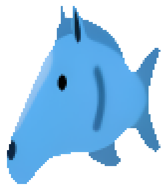
 i dont even care.

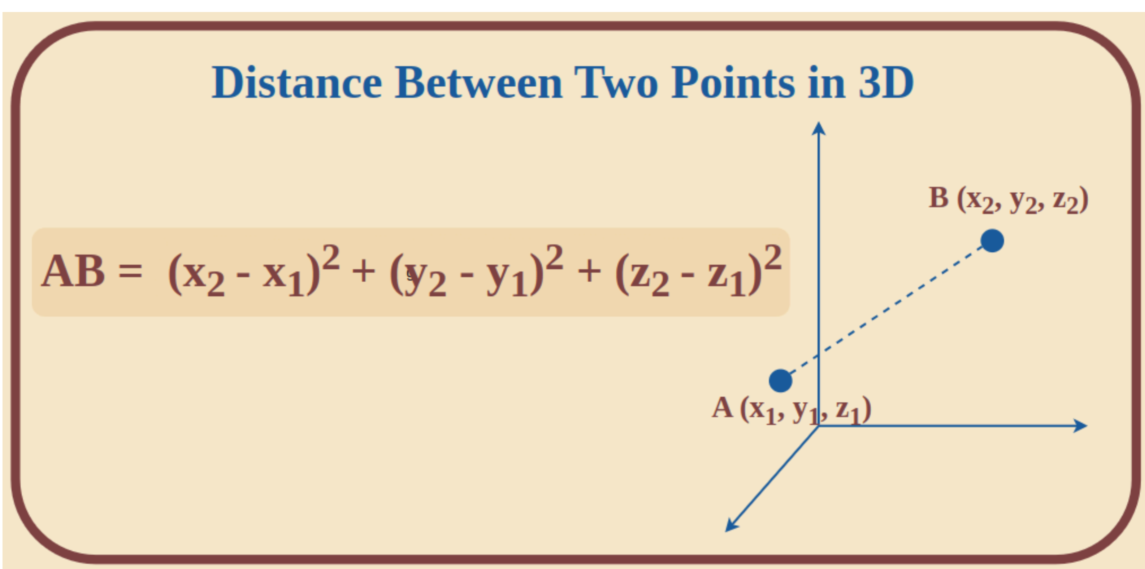
Why did you post your lecture notes in a publicly-shared document?

I'm not even in this class where did this come from

MODULE 1

LECTURE 1 3-Dimensional Rectangular System (8/23/24)

- 3d plane
- distance from point to xy plane = |z|
- distance from point to axis = find point on axis closest to it and use distance formula



LECTURE 2 Vectors (8/26/24)

- graphing plane through point
- Outline

- LECTURE 12: Partial Derivatives (9/25/24)

-
-
    - yz-plane through (x,y,z), just look at x
- circle in R^3 is a cylinder
- inequalities
- Vectors
    - represented by directed line segment (initial A terminal B) and length/magnitude |AB|
    - equal if same magnitude & direction, location doesn't matter
- Vector addition
    - tip to tail or parallelogram
- position vector has initial point at origin
- vector addition easy
- scalar multiplication easy
- properties of operations (vectors u,v,w scalars c,d)
    - u+v=v+u
    - (u+v)+w =u+(v+w)
    - u+0=u
    - u+(-u)=0
    - c(u+v)=cu+cv
    - (c+d)u=cu+du
    - 0u=0
    - c0=0
    - 1u=u
    - c(du)=(cd)u=d(cu)
- vector magnitude = sqrt(x^2+y^2+z^2)
- unit vector has length 1

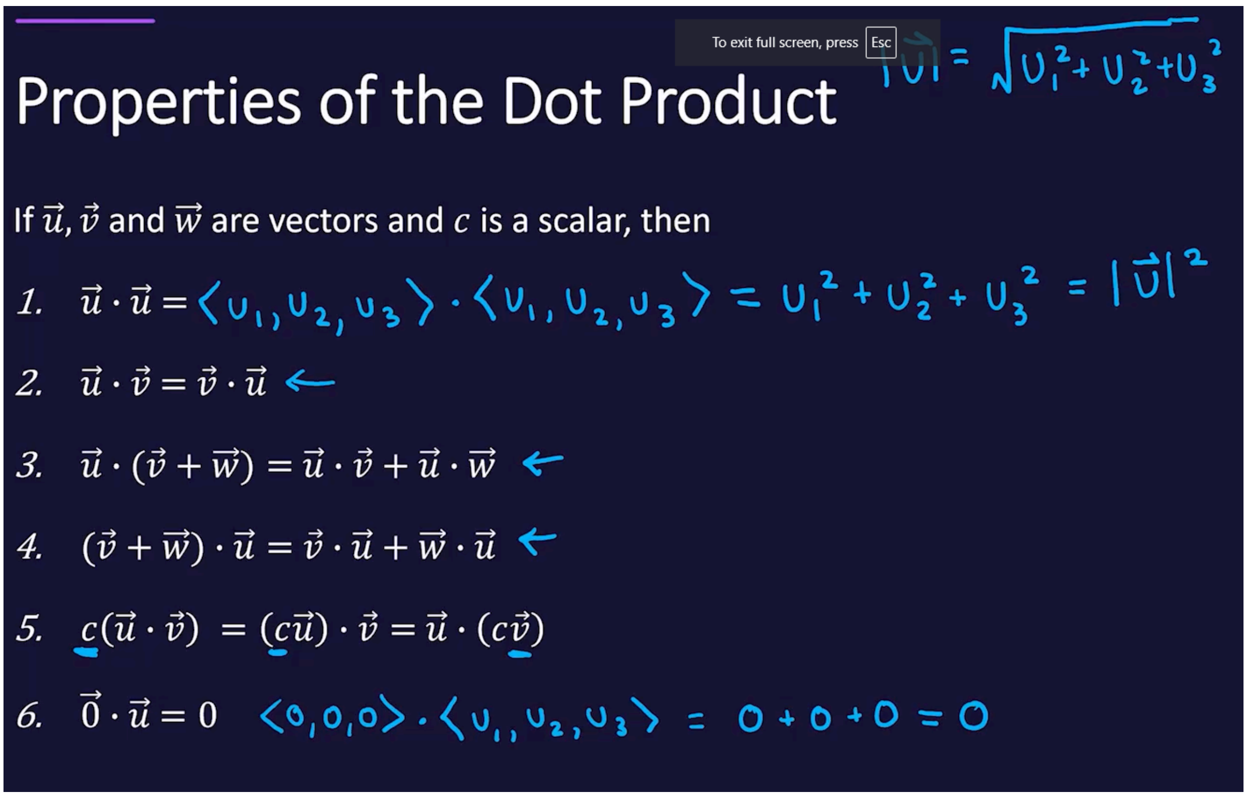
- to find unit vector, divide vector by magnitude
- standard basis vector notation
    - i=<1,0,0>vxcvxcvb
    - j=<0,1,0>
    - k=<0,0,1>
    - 487932i + 6824j - 2993k

