact we can prove $k = m$ for any pair of bases.



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$\vec{v}_1 \dots \vec{v}_k$ are linearly independent if $\sum_{i=1}^{k} a_i \vec{v}_i = 0 \Leftrightarrow$ all $a_i = 0$, $i = 1 \dots k$

$S = \langle \vec{v}_1, \vec{v}_2 \dots \vec{v}_k \rangle = \{ \sum_{i=1}^{k} a_i \vec{v}_i \mid a_i \in \mathbb{R} \}$ span

Defn: $\vec{v}_1 \dots \vec{v}_k$ are basis for S if $S = \langle \vec{v}_1 \dots v_k \rangle$ and $\vec{v}_1 \dots \vec{v}_k$ are lin. indep.

Defn The dimension of $S \subset \mathbb{R}^n$ is equal to k if it has basis with k vectors.

Thm: If $\vec{v}_1, \vec{v}_2 \dots \vec{v}_k$ is a basis for S and $\vec{w}_1, \vec{w}_2 \dots \vec{w}_m$ is a basis for S then $k = m$.

Thus dim is defined.

Proof by Contradiction:

① Assume on the contrary that $k \neq m$ ① Indirect Hypothesis

Examine the consequences (plan)

Reach a Contradiction ⊗

Thus $k = m$ QED

Thm: If $\vec{v}_1, \vec{v}_2 \dots \vec{v}_k$ is a basis for S and $\vec{w}_1, \vec{w}_2 \dots \vec{w}_m$ is a basis for S then $k = m$.

Proof ① Assume on the contrary that $k \neq m$ (Indirect Hypothesis)

② Without loss of generality $k > m$.

③ $\langle \vec{v}_1, \vec{v}_2 \dots \vec{v}_m \dots \vec{v}_k \rangle = S = \langle \vec{w}_1, \dots w_m \rangle$ by defn of basis. (span)

④ Since each $v_i \in S = \langle \vec{w}_1, \dots \vec{w}_m \rangle$ $\exists\, a_{i1} \dots a_{im} \in \mathbb{R}$ s.t. $\vec{v}_i = \sum_{j=1}^{m} a_{ij} \vec{w}_j$ for $i = 1 \dots m \dots k$

⑤ $\vec{0} = \sum_{i=1}^{k} t_i \vec{v}_i = \sum_{i=1}^{k} t_i \left(\sum_{j=1}^{m} a_{ij} \vec{w}_j \right)$

$\Updownarrow$

⑥ $\vec{0} = \sum_{j=1}^{m} \left(\sum_{i=1}^{k} t_i a_{ij} \right) \vec{w}_j$ by reordering sums

$\Updownarrow$

⑦ For $j = 1 \dots m$ $\left(\sum_{i=1}^{k} t_i a_{ij} \right) = 0$ because $\vec{w}_j$ are lin indep.

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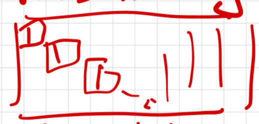
Linear Algebra

⑧
$$\begin{aligned}
t_1 a_{11} + t_2 a_{21} + \dots + t_k a_{k1} &= 0 \\
t_1 a_{12} + t_2 a_{22} + \dots + t_k a_{k2} &= 0 \\
&\dots \\
t_1 a_{1m} + t_2 a_{2m} + \dots + t_k a_{km} &= 0
\end{aligned}$$

Since $\vec{v}_1 \dots \vec{v}_k$ are lin indep this should have only one solution $\boxed{t_1 = \dots = t_k = 0.}$

Since $k > m$ we have more unknowns $t_1 \dots t_k$ than equations $j = 1$ to m.

So this homogeneous system



will have free variables.

So the solution set includes more than just $\vec{0}$ ⊗

So the indirect hyp is false, $k = m$ QED

HW1 Find the dimension of Null(A) for $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

hint: find a basis by writing Null(A) as a span of directions and verifying the directions are linearly independent, then count number of vectors in the basis. (see previous lesson)

HW2 Find the dimension of Null(A) for $A = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & 0 \end{pmatrix}$

same hint

HW3 Find the dimension of $S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : \begin{matrix} 2x + 1y + z = 0 \\ x + y + z = 0 \\ 3x + 2y + 2z = 0 \end{matrix} \right\}$

solve the system and use directions as in HW1 + HW2

HW4 Find the dimension of a plane $S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \, ax + by + cz = 0 \right\}$ assuming $a \neq 0$

Hint for HW1-HW4: see classwork in Part 4 of Lesson 10.

EC $\boxed{\text{HW5}}$ Suppose $\vec{v}_1, \vec{v}_2, \vec{v}_3 \in S$ and $S = \langle \vec{w}_1, \vec{w}_2 \rangle$ show $\vec{v}_1, \vec{v}_2$ and $\vec{v}_3$ are not linearly independent.

