An Introduction To Reasoning

Propositional & Categorical Reasoning

The Method Of Derivation

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A star (*) indicates that there are exercises covering this section and previous unmarked sections.

This piece in relation to others: The piece is a chapter covering the method of derivation or proof. It is advisable to have previously covered the sections concerning English-language expressions for the logical operators from the chapter on Logically Structured English.

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The Method Of Derivation

1 Introduction

1. The method of Derivation is a method for deriving (a.k.a proving, validly inferring) a conclusion from a set of one or more premises. This chapter presents a number of rules of inference, each of which describes a valid pattern of reasoning; that is, any conclusion reached by manipulating the information in the premises in accordance with these rules will be reached validly. We can thus use the method of derivation to evaluate the reasoning of others: if we begin with their premises and can reach their conclusion using the rules of inference, the reasoning is valid. The chapter also presents a number of rules of equivalence which allow us to transform how a proposition is formulated, so that the rules of inference can be used.

(Comparison with the Big 8 method: the Big 8's six valid rules are repeated here, but more rules of inference are added, some of which require only a single premise (whereas the rules in the Big 8 method involved two or three premises). The method of Derivation also allows for there to be more than three premises because the rules can be applied to a subset of the full set of premises, as we shall see shortly.)

2 Logically Structured Symbolic Propositions*

1. The method of Derivation requires that the propositions involved in an inference are *logically structured symbolic propositions*, or propositions *in symbolic,* for short.

2. Writing propositions in symbolic requires three kinds of components, two of which will be familiar from *Logically Structured English* propositions. First, it has lower-case letters. For example, "b", "c", and "d". In translating the propositions of an inference into symbolic, use letters to stand for (a) simple propositions and (b) compound propositions that are not negations, disjunctions, conjunctions or conditionals. (This is just like in logically structured English.)

Second, it has parentheses "(" and ")" to clear up ambiguous propositions (as in logically structured English).

Third, it uses symbols called "operators" in place of the logical words: the tilde (pronounced TIL-de) "~", the wedge (or vel) "v", the ampersand "&", and the horseshoe "⊃".

Use the tilde to stand for "it is not the case that" and "not" in negations. Use the wedge to stand for "or" in disjunctions. Use the ampersand to stand for "and" in conjunctions. And use the horseshoe to stand for "if ..., then ..." in conditionals. (The horseshoe should be placed where the "then" would go in logically structured English; the "if" disappears.) The chapter on *Logically Structured English* shows how to understand the logical structure of various English words and phrases.

3. Make sure your propositions in symbolic are well-formed. The simplest proposition is just a single lower-case letter. "b", for example, is well-formed. "B", in contrast, is ill-formed; this is a capital letter.

To express negation, a tilde goes to the left of the letter standing for a simple proposition or to the left of a complex proposition in parentheses. Examples of negating a simple proposition would be "~b" and "~k". These are well-formed, whereas "b~" and "k~" are ill-formed; the tilde is on the wrong side of the letter. "~(b v c)" and "~(p & q)" are also well-formed. The parentheses make it clear that the tilde is the main operator, in that they make it clear that the propositions have the form "~S" rather than "~S v T" or "~S & T". The proposition "(b v c)~" is also incorrectly formed.

The remaining symbols — the wedge, ampersand, and horseshoe — go between two propositions, whether simple or complex. Examples involving only simple propositions as parts are "b v c", "h & l", and "b ⊃ q". The examples "b &", "⊃ b g", and "jk v r", however, are not well-formed. The proposition "b &" has nothing to the right of the ampersand; the proposition "⊃ b g" fails to put the horseshoe between the letters; "jk v r" has two letters to the left of the wedge.

Complex propositions can also be placed on either side, or both sides, of the wedge, ampersand and horseshoe. For example, in the proposition "b ⊃ ((d v c) & e)", the antecedent is the simple proposition "b", while the consequent is the complex proposition "(d ∨ c) & e". And the proposition "(b ∨ c) & (d ⊃ e)" has complex propositions on both sides of the ampersand: the first conjunct is "b ∨ c" and the second is "d ⊃ e". The complexity of the parts does not change the basic structure of the propositions. Both of them are well-formed. In the first, the parentheses around the consequent make it clear that the horseshoe is the main operator, in that they make it clear that the proposition has the form "S ⊃ T". In the second, the parentheses around the two conjuncts make it clear that the ampersand is the main operator; that is, the basic structure of the proposition is "S & T".

4. For propositions with two or more operators, use parentheses to avoid ambiguity. Consider, for example, the proposition "p ∨ q & r". On one reading the wedge is the fundamental operator and, thus, it has the form "S ∨ T", where "T" is "q & r". On another reading the ampersand is the main operator and, thus, it has the form "S & T", where "S" is "p ∨ q". Adding parentheses like this "(p ∨ q) & r" would make the ampersand the fundamental operator, whereas adding parentheses like this "p ∨ (q & r)" would make the wedge the main operator. As it stands, "p ∨ q & r" is ill-formed.

Note: there are two exceptions, where parentheses are not strictly necessary but can be added if desired. First, a letter with two or more tildes to the left of it is a well-formed proposition. The first tilde (i.e., the one furthest to the left) is the main operator. "~~~d", for example, is well-formed, and the tilde furthest from "d" is the main operator. If parentheses were used, the proposition would look like this: ~(~(~d)); only the leftmost tilde is outside any parentheses.

Second, the tilde is understood to modify whatever follows it immediately. So, in the proposition "~b & c" the tilde modifies "b" and "~b" is the left-hand conjunct in the conjunction; the ampersand is the main operator. We could make this abundantly clear by writing "(~b) & c". On the other hand, in the proposition "~(b & c)" the negation applies to "b & c" and not just to "b"; since parentheses immediately follow the tilde, the tilde is the fundamental operator.