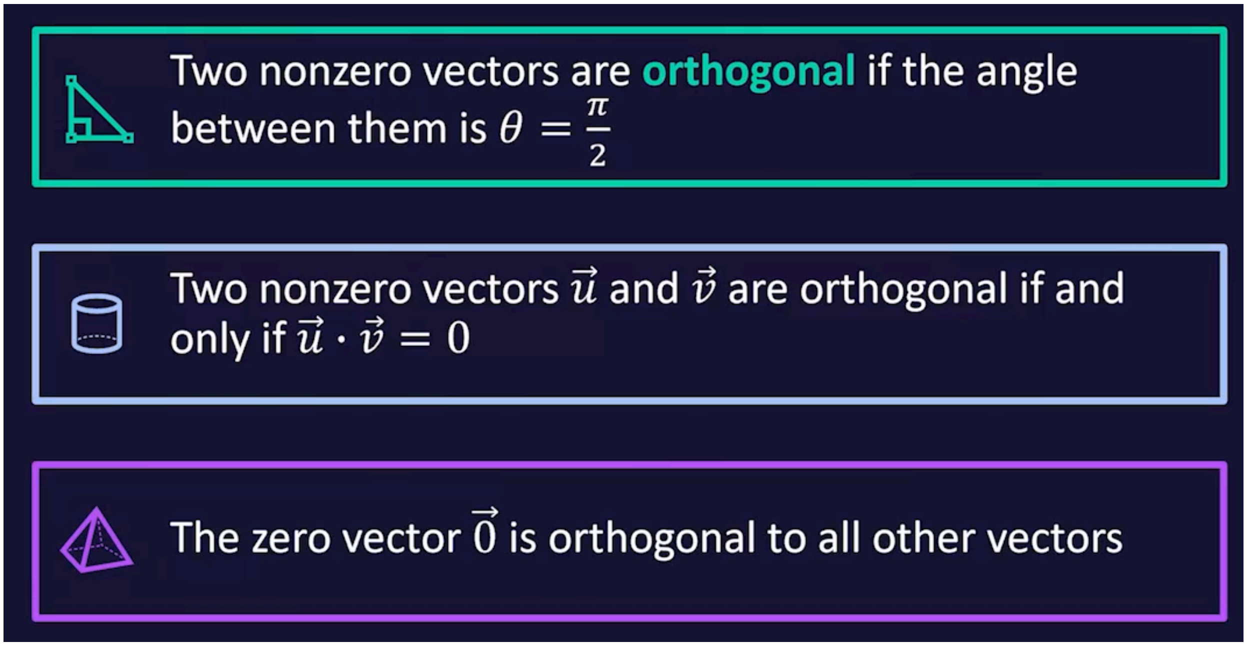
LECTURE 3 Dot Product (8/

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \cdot \|\vec{v}\|}$$

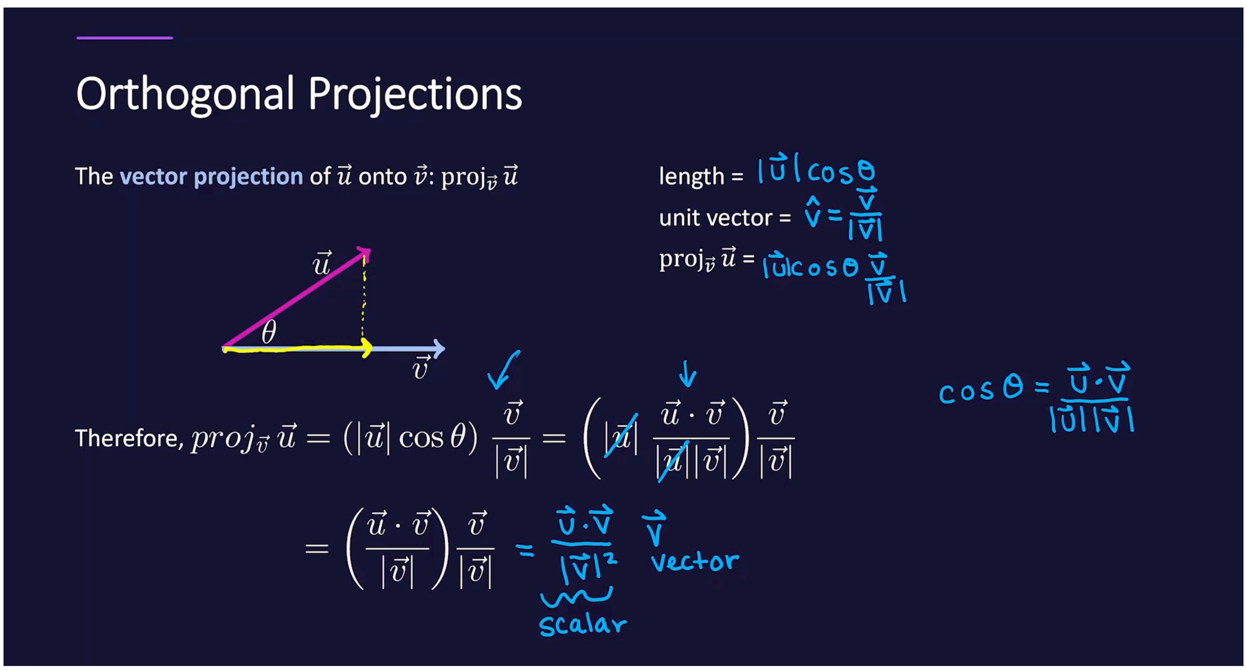
26/24)