(**hint:** imitate the proof in the video above but do not use sum notation)

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Part II **Finding a basis using pivot columns (required)**

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Watch [Video 313F20-11-4]()

Part II

Given

$$S = \langle \vec{v}_1, \vec{v}_2, \dots, \vec{v}_m \rangle$$

Find a basis for S and find the $\dim(S)$.

Goal To select out of our original list of vectors, which are needed for a basis.

Linear Algebra

Example Find a basis for S

$$S = \left\langle \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \\ 6 \\ 0 \end{pmatrix} \right\rangle$$

(1)

$$t_1 \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix} + t_3 \begin{pmatrix} 3 \\ 6 \\ 6 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Solved the system

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 2 & 2 & 6 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow[\rho_2 \to \rho_2 - 2\rho_1]{} \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow[\rho_2 \leftrightarrow \rho_3]{} \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow[\rho_2 \to \frac{\rho_2}{3}]{} \left(\begin{array}{ccc|c} \boxed{1} & 1 & 3 & 0 \\ 0 & \boxed{1} & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

t_1 and t_2 are leaders
t_3 is free

It seems like the third vector is extra

(2) If we remove the third vector do we still get the same span?

$$S = \left\langle \begin{pmatrix}1\\2\\0\\0\end{pmatrix}, \begin{pmatrix}1\\2\\3\\0\end{pmatrix} \right\rangle \text{ ?} \quad \underline{\text{Yes}}$$

Check $\begin{pmatrix}3\\6\\6\\0\end{pmatrix} \in \left\langle \begin{pmatrix}1\\2\\0\\0\end{pmatrix}, \begin{pmatrix}1\\2\\3\\0\end{pmatrix} \right\rangle$

Solve the system

$$t_1\begin{pmatrix}1\\2\\0\\0\end{pmatrix} + t_2\begin{pmatrix}1\\2\\3\\0\end{pmatrix} = \begin{pmatrix}3\\6\\6\\0\end{pmatrix}$$

$$\left(\begin{array}{cc|c} 1 & 1 & 3\\ 2 & 2 & 6\\ 0 & 3 & 6\\ 0 & 0 & 0 \end{array}\right)$$ same row reduction

$$\rightarrow \left(\begin{array}{cc|c} \boxed{1} & 1 & 3\\ 0 & \boxed{1} & 2\\ 0 & 0 & 0\\ 0 & 0 & 0 \end{array}\right)$$ has a solution!

(3) Do the vectors with leaders form a basis?

We see they do span S. (Their span is S).

Must check: lin indep?

Solve $t_1\begin{pmatrix}1\\2\\0\\0\end{pmatrix} + t_2\begin{pmatrix}1\\2\\3\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$

Solve the system

$$\left(\begin{array}{cc|c} 1 & 1 & 0\\ 2 & 2 & 0\\ 0 & 3 & 0\\ 0 & 0 & 0 \end{array}\right) \xrightarrow[\text{red}]{\text{same row}} \left(\begin{array}{cc|c} \boxed{1} & 1 & 0\\ 0 & \boxed{1} & 0\\ 0 & 0 & 0\\ 0 & 0 & 0 \end{array}\right)$$

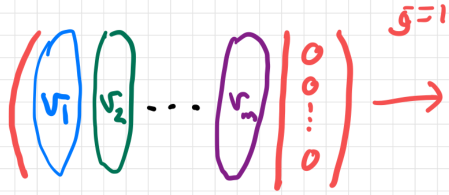
The only solution is

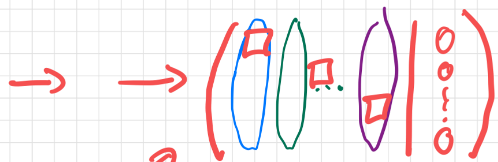
$t_1 = 0$
$t_2 = 0$

So yes lin indep.

Pivot Column Method of Finding a basis for $S = \langle v_1\ v_2 \ldots v_m \rangle$

(1) Solve the system $\sum_{j=1}^{m} t_j \vec{v}_j = \vec{0}$





Echelon Form

"Pivot columns" have leaders

(2) Remove vectors whose columns do not have leaders

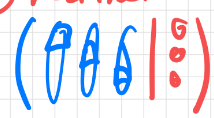
The remaining vectors span S because if we solve the system (remove a vector without a leader)

same reduction

will have a solution

all columns are vectors with leaders

(3) Furthermore These are linearly indep



← has only one solution.

Example

$$S=\left\langle \begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}1\\3\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}0\\3\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right\rangle$$

Find a basis for S.

① Solve the system

$$\left(\begin{array}{ccccc|c}1&1&0&0&1&0\\2&3&1&3&1&0\end{array}\right) \xrightarrow[\rho_2\to\rho_2-2\rho_1]{} \left(\begin{array}{ccccc|c}\boxed{1}&1&0&0&1&0\\0&\boxed{1}&1&3&-1&0\end{array}\right)$$

first two columns are pivot columns

Fast Answer

The basis for S is the first two: $\begin{pmatrix}1\\2\end{pmatrix}$ & $\begin{pmatrix}1\\3\end{pmatrix}$

We can say $S=\left\langle \begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}1\\3\end{pmatrix}\right\rangle$ and $\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}1\\3\end{pmatrix}$ are a basis.