Note also that a tilde can follow another symbol if it modifies a proposition-letter. All of the following are well-formed: "b & ~c", "b ∨ ~c", "b ⊃ ~c". All other combinations of two or more symbols are ill-formed.

The horseshoe requires the use of parentheses more than "if ... then ..." does in Logically Structured English. This is because the word "if" is not used in Symbolic. For example, the proposition "Smith is ambitious and if she works hard then she can get a promotion" in Logically Structured English would not need parentheses at all: "a and if w then p". In Symbolic, however, "a & w ⊃ p" is ambiguous; it could mean "a & (w ⊃ p)" (which is the correct translation) or "(a & w) ⊃ p".

5. Consider, for example, the following propositions:

(a) The cat is on the mat.

(b) The cat is on the mat or my grandma is in Barcelona.

(c) The cat isn't on the mat.

(d) Jack is slightly amused if either the cat is on the mat or the dog is chasing the frog.

(e) Nixon served as the President of the United States before Reagan served as the President of the United States.

(f) It is well known that the earth is not flat.

(a) is simple and, thus, gets represented as "m", where "m" stands for "The cat is on the mat.". (b) is a disjunction and, thus, gets represented as "m ∨ b", where "b" stands for "My grandma is in Barcelona.". (c) is a negation and, therefore, gets represented as "~m". (d) is a conditional and, so, gets represented as "(m ∨ c) ⊃ a", where "c" stands for "The dog is chasing the frog." and "a" stands for "Jack is slightly amused.". Although (e) and (f) are complex propositions, they get represented by single letters — "s" and "k", for example — since neither is a negation, a disjunction, a conjunction, nor a conditional.

Logically Structured Symbolic Propositions

Use lower-case letters to stand for simple propositions.
Use symbols called "operators" in place of the logical words:

- tilde "~" for negation
- wedge "v" for disjunction
- ampersand "&" for conjunction
- horseshoe "⊃" for conditional

Each proposition must have one and only main operator and the hierarchy of operators must be clear; use parentheses as necessary.

Exercise Set (1) | Exercise Set (2)

3 The Method Of Derivation & The Rules Of Inference*

1. The method of derivation shows that a conclusion follows validly from a set of premises when the conclusion can be derived from the premises. A conclusion is derived from a set of premises when the conclusion is the result of a connected series of inferences using one rule of inference at a time. The method of derivation works as follows:

If you are starting from an English-language passage, make a translation key and translate the propositions into symbolic.

Set out the premises and conclusion as follows: Write one premise on each line. Number it and mark each one as a premise by writing the word "Premise" to the right. The conclusion (and any interim conclusions) are written off to the right along with the word "Conclusion" (or "Interim conclusion").

Then, using *one rule of inference at a time* and using *only* as many propositions as each rule requires, attempt to derive (or: infer, prove) the conclusion, taking multiple steps if necessary.

If a derivation of the conclusion can be found, using only the premises and the rules of inference, the inference is valid. If no derivation can be found, we can conclude nothing, as the failure might be due to our limited abilities.

2. As a simple example, consider the following passage:

> If logic is beneficial, then mathematics is beneficial. Logic is beneficial. So, mathematics is beneficial.

Using "l" for "Logic is beneficial." and "m" for "Mathematics is beneficial." we translate into symbolic and set out the premises and conclusion(s). We write down and number the premises, and to the right of each premise we write down the word "Premise" to indicate that the proposition on that line is a premise. The word "Conclusion" and the conclusion are written down off to the right, on the same line as the final premise. After the set-up stage, we get this:

1.	$l \supset m$	Premise	
2.	l	Premise	Conclusion: m

We now think about how to derive the conclusion using the premises and the rules of inference. Even though we haven't introduced the rules of inference, you might see that "m" follows validly from premises (1) and (2) right away: when we put together the claim that logic is beneficial ("l") and the conditional linking "l" and "m", we can validly conclude "m". The rule that allows us to conclude the consequent of a conditional when its antecedent is given is called *matching the antecedent*, or MA. On line (3) we write down the result of the inference ("m") and indicate that it was inferred from premises (1) and (2) using MA.

1.	l ⊃ m	Premise	
2.	l	Premise	Conclusion: m
3.	m	1, 2 MA	

Line (3) is the same as the conclusion on the far right, so the conclusion has been reached and the derivation is complete.

In this example there is only one more line beyond the premises. Longer derivations will require additional lines, but the procedure for each line is the same: number the line, write down the proposition being inferred, and write down the lines of the propositions from which it was inferred and the rule used to infer it. In this case, on line (3), "m" was derived from the propositions on lines (1) and (2) by MA.

(If you have already studied standard form or the Big 8 method, notice that there is no horizontal line dividing the premises from the results of applying the rules of inference as there is when we write inferences in standard form. There is no need for a dividing line between premises and conclusion because each line is labeled as either a premise or the result of an application of a rule of inference.)

3. Matching the antecedent (MA) can be understood as a rule which says it is valid to infer any proposition "T" from the propositions "S ⊃ T" and "S" if they appear anywhere in the derivation. (As always, capital "S" and "T" are variables standing for any proposition, either simple or complex.) Here are six basic rules of inference: *matching the antecedent, contradicting the consequent*, *hypothetical syllogism*, *constructive dilemma*, *destructive dilemma*, and *disjunctive syllogism*. (These will be very familiar if you worked through the chapter on the *Big 8 Method*; they are the six valid inference patterns; MC or CA are not used, since derivation proceeds only by valid inference.)

Matching the Antecedent (MA)
k. $S \supset T$
.
.
l. S
.
.
m. T k, l MA

Contradicting the Consequent (CC)
k. $S \supset T$
.
.
l. $\sim T$
.
.
m. $\sim S$ k, l CC

Chain Inference (Chain)
k. $S \supset T$
.
.
l. $T \supset U$
.
.
m. $S \supset U$ k, l Chain

Elimination Inference (Elim.)
k. $S \vee T$
.
.
l. $\sim S$
.
.
m. T k, l Elim.