- dot product - multiply the components then add it all up





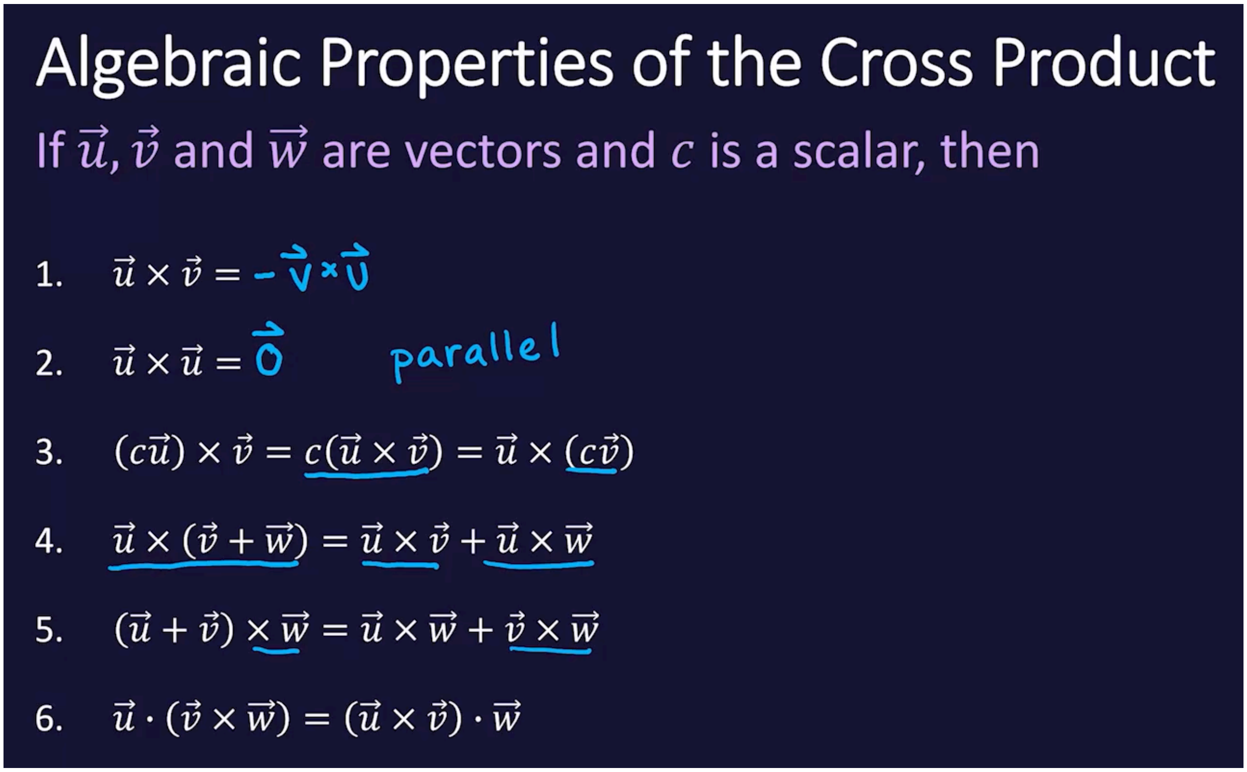
- vector projection (yellow arrow)



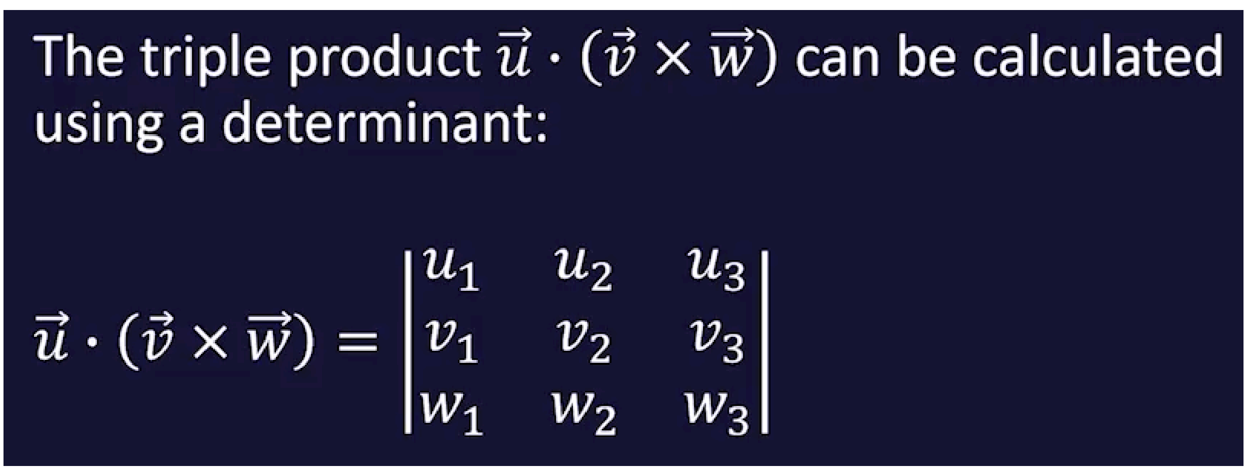
- WORK =
  - F • D
  - |F||D|cosθ

LECTURE 4 Cross Product (8/29/24)