② Check the other vectors are already in $\left\langle \begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}1\\3\end{pmatrix}\right\rangle$

$$\left(\begin{array}{cc|c}1&1&0\\2&3&1\end{array}\right) \Rightarrow \left(\begin{array}{cc|c}1&1&0\\0&1&1\end{array}\right)$$

This has a solution

$$\left(\begin{array}{cc|c}1&1&0\\2&3&3\end{array}\right) \to \left(\begin{array}{cc|c}1&1&0\\0&1&3\end{array}\right)$$

$$\left(\begin{array}{cc|c}1&1&1\\2&3&1\end{array}\right) \to \left(\begin{array}{cc|c}1&1&1\\0&1&-1\end{array}\right)$$

③ Check $\begin{pmatrix}1\\2\end{pmatrix}\begin{pmatrix}1\\3\end{pmatrix}$ are lin indep

$$\left(\begin{array}{cc|c}1&1&0\\2&3&0\end{array}\right) \to \left(\begin{array}{cc|c}\boxed{1}&1&0\\0&\boxed{1}&0\end{array}\right)$$

So the only solution is 0

To see this next example explained see [Video 313F20-11-5]()

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Linear Algebra

Example: Use the pivot column method to find a basis for $S = \left\langle \begin{pmatrix}1\\2\\0\end{pmatrix}, \begin{pmatrix}2\\4\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix} \right\rangle$

Solution:

$$\left(\begin{array}{ccc|c}1&2&1&0\\2&4&0&0\\0&0&1&0\end{array}\right) \xrightarrow{\rho_2 \to \rho_2 - 2\rho_1} \left(\begin{array}{ccc|c}1&2&1&0\\0&0&-2&0\\0&0&1&0\end{array}\right) \xrightarrow{\rho_2 \to \frac{\rho_2}{-2}} \left(\begin{array}{ccc|c}1&2&1&0\\0&0&1&0\\0&0&1&0\end{array}\right)$$

$$\xrightarrow{\rho_3 - \rho_2} \left(\begin{array}{ccc|c}\boxed{1}&2&1&0\\0&0&\boxed{1}&0\\0&0&0&0\end{array}\right)$$

box the leaders

columns 1 and 3 are pivot columns

So $\left\langle \begin{pmatrix}1\\2\\0\end{pmatrix}, \begin{pmatrix}2\\4\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix} \right\rangle = \left\langle \begin{pmatrix}1\\2\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix} \right\rangle$

This is the basis for S

Quick Answer

Linear Algebra

Longer Answer.

② Check the extra vector $\begin{pmatrix}2\\4\\0\end{pmatrix}$ is in the span

$$\left(\begin{array}{cc|c}1&1&2\\2&0&4\\0&1&0\end{array}\right) \xrightarrow[\text{actions}]{\text{same row}} \left(\begin{array}{cc|c}\boxed{1}&1&2\\0&\boxed{1}&0\\0&0&0\end{array}\right)$$

has a solution

③ Check the pivot column vectors are linearly indep.

$$\left(\begin{array}{cc|c}1&1&0\\2&0&0\\0&1&0\end{array}\right) \xrightarrow[\text{actions}]{\text{same row}} \left(\begin{array}{cc|c}\boxed{1}&1&0\\0&\boxed{1}&0\\0&0&0\end{array}\right)$$

only soln is $t_1 = 0$ $t_2 = 0$

So they are linearly indep

Yes they are a basis.

HW6 $S=\left\langle \begin{pmatrix}1\\0\\2\end{pmatrix}, \begin{pmatrix}2\\0\\4\end{pmatrix}, \begin{pmatrix}1\\0\\3\end{pmatrix}, \begin{pmatrix}3\\0\\5\end{pmatrix} \right\rangle$

Find a basis for S using the pivot column method. Then check the extra vectors are in the span of the basis and check the basis vectors are linearly independent. What is $\dim(S)$? Hint: the dimension is the number of vectors in the basis.

HW7 $S=\left\langle \begin{pmatrix}1\\2\\3\\4\end{pmatrix}, \begin{pmatrix}1\\2\\3\\0\end{pmatrix}, \begin{pmatrix}3\\4\\6\\4\end{pmatrix} \right\rangle$

Find a basis for S using the pivot column method. What is $\dim(S)$?