Constructive Dilemma (CD)
k. $S \supset T$
.
l. $U \supset V$
.
m. $S \vee U$
.
.
n. $T \vee V$ k, l, m CD

Destructive Dilemma (DD)
k. $S \supset T$
.
l. $U \supset V$
.
m. $\sim T \vee \sim V$
.
.
n. $\sim S \vee \sim U$ k, l, m DD

Here, as an example you can follow in the other cases, is how to read the rule of inference MA. On the left-hand side of each line "k", "l", "m" stand for the numbers of the lines. The dots between lines indicate that the propositions which are used in the inference can appear anywhere in the derivation (and indeed, they can appear *in any order*). The rule thus says that "T" can be derived (on line (m)) if the propositions "$S \supset T$" and "S" appear *anywhere* above. Each of "$S \supset T$" and "S" (on lines (k) and (l)) might be either a premise from the original passage or a proposition previously derived which can be used in a new inference. If a proposition on a line is a premise given in the original inference it will have the word "Premise" written to its right; if a proposition on a line has been inferred from the original premises, to its right will be written the numbers of the lines of the propositions used and the rule used to derive it. When we use the rule MA to derive the conclusion, to the right of this conclusion we write 'k, l, MA', which means that "T" was inferred on line (m) from the propositions on lines (k) and (l) using MA.

Notice that (as with the Big 8 method) Elim. involves negating the left-hand disjunct ("~S" on line (l).) and deriving the right-hand disjunct ("T" on line (m).). As part of the method of derivation we will soon introduce a rule of equivalence called *commutation* which states that "S ∨ T" is equivalent to "T ∨ S". This will enable us to infer "S" when we have "~T" on line (l).

4. For a more complex case, consider the following example:

> Either I failed History 101 or I passed it. I know I didn't fail it. So, I passed. And if I passed 101 I can register for History 201. So, I can register for History 201.

We begin by translating the propositions into symbolic and setting out the premises and conclusions. Using obvious proposition-letters, we get:

1.	f ∨ p	Premise	
2.	~f	Premise	Interim Conclusion: p
3.	p ⊃ r	Premise	Conclusion: r

The first thing to notice is that the passage has an interim conclusion "So, I passed.". The interim conclusion is *not* written in the same column as the premises; it is written to the right of premise (2), since it is a conclusion and on the same line as premise (2), since this is where it occurs in the English-language passage. The speaker's inclusion of this interim conclusion in fact gives us a hint for our derivation; the speaker suggests that the interim conclusion somehow follows from (1) and (2). We want to check whether the speaker reasoned well at each step, so we will check whether or not the speaker validly inferred the interim conclusion, after we have set up the rest of the propositions, which we do by writing down the third premise in the list of premises and the (ultimate) conclusion off to the right.

After we have set out the premises and conclusion(s), we write the conclusion of each successive inference on the next line below the final premise and number the lines successively. The result of the final inference, written on the last line of the complete derivation, should be the ultimate conclusion we are seeking.

In this case, we can infer the interim conclusion "I passed." by using lines 1 and 2 and the rule of elimination (Elim.). The speaker began (line 1) by presenting a disjunction and then denied one of the disjuncts (line 2). Using these two premises and the Elim. rule, "p" can be inferred. In the derivation, we add this on line (4). At this stage, the derivation looks like this:

1.	$f \vee p$	Premise	
2.	$\sim f$	Premise	Interim Conclusion: p
3.	$p \supset r$	Premise	Conclusion: r
4.	p	1, 2 Elim.	

When a derivation takes more than one step, the crucial thing to remember is that each inferred proposition is then available for use in further inferences. Thus, in the proof above, proposition (4) can be used in subsequent inferences, and indeed it can now be used in reaching the ultimate conclusion: lines (3) and (4) can be used to infer the ultimate conclusion, "r", using MA. This is the (ultimate) conclusion we are seeking — it is the same proposition as we wrote down off to the right — and so the derivation is complete. The complete derivation is as follows:

1.	$f \vee p$	Premise	
2.	$\sim f$	Premise	Interim Conclusion: p
3.	$p \supset r$	Premise	Conclusion: r
4.	p	1, 2 Elim.	
5.	r	3, 4 MA	

Derivation – Six Rules Of Inference

Matching the Antecedent (MA)
k. $S \supset T$
.
.
l. S
.
.
m. T k, l MA

Contradicting the Consequent (CC)
k. $S \supset T$
.
.
l. $\sim T$
.
.
m. $\sim S$ k, l CC

Chain Inference (Chain)
k. $S \supset T$
.
.
l. $T \supset U$
.
.
m. $S \supset U$ k, l Chain

Elimination Inference (Elim.)
k. $S \vee T$
.
.
l. $\sim S$
.
.
m. T k, l Elim.

Constructive Dilemma (CD)
k. $S \supset T$
.
l. $U \supset V$
.
m. $S \vee U$
.
.
n. $T \vee V$ k, l, m CD

Destructive Dilemma (DD)
k. $S \supset T$
.
l. $U \supset V$
.
m. $\sim T \vee \sim V$
.
.
n. $\sim S \vee \sim U$ k, l, m DD

Exercise Set (1) | Exercise Set (2)

4 The "Forwards" & "Backwards" Strategies Of Derivation*

1. Speakers who reason step by step, making interim conclusions explicit, help us in checking the validity of their inferences. However, sometimes the premises and only the ultimate conclusion are presented, without any interim conclusions. Here is an example:

> If Spurs win this weekend then they will top the league table. If Spurs are on top, attendance figures will rise. If attendance rises, then the club will be able to fix the stadium. So, if they win this weekend, repairs can be made.