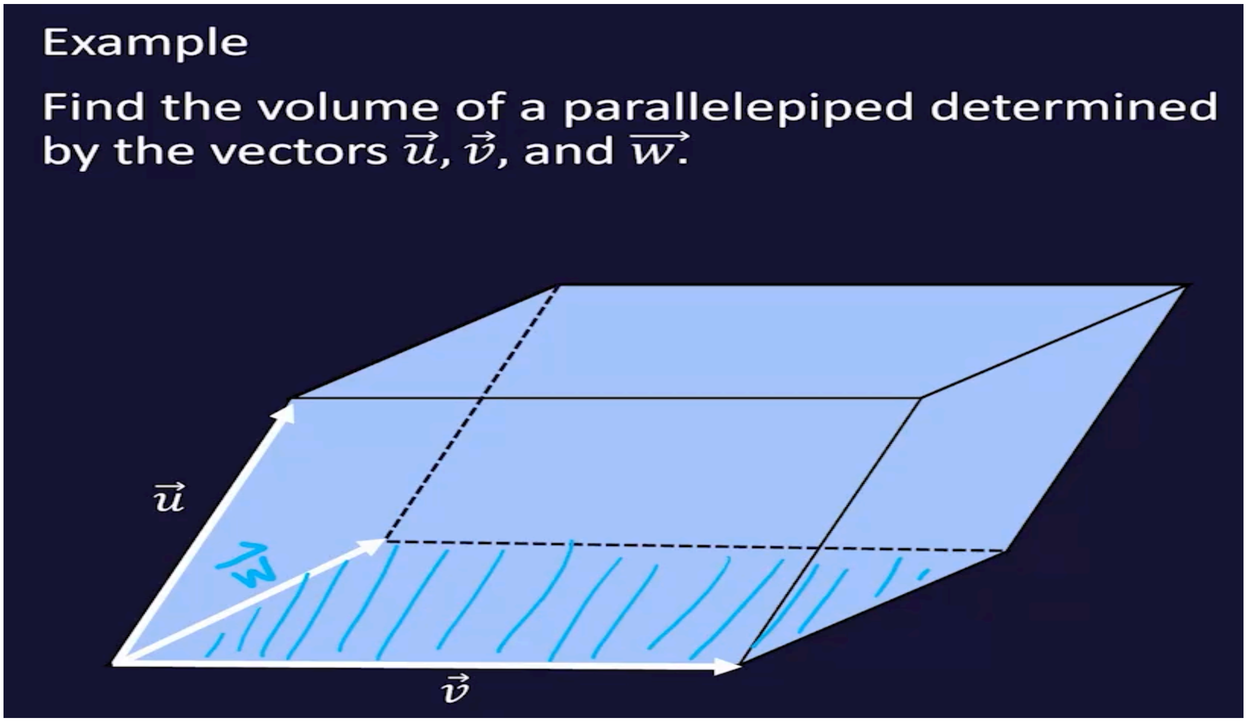
- cross product - easy



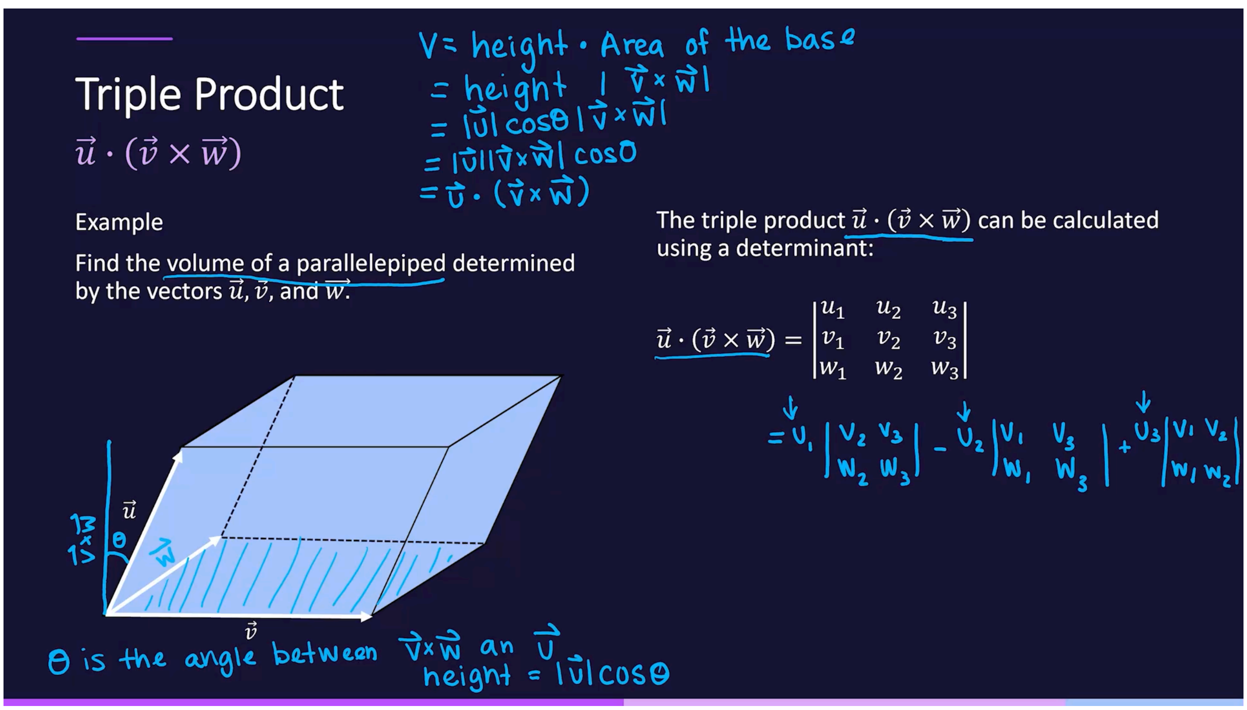
- triple product - not easy



- V = height x area of base



- the area of base is just area of parallelogram or magnitude of cross product |v x w|
- the height is perpendicular to v and w so it is v x w