HW8 Suppose

$\vec{v}_0 \in \langle \vec{v}_1, \vec{v}_2, \dots \vec{v}_k \rangle = S$

Prove

$S = \langle \vec{v}_0, \vec{v}_1, \dots \vec{v}_k \rangle$

Hint: Let $\vec{w} \in \langle \vec{v}_0, \vec{v}_1, \dots \vec{v}_k \rangle$

So there are $t_0, \dots, t_k \in \mathbb{R}$ s.t. $\vec{w} = \sum_{i=0}^{k} t_i \vec{v}_i$

Find $s_1, \dots, s_k \in \mathbb{R}$ s.t. $\vec{w} = \sum_{i=1}^{k} s_i \vec{v}_i$.

Then do the converse: Start knowing you have s_i and find t_i. (Extra Tricky)

Hint for HW6 and HW7: Remember the dimension is the number of vectors in the basis. So once you find the basis, just count how many vectors there are in that basis.

Do HW6-HW7 and read HW8 before starting Part 3. HW8 is extra credit.

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Part 3 Orthonormal Basis and Gram Schmidt Process (extra credit)

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Students who are behind schedule may skip this part and its homework. HW9-14 is an extra credit assignment.

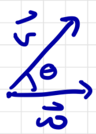
Watch [Video 313F20-11-6] on Orthonormal Basis pausing and doing HW as you progress through the video.

Part III Orthonormal Basis and Gram-Schmidt

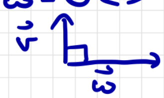
Review: $|\vec{v}| = \sqrt{v_1^2 + v_2^2 + \dots + v_m^2}$

$$|\vec{v}|^2 = \vec{v} \cdot \vec{v}$$

$\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}|\cos\theta$ proven in trig.



So $\vec{v} \cdot \vec{w} = 0 \Leftrightarrow \vec{v} \perp \vec{w}$



Defn $\vec{v}_1 \dots \vec{v}_k$ are an orthonormal basis for S

if $\vec{v}_i \cdot \vec{v}_i = 1$ (so length is 1)

$\vec{v}_i \cdot \vec{v}_j = 0$ (so perpendicular) when $i \neq j$

Linear Algebra

The standard basis is orthonormal

HW 9a

Check $\begin{pmatrix}1\\0\\0\end{pmatrix}$ $\begin{pmatrix}0\\1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\0\\1\end{pmatrix}$

HW 9b

Check $\begin{pmatrix}0\\1\\0\end{pmatrix}$ $\begin{pmatrix}-1\\0\\0\end{pmatrix}$ and $\begin{pmatrix}0\\0\\1\end{pmatrix}$

HW 9c

Check $\begin{pmatrix}\sqrt{2}/2\\\sqrt{2}/2\\0\end{pmatrix}$ $\begin{pmatrix}\sqrt{2}/2\\-\sqrt{2}/2\\0\end{pmatrix}$ $\begin{pmatrix}0\\0\\1\end{pmatrix}$

are orthonormal

HW 9d

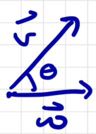
Check $\begin{pmatrix}1\\2\\3\end{pmatrix}$ $\begin{pmatrix}0\\1\\0\end{pmatrix}$ $\begin{pmatrix}0\\0\\1\end{pmatrix}$

Part III Orthonormal Basis and Gram-Schmidt

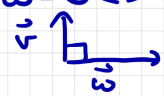
Review: $|\vec{v}| = \sqrt{v_1^2 + v_2^2 + \dots + v_m^2}$

$|\vec{v}|^2 = \vec{v} \cdot \vec{v}$

$\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}|\cos\theta$ proven in trig.



So $\vec{v} \cdot \vec{w} = 0 \Leftrightarrow \vec{v} \perp \vec{w}$



Defn $\vec{v}_1 \dots \vec{v}_k$ are an orthonormal basis for S

if $\vec{v}_i \cdot \vec{v}_i = 1$ (so length is 1)

$\vec{v}_i \cdot \vec{v}_j = 0$ (so perpendicular) when $i \neq j$

The standard basis is orthonormal

$\boxed{\text{HW 9a}}$ Check $\begin{pmatrix}1\\0\\0\end{pmatrix}$ $\begin{pmatrix}0\\1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\0\\1\end{pmatrix}$

$\boxed{\text{HW 9b}}$ Check $\begin{pmatrix}0\\1\\0\end{pmatrix}$ $\begin{pmatrix}-1\\0\\0\end{pmatrix}$ and $\begin{pmatrix}0\\0\\1\end{pmatrix}$

$\boxed{\text{HW 9c}}$ Check $\begin{pmatrix}\sqrt{2}/2\\\sqrt{2}/2\\0\end{pmatrix}$ $\begin{pmatrix}\sqrt{2}/2\\-\sqrt{2}/2\\0\end{pmatrix}$ $\begin{pmatrix}0\\0\\1\end{pmatrix}$ are orthonormal

$\boxed{\text{HW 9d}}$ Check $\begin{pmatrix}1\\2\\3\end{pmatrix}$ $\begin{pmatrix}0\\1\\0\end{pmatrix}$ $\begin{pmatrix}0\\0\\1\end{pmatrix}$

2:58 AM Thu Sep 17

Linear Algebra

What is useful about an orthonormal basis?