Translated into symbolic and set out in derivation format, the inference looks like this:

1.	$w \supset t$	Premise	
2.	$t \supset a$	Premise	
3.	$a \supset r$	Premise	Conclusion: $w \supset r$

In this passage, there is only one conclusion ("If Spurs win this weekend, repairs can be made to the stadium.") and three premises. The three premises are not all of the kinds needed for CD or DD and so the derivation will involve more than one step. That is, the derivation of the conclusion will involve at least one interim conclusion before arriving at the given conclusion. How can we work out the steps that will bring us to the conclusion?

One strategy (the "forward" strategy) is to look at the premises and see if any two of them fit the rules of derivation (or any three, in the case of CD and DD). In other words, if you see some premises that work together according to one of the rules of derivation, go ahead and make the derivation.

In this case, you might look at lines (1) and (2) and notice that they are both conditionals and that "t" is a shared term: it appears as the consequent in (1) and as the antecedent in (2). You could thus Chain them together to get "$w \supset a$". At this stage, the derivation looks like this:

1.	$w \supset t$	Premise	
2.	$t \supset a$	Premise	
3.	$a \supset r$	Premise	Conclusion: $w \supset r$
4.	$w \supset a$	1, 2 Chain	

Does line (4) help us get to the conclusion? If you look back at the other premises, you might see that lines (4) and (3) can be combined using Chain, and the result will be "$w \supset r$", which is the conclusion that we want to arrive at. The full derivation looks like this:

1.	$w \supset t$	Premise	
2.	$t \supset a$	Premise	
3.	$a \supset r$	Premise	Conclusion: $w \supset r$
4.	$w \supset a$	1, 2 Chain	
5.	$w \supset 4$	4, 3 Chain	

(Notice that to the right of line (5) we write "4, 3 Chain" rather than "3, 4 Chain". This is merely a piece of etiquette which indicates that line 4 contains the antecedent of the conclusion, while line 5 contains the consequent. If you write "3, 4" that is fine, too.)

2. This derivation is quite simple: having derived "$w \supset a$" on line (4) from (1) and (2), we combine it with line (3) in a new instance of Chain, which yields the desired conclusion "$w \supset r$".

Often, when trying to show an inference to be valid using the method of derivation, it is useful to *begin with the conclusion and work backwards*. Consider the following inference, whose conclusion is "b.". We write the conclusion some distance below the premises, as follows:

1.	$c \lor a$	Premise	
2.	$\sim c$	Premise	
3.	$a \supset b$	Premise	Conclusion: b
.			
.			
.			
.			
m.	b	__ ___	

The "m" on the last line is the number that this line *will* have in the completed derivation. We do not yet know what number it is. If we derive "b" on line (m) we will do so from some propositions above it and by using some rule of inference, but we do not currently know how, so the lines to the right are blank at the moment.

Working "backwards", we start by looking at the conclusion. We see that the conclusion is a single letter (it has no logical symbols), and it is "b". We then think as follows: "Could any of the premises give us "b"?". In this case, we see "b" in the third premise "a ⊃ b", where "b" is the consequent. Together, line (3) and line (m) are two parts of an instance of MA:

3.	a ⊃ b	
.		
m.	b	3,_ MA

Or, to put it another way, if we had "a" by itself, then we could combine "a" and line (3) in order to infer "b".' We write "a" in on line (l), above line (m).

1.	c ∨ a	Premise	
2.	~c	Premise	
3.	a ⊃ b	Premise	Conclusion: b
.			
.			
.			
l.	a		
m.	b	3, l MA	

Now we ask the same question again: 'Which of the premises could give us "a"?' We see "a" in line (3), of course, but we want to use line (3) to get the conclusion on line m, and this suggests that line 3 will not be used on line l (though as we will see in a moment, sometimes the same proposition is used more than once in a derivation). We also see "a" in line (1) "c ∨ a". Together, line (1) and line (l) are a partial match for Elim.:

1.	c ∨ a	
.		
l.	a	1,_ Elim.

If we had "~c" we could derive "a" from "~c" and line 1 using Elim.. So we write "~c" on line (k), above line (l).

1.	c ∨ a	Premise	
2.	~c	Premise	
3.	a ⊃ b	Premise	Conclusion: b
.			
.			

k.	~c	
l.	a	1, k Elim.
m.	b	3, l MA

Now our question is: 'Which of our premises could give us "~c"?' We see it on line (2) and, what's more, it is present just by itself and so we do not need to perform any further inferences. Link (k) is just line (2), and we can remove line (k).

Our complete derivation is written as follows. When read from the bottom up, it says that "b" was derived from (3) and (4) using MA. MA requires the antecedent of the condition ("a") by itself, and this was derived (on line (4)) from (1) and (2), using Elim.

1.	c ∨ a	Premise	
2.	~c	Premise	
3.	a ⊃ b	Premise	Conclusion: b
4.	a	1, 2 Elim.	
5.	b	3, 4 MA	

3. The "backwards" strategy is as follows: Look at the conclusion and think about what rule is partially matched by the conclusion plus one or more of the premises. If the conclusion is found among the premises as ...

... the consequent to a conditional, think of using MA and search for the antecedent. Look at the conclusion and line (3) in the following; we would next be searching for "a".

1.	...	Premise	
2.	...	Premise	
3.	a ⊃ (g ∨ a)	Premise	Conclusion: g ∨ a

... a un-negated antecedent, think of using CC and search for the negation of the consequent. Look at the conclusion and line (2) in the following; we would next be searching for "~d".

1.	...	Premise	
2.	c ⊃ d	Premise	
3.	...	Premise	Conclusion: ~c

... a conditional, think of using Chain and search for a conditional having either the antecedent of the conclusion as an antecedent, or having the consequent of the conclusion as a consequent. Look at the conclusion and line (2) in the following; we would next be searching for "r ⊃ f".