LECTURE 5 Lines and Planes (9/3/24)

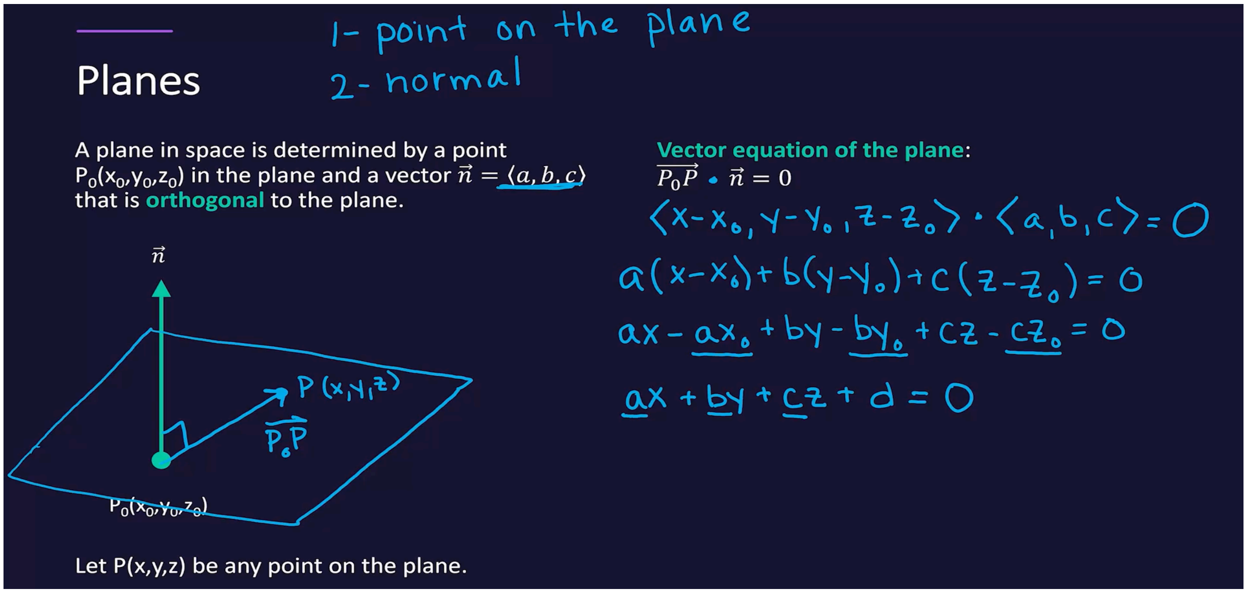
- to define line, we need point $P_0<x_0,y_0,z_0>$ and vector v = $<a,b,c>$ parallel to the line
    - $r = r_0+vt$ (t is a scalar multiple)
    - $<x_0,y_0,z_0>+t<a,b,c>$ (called vector equation)
- Line segment
    - $r(t) = (1-t)<x_0,y_0,z_0>+t<x_1,y_1,z_1>$
        - initial point at t=0, terminal point at t=1
        - will not be used until line integrals (module 4)
- Parametric equation - component by component