For any basis $S = \langle \vec{v}_1 \cdots \vec{v}_k \rangle$ and take $\omega \in S$ we can find $t_1\ t_2 \cdots t_k \in \mathbb{R}$ such that $\omega = \sum_{i=1}^{k} t_i \vec{v}_i$ by the defn of span.

But to find the t_i we solve

$$\left(\begin{array}{cccc|c} v_1 & v_2 & \cdots & v_k & \omega \end{array}\right)$$

that can take awhile.

Linear Algebra

If $\vec{v}_1, \cdots \vec{v}_k$ are an orthonormal basis for S we can find $t_1 \cdots t_k$ very quickly. {Case $k=3$}

① $\vec{\omega} = t_1\vec{v}_1 + t_2\vec{v}_2 + t_3\vec{v}_3$

Take dot product with $\vec{v}_1$

$$\vec{\omega}\cdot\vec{v}_1 = (t_1\vec{v}_1 + t_2\vec{v}_2 + t_3\vec{v}_3)\cdot\vec{v}_1$$

$$= t_1\vec{v}_1\cdot\vec{v}_1 + t_2\vec{v}_2\cdot\vec{v}_1 + t_3\vec{v}_3\cdot\vec{v}_1$$

by distribution

$$= t_1 \, 1 + t_2 \, 0 + t_3 \, 0$$

$$\vec{\omega}\cdot\vec{v}_1 = t_1$$

No row reduction needed!

3:00 AM Thu Sep 17

Linear Algebra

If $\vec{v}_1, \dots \vec{v}_k$ are an orthonormal basis for S we can find $t_1 \dots t_k$ very quickly.

Case $k=3$

(1) $\vec{w} = t_1\vec{v}_1 + t_2\vec{v}_2 + t_3\vec{v}_3$

Take dot product with $\vec{v}_2$

$$\vec{w}\cdot\vec{v}_2 = (t_1\vec{v}_1 + t_2\vec{v}_2 + t_3\vec{v}_3)\cdot\vec{v}_2$$

$$= t_1\vec{v}_1\cdot\vec{v}_2 + t_2\vec{v}_2\cdot\vec{v}_2 + t_3\vec{v}_3\cdot\vec{v}_2$$

by distribution

$$= t_1 \, 0 + t_2 \, 1 + t_3 \, 0$$

$$\vec{w}\cdot\vec{v}_2 = t_2$$

No row reduction needed!

Linear Algebra

Thm If $\vec{v}_1 \dots \vec{v}_k$ is an orthonormal basis for a space S and $\vec{w} \in S$ then $\vec{w} = \sum_{i=1}^{k} t_i \vec{v}_i$

where $t_i = \vec{w}\cdot\vec{v}_i$

(no need to solve a system to find t_i)

So it is important to find orthonormal basis for S

3:10 AM Thu Sep 17

Linear Algebra

$\boxed{HW10}$ Given $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \in \mathbb{R}^3$ find t_i such that $\vec{w} = t_1\vec{v}_1 + t_2\vec{v}_2 + t_3\vec{v}_3$ and check your answer.

$\boxed{HW10a}$ $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\boxed{HW10b}$ $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\boxed{HW10c}$ $\begin{pmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \\ 0 \end{pmatrix}$ $\begin{pmatrix} \sqrt{2}/2 \\ -\sqrt{2}/2 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\boxed{HW10d}$ $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Linear Algebra

Hint

try $t_i = \vec{w} \cdot \vec{v}_i$ which works for the ones that are orthonormal

if it fails then that is not orthonormal so instead find t_i the usual way solving the system

$$\left(\begin{pmatrix} v_1 \end{pmatrix} \begin{pmatrix} v_2 \end{pmatrix} \begin{pmatrix} v_3 \end{pmatrix} \middle| \begin{pmatrix} w \end{pmatrix} \right)$$

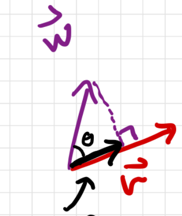
Watch Video 313F20-11-7 **on projections**