1.	...	Premise	
2.	g ⊃ r	Premise	
3.	...	Premise	Conclusion: g ⊃ f

... the right-hand disjunct, think of using Elim. and search for the negation of the left-hand disjunct. Look at the conclusion and line (1) in the following; we would next be searching for "~b".

1.	b ∨ (a ⊃ f)	Premise	
2.	...	Premise	
3.	...	Premise	Conclusion: a ⊃ f

... a disjunction, think of using CD or DD and search for the appropriate conditionals and/or the appropriate disjunction. Look at the conclusion and lines (2) and (3) in the following:

1.	...	Premise	
2.	b ⊃ g	Premise	
3.	r ⊃ f	Premise	
4.	...	Premise	Conclusion: g ∨ f

For each proposition sought, repeat this process until the proposition(s) needed are among the original premises.

4. The two strategies are similar in that they both try to find partial matches with the rules of derivation. The difference between the two strategies is that in the "forwards" strategy, you look at the premises (only) and try to find two (or possibly three) premises which look like the premises from the rules. You then derive a new proposition from them and hope it matches the "missing" conclusion. If it doesn't you keep working forwards, and in particular you look at the new proposition along with the given premises.

In the "backwards" strategy, by contrast, you try to find a match between the rules and the conclusion plus one or more of the premises. You then think about the "missing" premise and hope it appears in the premises. If it doesn't, you continue working backwards, and in particular you look for the needed premise along with the given premises.

5. A proposition *can be used more than once* in the course of a complete derivation. Consider the following inference, in standard form using symbolic:

1. a ⊃ (~c ⊃ b)
2. ~c
3. c ∨ a

   ----

4. b

The derivation of "b" from the premises requires the use of the premise on line (2) twice, as follows:

1.	a ⊃ (~c ⊃ b)	Premise	
2.	~c	Premise	
3.	c ∨ a	Premise	Conclusion: b
4.	a	3, 2 Elim.	
5.	~c ⊃ b	1, 4 MA	
6.	b	5, 2 MA	

The Forwards & Backwards Strategies Of Derivation

Forwards strategy: look at the premises (only) and try to find two or three premises which match the premise(s) from one of the rules, and make the inference. Hopefully the newly inferred proposition can then be used (with the existing premises) in order to reach the desired conclusion. If not, keep looking for matches with the premises of the rules until the conclusion is found and the derivation is complete.

Backwards strategy: try to find a partial match between the desired conclusion plus one of the premises and one of the rules. You then think about the "missing" premise and hope it appears in the premises. If it doesn't you continue working backwards, and in particular look at the needed premise along with the given premises. Once the needed premise is found in the given premises, the derivation is complete.

Exercise Set (1) | Exercise Set (2)

5 Three Additional Rules Of Inference*

1. Here is a piece of very simple reasoning, one whose conclusion might be too obvious to put into words:

> The Hammers and Spurs won. So, the Hammers won.

This type of inference is called *simplification*. When two pieces of information are presented together, you can infer one of them. This is a loss of information, but it can be useful, when, for example, you are given two pieces of information together but only need one. If you had bet someone that Spurs would win their game, and then learned that both the Hammers and Spurs had won, you would immediately use simplification to extract the information you need (that Spurs had won) and claim your winnings.

2. Another very obvious piece of reason is to conjoin two pieces of information. For example:

> Jack is in the kitchen. Jim is in the garden. So, Jack is in the kitchen and Jim is in the garden.

This one is called *conjunction*. It is useful when you are presented with pieces of information separately but need to combine them in order to make a further inference.

3. We can consider simplification and conjunction as additional rules of inference. They thus expand the range of inferences that can be shown to be valid by the method of derivation. As rules of inference, they are:

Simplification (Simp.)

k.	S & T	
.		
l.	S	k, Simp.

Conjunction (Conj.)

k.	S	
.		
l.	T	
.		
m.	S & T	k, l Conj.

Notice that, as written above, Simp. only allows us to derive the left hand conjunct. As part of the method of derivation we will introduce in the next section a rule of equivalence called "commutation" which states that "S & T" is equivalent to "T & S" (and also that "S ∨ T" is equivalent to "T ∨ S".)

4. Now for a simple but rather strange rule, called *addition* (Add.).

Addition (Add.)
k. S
.
.
l. S ∨ T k, Add.

Add. allows us to turn an existing proposition into a disjunction by adding *anything at all*. This might seem a strange rule at first, as it throws away the 'definiteness' of the original proposition. Compare the information value of "Spurs have won." with "Spurs have won or the Hammers have won.".

But this rule does have an important use when a disjunction is part of a more complex proposition, such as a conditional. For example, imagine that you know "If either the Colts or the Jets wins, then the Patriots will not qualify for post-season play." and now you find out that the Colts have won. It follows that the Patriots do not qualify. But how can this be shown by the method of derivation? Using the obvious letters, the initial inference looks like this:

1.	(c ∨ j) ⊃ ~p	Premise	
2.	c	Premise	Conclusion: ~p

The conclusion appears amongst the premises as the consequent in line 1, the antecedent being "c ∨ j". If we had the antecedent, "c ∨ j", we could use MA to infer the conclusion. We can use addition of "j" to line 2 to get the necessary antecedent, as follows:

1.	(c ∨ j) ⊃ ~p	Premise	
2.	c	Premise	Conclusion: ~p
3.	c ∨ j	2 Add.	
4.	~p	1, 3 MA	

Note that Add. is valid. If we grant that "S" is true, then it follows necessarily that "S ∨ T" would be true, for the supposed truth of "S" would be sufficient to make "S ∨ T" true.

Do not confuse Add. and Conj.. Their names suggest something quite similar, but one (Add.) produces a disjunction while the other (Conj.) produces a conjunction.

5. These new rules suggest certain additional strategies when trying to work backwards from the conclusion. If the conclusion is found among the premises as ...

... a left-hand conjunct, think of using Simp.;

... a conjunction, think of using Conj.. And ...