$$x = x_0 + at$$
$$y = y_0 + bt$$
$$z = z_0 + ct$$

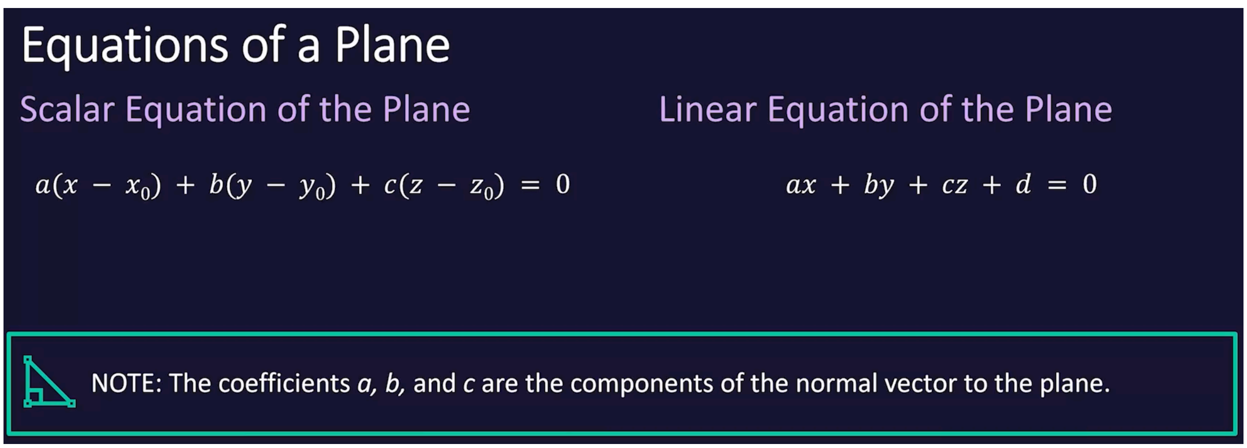
- Symmetric equation - solve for t

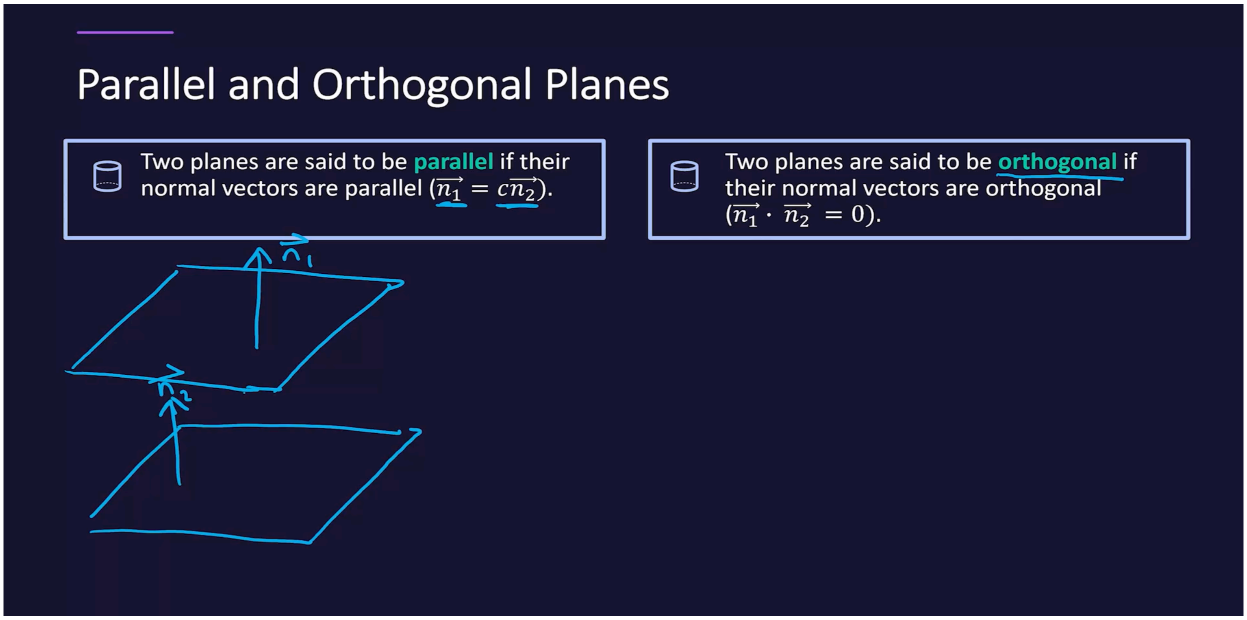
$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

- to define a plane, we need a point on the plane and a normal vector



- PRO STRAT FOR FIND PLANE WITH NORMAL & POINT
  - to find d take each component of normal and multiply it by each component of the point *-1

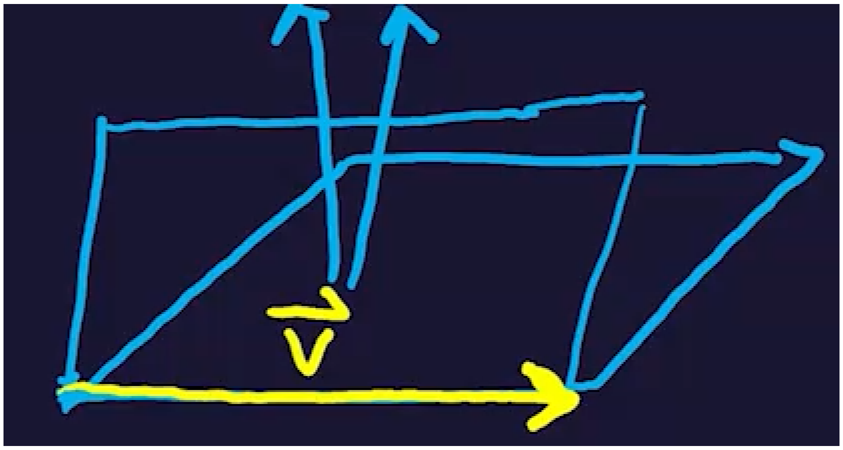




- literally just look at the normal vector to determine parallel/orthogonal/neither
  - if it's line vs plane it'll be reversed if using the vector of the line, so if the dot product is 0 they're parallel and if it looks parallel it's orthogonal. kinda stupid lol
- find the angle between two planes
  - just find angle between normal vectors using:

$$\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||n_2|}$$

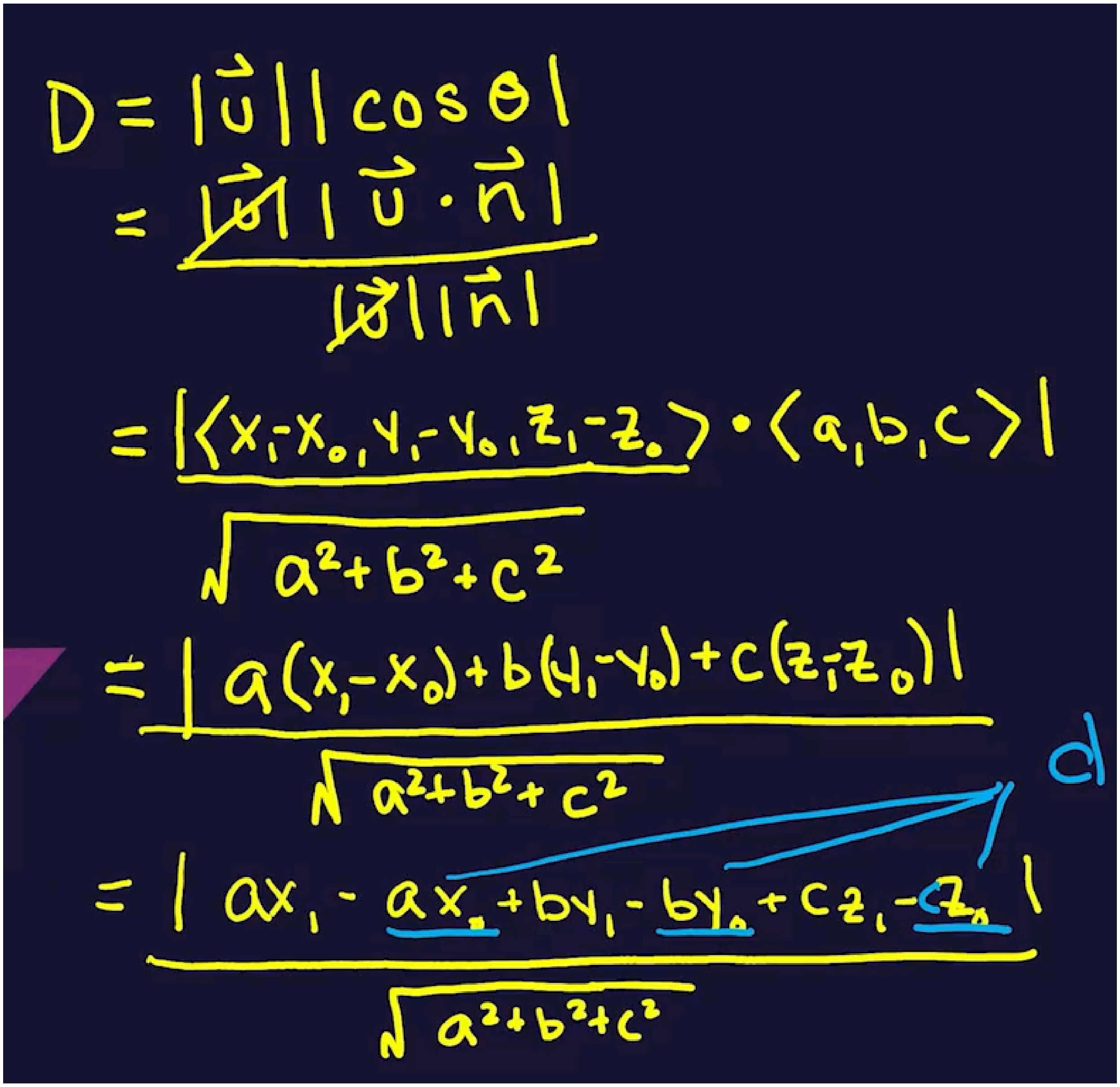
- find parametric equations for line of intersection of two planes



  - v is perpendicular to both normal vectors so cross both normal vectors to find v
  - to find a point on the line, fix a value (x/y/z) and solve for the other two (x/y/z) (this might not work maybe)
- find distance from point to plane

$$D = \frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$$

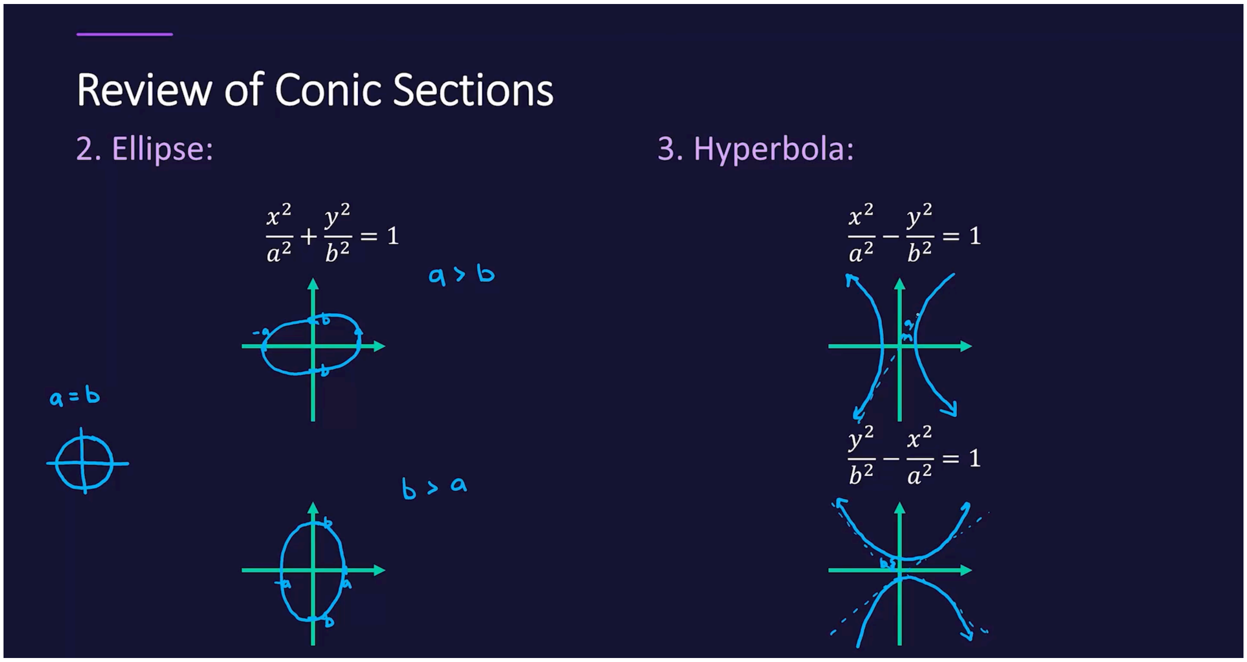
  - Proof:



- x_1,y_1,z_1 is the point and a,b,c,d is the plane
- find distance from two parallel planes
  - find a point on one of the planes then the problem is the same as point to plane

LECTURE 6 Cylinders and Quadric Surfaces (9/6/24)

- conic sections because i forgot



- whatever the - is on is the axis of the vertical/horizontal asymptote
  - technically they have diagonal asymptotes or something
- in this world a cylinder doesn't have to have a circle as a base, it can be a triangle or whatever tf
  - this makes any 2d graph in R^3 a cylinder i think
- NO DRAWING ON EXAMS!!!

A **quadric surface** is a 3D surface whose equation is of the second degree.