3:19 AM Thu Sep 17

Linear Algebra

Projection of a Vector

$\text{Proj}_{\vec v}\vec w$ — Project $\vec w$ to $\vec v$



$\text{Proj}_{\vec v}\vec w$

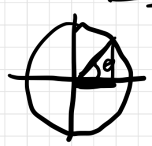
$\vec v$ has length 1

$|\vec v| = 1$

$\text{Proj}_{\vec v}\vec w$ is a vector in the same direction as $\vec v$

$$\vec w \cdot \vec v = |\vec w|\,|\vec v|\cos(\theta)$$

$$= |\vec w|\cos\theta \leftarrow \text{length of } \text{Proj}_{\vec v}\vec w$$



$$\text{Proj}_{\vec v}\vec w = (\vec w \cdot \vec v)\,\vec v$$

Linear Algebra

Projection of $\vec w$ to $\vec v$ when $|\vec v| = 1$ is

$$\text{Proj}_{\vec v}\vec w = (\vec w \cdot \vec v)\,\vec v$$

Suppose $\vec v_1, \dots, \vec v_k$ orthonormal

$$\vec w = t_1\vec v_1 + \dots + t_k\vec v_k$$

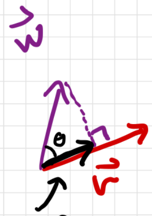
$$t_i = \vec w \cdot \vec v_i$$

$$\vec w = (\vec w \cdot \vec v_1)\,\vec v_1 + \dots + (\vec w \cdot \vec v_k)\,\vec v_k$$

$$= \text{Proj}_{v_1} w + \dots + \text{Proj}_{v_k}\vec w$$

Projection of a Vector

$\text{Proj}_{\vec{v}} \vec{w}$ Project $\vec{w}$ to $\vec{v}$

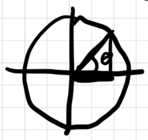


$\text{Proj}_{\vec{v}} \vec{w}$

$\vec{v}$ has length 1
$|\vec{v}| = 1$

is a vector in the same direction as $\vec{v}$

$$\vec{w} \cdot \vec{v} = |\vec{w}|\,|\vec{v}| \cos(\theta)$$
$$= |\vec{w}| \cos\theta \leftarrow \text{length of } \text{Proj}_{\vec{v}} \vec{w}$$



$$\text{Proj}_{\vec{v}} \vec{w} = (\vec{w} \cdot \vec{v})\, \vec{v}$$

Projection of $\vec{w}$ to $\vec{v}$ when $|\vec{v}| = 1$ is

$$\text{Proj}_{\vec{v}} \vec{w} = (\vec{w} \cdot \vec{v})\, \vec{v}$$

Suppose $\vec{v}_1, \dots, \vec{v}_k$ orthonormal

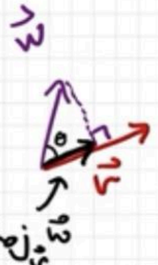
$$\vec{w} = t_1 \vec{v}_1 + \dots + t_k \vec{v}_k$$

$$t_i = \vec{w} \cdot \vec{v}_i$$

$$\vec{w} = (\vec{w} \cdot \vec{v}_1)\vec{v}_1 + \dots + (\vec{w} \cdot \vec{v}_k)\vec{v}_k$$
$$= \text{Proj}_{v_1} w + \dots + \text{Proj}_{v_k} \vec{w}$$

Projection of a Vector

$\mathrm{Proj}_{\vec{v}} \vec{w}$ — Project $\vec{w}$ to $\vec{v}$



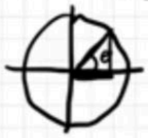
$\vec{v}$ has length 1

$|\vec{v}| = 1$

$\mathrm{Proj}_{\vec{v}} \vec{w}$ is a vector in the same direction as $\vec{v}$

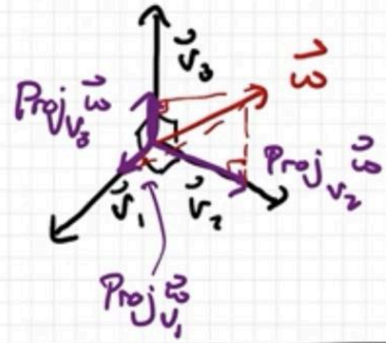
$\vec{w} \cdot \vec{v} = |\vec{w}||\vec{v}|\cos(\theta)$

$= |\vec{w}|\cos\theta \leftarrow$ length of $\mathrm{Proj}_{\vec{v}} \vec{w}$



$\mathrm{Proj}_{\vec{v}} \vec{w} = (\vec{w} \cdot \vec{v})\, \vec{v}$

$$\boxed{\vec{w} = \mathrm{Proj}_{\vec{v}_1} \vec{w} + \mathrm{Proj}_{\vec{v}_2} \vec{w} + \mathrm{Proj}_{\vec{v}_3} \vec{w}}$$



If $\vec{v}$ does not have $|\vec{v}| = 1$

$$\mathrm{Proj}_{\vec{v}} \vec{w} = \frac{\vec{v} \cdot \vec{w}}{\vec{v} \cdot \vec{v}} \vec{v}$$

Projection of $\vec{w}$ to $\vec{v}$ when $|\vec{v}| = 1$ is

$\mathrm{Proj}_{\vec{v}} \vec{w} = (\vec{w} \cdot \vec{v})\, \vec{v}$

Suppose $\vec{v}_1 \ldots \vec{v}_k$ orthonormal

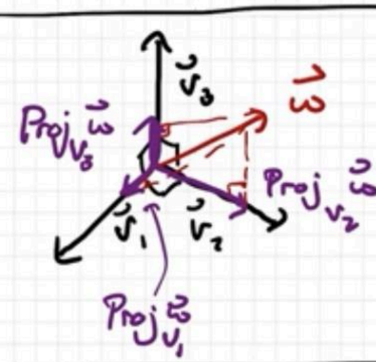
$\vec{w} = t_1 \vec{v}_1 + \ldots + t_k \vec{v}_k$