... if the conclusion is a disjunction whose right-hand disjunct is not among the premises, think of using Add..

(And if the conclusion *is* a proposition which is not among the premises, think of using Add. followed by Elim.. (This is equivalent to finding a contradiction among, or deriving a contradiction from, the premises.))

Derivation – Nine Rules Of Inference

Matching the Antecedent (MA)
k. $S \supset T$
.
.
l. S
.
.
m. T k, l MA

Contradicting the Consequent (CC)
k. $S \supset T$
.
.
l. $\sim T$
.
.
m. $\sim S$ k, l CC

Chain Inference (Chain)
k. $S \supset T$
.
.
l. $T \supset U$
.
.
m. $S \supset U$ k, l Chain

Elimination Inference (Elim.)
k. $S \vee T$
.
.
l. $\sim S$
.
.
m. T k, l Elim.

Constructive Dilemma (CD)
k. $S \supset T$
.
l. $U \supset V$
.
m. $S \vee U$
.
.
n. $T \vee V$ k, l, m CD

Destructive Dilemma (DD)
k. $S \supset T$
.
l. $U \supset V$
.
m. $\sim T \vee \sim V$
.
.
n. $\sim S \vee \sim U$ k, l, m DD

Simplification (Simp.)
k. $S \& T$
.
l. S k, Simp.

Addition (Add.)
k. S
.
.
l. $S \vee T$ k, Add.

Conjunction (Conj.)
k. S
.
l. T
.
m. $S \& T$ k, l Conj.

Exercise Set (1) | Exercise Set (2)

6 Four Rules Of Equivalence*

1. We can expand the Method Of Derivation by adding *rules of equivalence* to the foregoing rules of derivation. Rules of equivalence do not generate new propositions. Instead, they allow us to restate a proposition in a different but logically equivalent way. Rules of equivalence are different from rules of inference because they can be used to replace a proposition *or any part of a proposition*.

2. The most obvious such equivalence is *double negation*:

Double Negation (DN)
S <--> ~~S

The double arrow "<-->" indicates an equivalence. Equivalence-rules permit us to substitute what appears on the left side of the double arrow with what is on the right, *and vice versa*. This means that we can convert in either direction.

Consider the following inference:

Since sensory experiences cannot serve as good reasons only if coherentism is true, and coherentism is not true, thus sensory experiences can serve as good reasons.

Using an obvious translation key, we can set the derivation up as follows:

1.	~s ⊃ c	Premise	
2.	~c	Premise	Conclusion: s

The desired conclusion is "s". The relationship between "~c" in premise (2) and "c" in the consequent of premise (1) suggests CC, but using CC yields "~~s", rather than "s". So, we convert "~~s" to "s" using the equivalence DN. The full derivation is written as follows:

1.	~s ⊃ c	Premise	
2.	~c	Premise	Conclusion: s
3.	~~s	1, 2 CC	
4.	s	3 DN	

3. To repeat, we can employ equivalences on whole propositions *or on parts of propositions*. In this respect equivalences are importantly different from rules of derivation. So, for example, if an inference contained the line ...

4. $\sim\sim S \supset T$

... we could employ DN on the antecedent only to convert to ...

5. $S \supset T$ 4 DN

4. Another rule of equivalence is *commutation*.

Commutation (Comm.)
S v T <--> T v S and S & T <--> T & S

Commutation states, in effect, that the order of appearance of propositions in a disjunction or conjunction makes no logical difference. It is important because we have defined only one ("left-hand") version of each of Elim. and Simp.. Given these definitions, the following is *not* a valid derivation:

1.	s v t	Premise	
2.	~t	Premise	Conclusion: s
3.	s	1, 2 Elim.	

Rather we must proceed as follows:

1.	s v t	Premise	
2.	~t	Premise	Conclusion: s
3.	t v s	1 Comm.	
4.	s	3, 2 Elim.	

Similarly, the following derivation is not valid:

1.	a & b	Premise	Conclusion: b
2.	b	1 Simp.	

Rather we must derive as follows:

1.	a & b	Premise	Conclusion: b
2.	b & a	1 Comm.	
3.	b	2 Simp.	

5. *Association* is defined as follows:

Association (Ass.)

(S v T) v U <--> S v (T v U)   and   (S & T) & U <--> S & (T & U)

Association states, in effect, that when there are three propositions concatenated by disjunction, or by conjunction, it makes no logical difference whether we treat the first wedge or ampersand as the main operator, or the second. Indeed, when combined with commutation, we see that it would make no logical difference if we took the first and third as a pair, as follows:

(1) (a v b) v c	
(2) a v (b v c)	1 Ass.
(3) (b v c) v a	2 Comm.
(4) b v (c v a)	3 Ass.

6. Another equivalence which is fairly common in English is *De Morgan's rule*. There are two versions, as follows:

De Morgan's Rule (DM)
~(S & T) <--> ~S v ~T   and   ~(S v T) <--> ~S & ~T

These look frightening, but they are quite familiar. Here are English-language examples that might help. The first version says:

It can't both be raining and warm.	~(S & T)
... is equivalent to ...	... is equivalent to ...
It is either not raining or not warm.	~S v ~T

The second version says:

It is not either raining or warm.	~(S v T)
... is equivalent to ...	... is equivalent to ...
It is not raining and it is not warm.	~S & ~T

(These are often expressed as: It is neither raining nor warm.)

Derivation – Nine Rules Of Inference
& Four Rules Of Equivalence

Matching the Antecedent (MA)

k. $S \supset T$
.
.
l. S
.
.
m. T k, l MA

Contradicting the Consequent (CC)

k. $S \supset T$
.
.
l. $\sim T$
.
.
m. $\sim S$ k, l CC

Chain Inference (Chain)

k. $S \supset T$
.
.
l. $T \supset U$
.
.
m. $S \supset U$ k, l Chain

Elimination Inference (Elim.)

k. $S \vee T$
.
.
l. $\sim S$
.
.
m. T k, l Elim.

Constructive Dilemma (CD)

k. $S \supset T$
.
l. $U \supset V$
.
m. $S \vee U$
.
.
n. $T \vee V$ k, l, m CD

Destructive Dilemma (DD)

k. $S \supset T$
.
l. $U \supset V$
.
m. $\sim T \vee \sim V$
.
.
n. $\sim S \vee \sim U$ k, l, m DD

Simplification (Simp.)

k. S & T
.
l. S k, Simp.

Addition (Add.)

k. S
.
.
l. $S \vee T$ k, Add.

Conjunction (Conj.)

k. S
.
l. T
.
m. S & T k, l Conj.

Double Negation (DN)
S <--> ~~S

Commutation (Comm.)
S v T <--> T v S
S & T <--> T & S

De Morgan's Rule (DM)
~(S & T) <--> ~S v ~T
~(S v T) <--> ~S & ~T

Association (Ass.)
(S v T) v U <--> S v (T v U)
(S & T) & U <--> S & (T & U)

Exercise Set (1) | Exercise Set (2)

7 Three Additional Rules Of Equivalence*

1. Other rules of equivalence are *exportation, transposition,* and *material implication.*

Exportation (Exp.)
$S \supset (T \supset U) \text{<-->} (S \,\&\, T) \supset U$

Transposition (Trans.)
$S \supset T \text{<-->} {\sim}T \supset {\sim}S$

Material Implication (MI)
$S \supset T \text{<-->} {\sim}S \vee T$ and $S \supset T \text{<-->} {\sim}(S \,\&\, {\sim}T)$

(Note: The rule of equivalence called "material implication" should not be confused with the use of the term to mean a conditional proposition.)

These equivalences (especially Exportation) occur in ordinary English and can be intuitively seen to be logical equivalences. Compare the following sets of propositions:

Exp. If it rains, then, if the field is water-logged, then the game is canceled.
If it rains and the field is water-logged, then the game is canceled.

Trans. If salt is soluble in water, it (salt) is ionic.
If salt is not ionic, it is not soluble in water.

MI If it's raining, the river will burst its banks.
Either it's not raining or the river will burst its banks.

MI If it's raining, the river will burst its banks.
It can't rain and the river not burst its banks.
(Often expressed as: It can't rain without the river bursting its banks.)

The propositions in each set of propositions are equivalent to one another. (The *Truth Table Method* can be used to verify these equivalences.)

2. These rules of equivalence, especially MI and DM, suggest additional strategies for reaching a conclusion in a derivation. We might try using a rule of equivalence to convert the conclusion into a proposition with a different operator. MI converts conditionals into negated conjunctions or disjunctions, and vice versa, while DM converts negated conjunctions into disjunctions, and vice versa, or negated disjunctions into conjunctions, vice versa. Consider the following example:

Gill will go out with Jack only if he is a bachelor, and, he will go out with her only if she likes orange juice. But Jack is not a bachelor. And Gill does not like orange juice. So it's not the case that she will go out with him or he with her.

Let the inference be rendered as follows:

1. $g \supset b$	Premise	
2. $j \supset o$	Premise	
3. $\sim b$	Premise	
4. $\sim o$	Premise	Conclusion: $\sim(g \vee j)$

The conclusion is "$\sim(g \vee j)$". Since the conclusion is a negation, we might hope to derive it by using CC and so look for "$g \vee j$" as the antecedent of a conditional. But we do not see this among the premises. We suspect that an equivalence is necessary. We can convert the conclusion into the conjunction "$\sim g \& \sim j$" by using DM. The conjunction in turn suggests a conjunction of "$\sim g$" and "$\sim j$". "$\sim g$" can be derived by CC from lines 1 and 3; "$\sim j$" can also be derived by CC, from lines (2) and (4). The derivation can be written as follows:

1. $g \supset b$	Premise	
2. $j \supset o$	Premise	
3. $\sim b$	Premise	
4. $\sim o$	Premise	Conclusion: $\sim(g \vee j)$
5. $\sim g$	1, 3 CC	
6. $\sim j$	2, 4 CC	
7. $\sim g \& \sim j$	5, 6 Conj.	
8. $\sim(g \vee j)$	7 DM	

Derivation – Nine Rules Of Inference
& Seven Rules Of Equivalence

Matching the Antecedent (MA)
k. S ⊃ T
.
.
l. S
.
.
m. T k, l MA

Contradicting the Consequent (CC)
k. S ⊃ T
.
.
l. ~T
.
.
m. ~S k, l CC

Chain Inference (Chain)
k. S ⊃ T
.
.
l. T ⊃ U
.
.
m. S ⊃ U k, l Chain

Elimination Inference (Elim.)
k. S v T
.
.
l. ~S
.
.
m. T k, l Elim.

Constructive Dilemma (CD)
k. S ⊃ T
.
l. U ⊃ V
.
m. S v U
.
.
n. T v V k, l, m CD

Destructive Dilemma (DD)
k. S ⊃ T
.
l. U ⊃ V
.
m. ~T v ~V
.
.
n. ~S v ~U k, l, m DD

Simplification (Simp.)
k. S & T
.
l. S k, Simp.

Addition (Add.)
k. S
.
.
l. S v T k, Add.

Conjunction (Conj.)
k. S
.
l. T
.
m. S & T k, l Conj.