$t_i = \vec{w} \cdot \vec{v}_i$

$\vec{w} = (\vec{w} \cdot \vec{v}_1)\vec{v}_1 + \ldots + (\vec{w} \cdot \vec{v}_k)\vec{v}_k$

$= \mathrm{Proj}_{v_1} w + \ldots + \mathrm{Proj}_{v_k} \vec{w}$

$$\boxed{\vec{w} = \mathrm{Proj}_{\vec{v}_1} \vec{w} + \mathrm{Proj}_{\vec{v}_2} \vec{w} + \mathrm{Proj}_{\vec{v}_3} \vec{w}}$$



If $\vec{v}$ does not have $|\vec{v}| = 1$

$$\mathrm{Proj}_{\vec{v}} \vec{w} = \frac{\vec{v} \cdot \vec{w}}{\vec{v} \cdot \vec{v}} \vec{v}$$

$\boxed{\text{HW11 a-d}}$ Let $\vec{w} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Find $\text{proj}_{\vec{v}_i} \vec{w}$ for each $\vec{v}_i$ in HW10 and check

$$\vec{w} = \sum_{i=1}^{3} \left(\text{Proj}_{v_i} \vec{w} \right) v_i$$

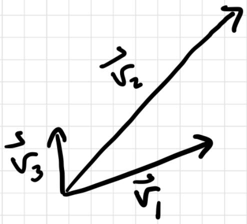
If this fails note (v_1, v_2, v_3) are not orthonormal.

Watch [Video 313F20-11-8] on the Gram Schmidt process

Gram-Schmidt Process

for finding an **orthonormal basis** for a span

$$S = \langle \vec{v}_1, \vec{v}_2, \vec{v}_3, \dots \vec{v}_k \rangle$$



We need to find $\vec{w}_1, \vec{w}_2 \dots$ such that

$S = \langle \vec{w}_1, \dots \vec{w}_\ell \rangle$

and $\vec{w}_i \cdot \vec{w}_j = 0$ if $i \neq j$

$\vec{w}_i \cdot \vec{w}_i = 1$

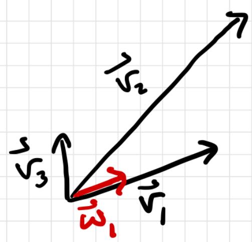
Linear Algebra

$\vec{w}_1 = \dfrac{\vec{v}_1}{|\vec{v}_1|}$ has length 1 "unit length"

Gram-Schmidt Process

for finding an orthonormal basis for a span

$S = \langle \vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots \vec{v}_k \rangle$



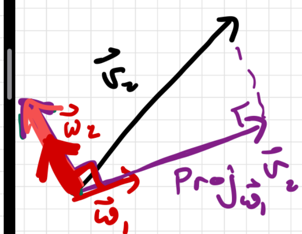
We need to find $\vec{\omega}_1, \vec{\omega}_2 \ldots$ such that $S = \langle \vec{\omega}_1, \ldots \vec{\omega}_\ell \rangle$

and $\vec{\omega}_i \cdot \vec{\omega}_j = 0$ if $i \neq j$

$\vec{\omega}_i \cdot \vec{\omega}_i = 1$

$\vec{\omega}_1 = \dfrac{\vec{v}_1}{|\vec{v}_1|}$ has length 1 "unit length"

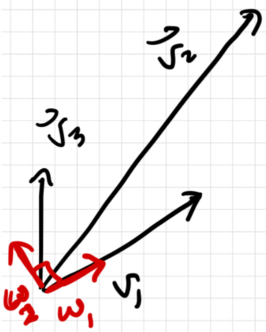
$\vec{\omega}_2 = \dfrac{\vec{v}_2 - Proj_{\vec{\omega}_1}\vec{v}_2}{|\vec{v}_2 - Proj_{\vec{\omega}_1}\vec{v}_2|}$ has unit length and is $\perp$ to $\vec{\omega}_1$



$\boxed{\text{HW 12}}$ Check that $\vec{v}_2 - Proj_{\vec{\omega}_1}\vec{v}_2 \perp \vec{\omega}_1$

(take dot product and use defn of Proj to check this)

Next find w_3



is $\perp$ to $\vec{w}_1$ and $\vec{w}_2$

$$w_3 = \frac{v_3 - \mathrm{Proj}_{w_1} v_3 - \mathrm{Proj}_{w_2} v_3}{\left| v_3 - \mathrm{Proj}_{w_1} v_3 - \mathrm{Proj}_{w_2} v_3 \right|}$$

divide to guarantee that w_3 has unit length

$$w_i = \frac{v_i - \sum_{l=1}^{i-1} \mathrm{Proj}_{w_l} v_i}{\left| v_i - \sum_{l=1}^{i-1} \mathrm{Proj}_{w_l} v_i \right|}$$

"Inductive defn"

using previous $w_1 \; w_2 \; \cdots \; w_{i-1}$ to find the next w_i

One concern
DIV BY 0 Error?!!
If denominator is 0, we have a problem!

Linear Algebra

For example suppose

$$v_3 - \mathrm{Proj}_{w_1} v_3 - \mathrm{Proj}_{w_2} v_3 = \vec{0}$$

we cannot divide by $|\vec{0}| = 0$

At each step confirm it is not zero before dividing.

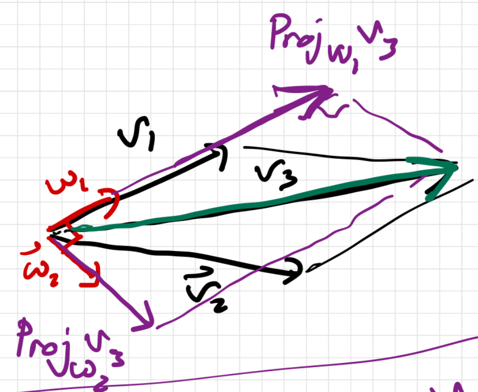
If it is zero, skip v_3 (toss it out) unnecessary anyway

$$v_3 = \mathrm{Proj}_{w_1} v_3 + \mathrm{Proj}_{w_2} v_3$$

$$= (w_1 \cdot v_3) w_1 + (w_2 \cdot v_3) w_2$$

$$\in \langle w_1, w_2 \rangle$$

Linear Algebra



$$\vec{v}_3 = \mathrm{Proj}_{w_2} v_3 + \mathrm{Proj}_{w_1} v_3$$

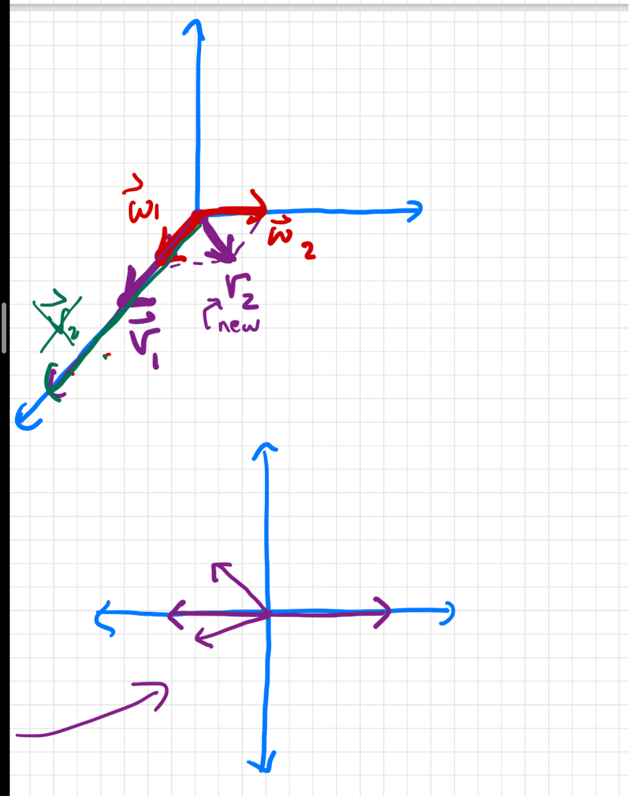
$$\in \langle w_1, w_2 \rangle$$

4:01 AM Thu Sep 17 — Linear Algebra

$\boxed{\text{HW13}}$ Use Gram-Schmidt to find an orthonormal basis for $\left\langle \begin{pmatrix}2\\0\\0\end{pmatrix}, \begin{pmatrix}4\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\2\\0\end{pmatrix} \right\rangle$

(see video for hints)

$\boxed{\text{HW14}}$ Use Gram Schmidt to find an orthonormal basis for $\left\langle \begin{pmatrix}0\\5\end{pmatrix}, \begin{pmatrix}-2\\2\end{pmatrix}, \begin{pmatrix}0\\-4\end{pmatrix}, \begin{pmatrix}-3\\-1\end{pmatrix} \right\rangle$ drawing the vectors and their projections



For help with the homework see [Video 313F20-11-9]. **You might also enjoy** [this video].

Share work for grading with Esteban Alcantara:
esteban.alcantara780@gmail.com
and also with Professor Sormani at sormanic@gmail.com.
Please email questions to Professor Sormani.

Many students have not been following the directions at the top of each lesson. You must include a selfie holding up your work. I need this for attendance purposes.

I first learned about dimension when I was a kid watching Carl Sagan's NOVA on TV in the 1970's: https://youtu.be/N0WjV6MmCyM