Double Negation (DN)
S <--> ~~S

Commutation (Comm.)
S v T <--> T v S
S & T <--> T & S

De Morgan's Rule (DM)
~(S & T) <--> ~S v ~T
~(S v T) <--> ~S & ~T

Exportation (Exp.)
S ⊃ (T ⊃ U) <--> (S & T) ⊃ U

Association (Ass.)
(S v T) v U <--> S v (T v U)
(S & T) & U <--> S & (T & U)

Material Implication (MI)
S ⊃ T <--> ~S v T
S ⊃ T <--> ~(S & ~T)

Transposition (Trans.)
S ⊃ T <--> ~T ⊃ ~S

Exercise Set (1) | Exercise Set (2)

8 Conditional Derivations & Indirect Derivations*

1. Oftentimes, people suppose something for the sake of argument and from this assumption infer something further. They can then conclude that, *if* the supposition is (or turns out to be) true, the conclusion follows. Such assumptions are said to be made "for the sake of argument" and can be used to think through the consequences of the supposition. Consider the following example:

> Let's suppose that the El (train) went by at midnight. If the El was going by, it would be really noisy. If there were a lot of noise, then the old man downstairs wouldn't have heard the struggle. So, if the El was going by, the old man wouldn't have heard the struggle. (Based on *12 Angry Men*)

2. The conclusion is a conditional, and it is inferred, in part, from an initial supposition of its antecedent. The consequent then follows from the supposition and the other premises. A conditional derivation takes the following general form:

```
.
.
    k.  S                           Assumption Cond.
        .
        .
    l.  U
m. S ⊃ U                            k-l Cond.
```

The initial dotted lines indicate that there might be additional lines prior to the conditional derivation. The conditional derivation might be only a portion of a complete derivation. Lines (k) through (l) are indented to indicate their conditional nature. That is, they are based on the assumption in line (k). Line (k) is labeled "Assumption Cond." which means that it is an assumption for the purpose of a conditional derivation. Line (m) is the conclusion of the conditional derivation. It is not indented because it is not dependent on any assumption. It is labeled with all of the numbers of the lines which are dependent on the assumption, and "Cond." for "conditional derivation".

The inference in the passage from *12 Angry Men* began ...

> Let's suppose that the El went by at midnight. If the El was going by, ...

The words "Let's suppose ..." indicate that the speaker is arguing conditionally, and the conclusion is appropriately a conditional one: *If* the El was going by, the old man wouldn't have heard the struggle.

The derivation which reaches this conclusion is written as follows, using obvious proposition letters:

1. e ⊃ n Premise
2. n ⊃ ~h Premise Conclusion: e ⊃ ~h
   3. e Assumption Cond.
   4. n 1, 3 MA
   5. ~h 2, 4 MA
6. e ⊃ ~h 3-5 Cond.

We write the other premises first, number them, and label them as premises. We indent the assumption, number it, and label it as an assumption for the purposes of a conditional derivation. The assumption is the antecedent ("e") of the conditional which appears in the conditional ("e ⊃ ~h") which will be derived. Once the derivation has reached the *consequent* of the conclusion ("~h") in the conditional derivation, we write the conditional on a non-indented line, number it, and label it with the *range* of lines used to arrive at it, and with the word "Cond." to indicate that it was arrived at by a conditional derivation.

3. Another conditional mode of inference is to derive a contradiction from a supposition and then infer the contradiction of the initial supposition. This mode of reasoning is called an *indirect inference*. Consider the following example:

> Home ownership will rise. Why so? Well, we know that if rates fall or gas prices fall, then home ownership rises. Now, suppose home ownership won't rise. So neither rates will fall nor gas prices. What's more, either gas prices will fall or inflation will rise. So, inflation will rise. And finally, if inflation rises then rates will fall and confidence will rise. So, rates will fall and confidence will rise, which is absurd. So, based on our assumption, rates will both rise and fall. There you have it: home ownership will rise.

Using an obvious translation key, the derivation goes as follows:

1. (r ∨ g) ⊃ h Premise
2. g ∨ i Premise
3. i ⊃ (r & c) Premise Conclusion: h
   4. ~h Assumption Ind.

5. ~(r v g)   1, 4 CC
6. ~r & ~g   5 DM
7. ~g & ~r   6 Comm.
8. ~g   7 Simp.
9. i   2, 8 Elim.
10. r & c   3, 9 MA
11. r   10 Simp.
12. ~r   6 Simp.
13. r & ~r   11, 12 Conj.
14. h   4-13 Ind.

As with derivations and conditional derivations, we begin by writing down the premises and conclusion. The assumption "~h" is then entered and labeled as an assumption. The assumption is indented, as are any derivations dependent upon the assumption. When a contradiction is reached, the un-negated assumption is entered on a non-indented line.

4. A indirect derivation has the following general pattern:

.
.
k. ~S   Assumption Ind.
.
.
l. T & ~T
m. S   k-l Ind.

The lines with dots indicate that there might be additional lines in the derivation. Lines (k) through (l) are indented to indicate their conditional nature. That is, they are based on the assumption in line (k). Line (k) is labeled "Assumption Ind." which means that it is an assumption for the purpose of an indirect derivation. The reason for assuming the negation of the conclusion is to show that it leads to a contradiction, which appears here on line (l). Line (m) is the conclusion of the indirect derivation. It is not indented because it is not dependent on any assumption. It is labeled with all of the numbers of the lines which are dependent on the assumption, and "Ind." for "indirect derivation".

5. An indirect derivation is also called a *reductio ad absurdum* because the general strategy is to show that something absurd or impossible (in the sense of contradictory) or undesirable follows if we grant some proposition. If a proposition can be shown to lead to something impossible or morally unconscionable or strongly undesirable, that is sufficient reason for accepting its denial. This method is used frequently in mathematics as, for example, in various of the proofs of Euclid, in order to show that a certain proposition is a necessity (given the other postulates and theorems of the system).

Conditional & Indirect Derivation

Conditional Derivation

.	
.	
k. S	Assumption Cond.
:	
l. U	
m. $S \supset U$	k-l Cond.

Indirect Derivation

.	
.	
k. ~S	Assumption Ind.
.	
.	
l. T & ~T	
m. S	k-l Ind.

Exercise Set (1) | Exercise Set (2)