The general equation is $Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$, given that $A^2 + B^2 + C^2 + D^2 + E^2 + F^2 \neq 0$.

With rotation and translation, these possibilities can be reduced to two distinct types.

1. $Ax^2 + By^2 + Cz^2 + J = 0$ (All three quadratic terms; no linear terms)
2. $Ax^2 + By^2 + Iz = 0$ (Only two quadratic terms plus one linear term)

type 1 surfaces

Basic Quadric Surface

0 negative signs

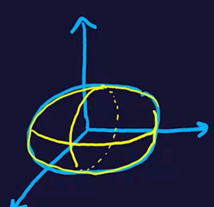
Ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$z=0$ $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ellipse

$y=0$ $\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1$ ellipse

$x=0$ $\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ ellipse



1 negative sign

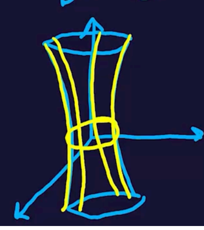
Hyperboloid of 1 sheet

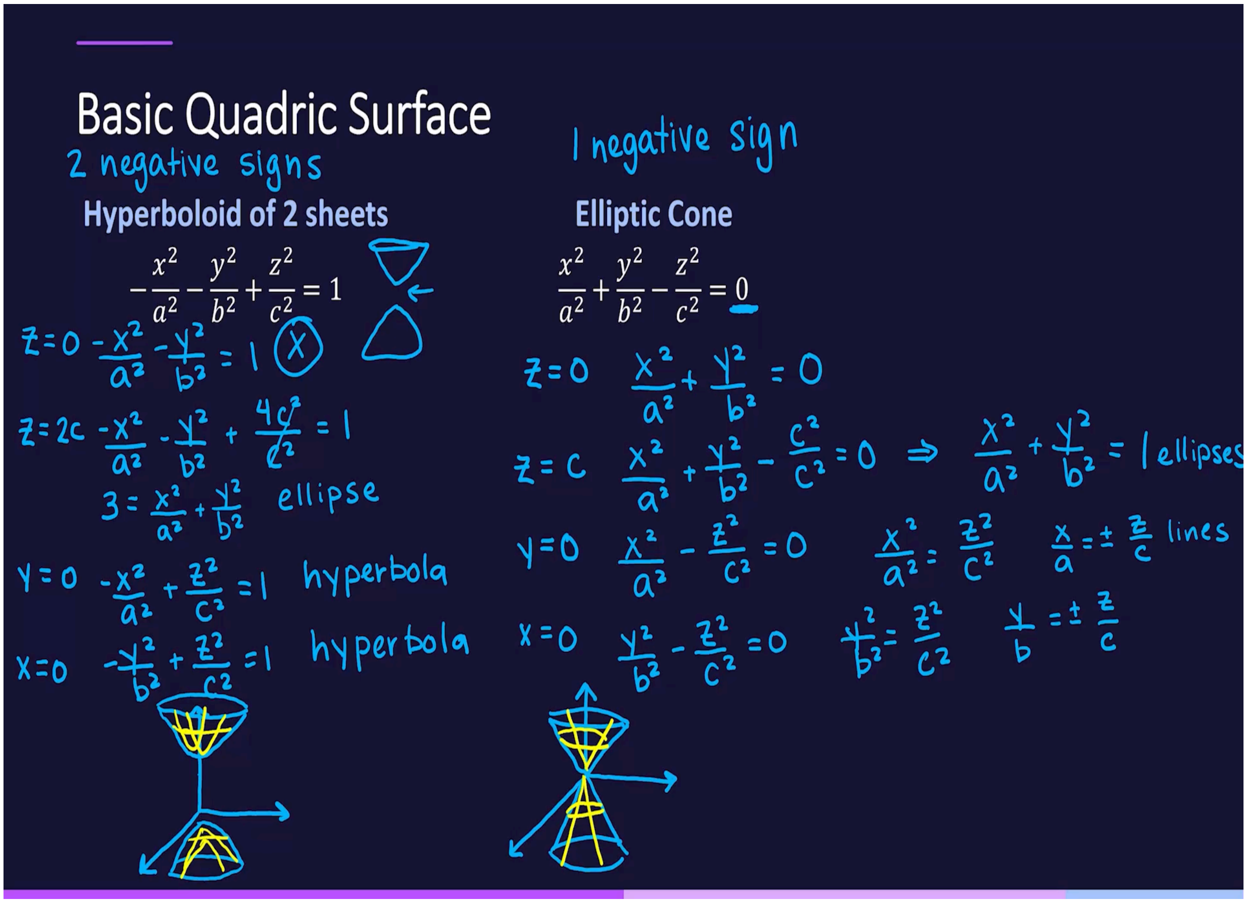
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

$z=0$ $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ellipse

$y=0$ $\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1$ hyperbola

$x=0$ $\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ hyperbola



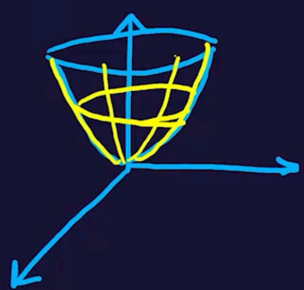


type 2 surfaces

Basic Quadric Surface

Elliptic Paraboloid

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$



Hyperbolic Paraboloid

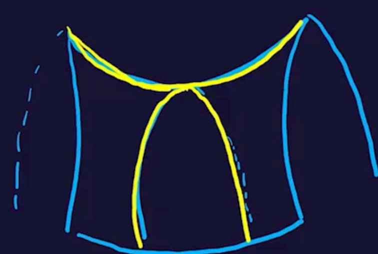
$$z = \frac{y^2}{b^2} - \frac{x^2}{a^2}$$

$z = c$

$c = \frac{y^2}{b^2} - \frac{x^2}{a^2}$ hyperbola

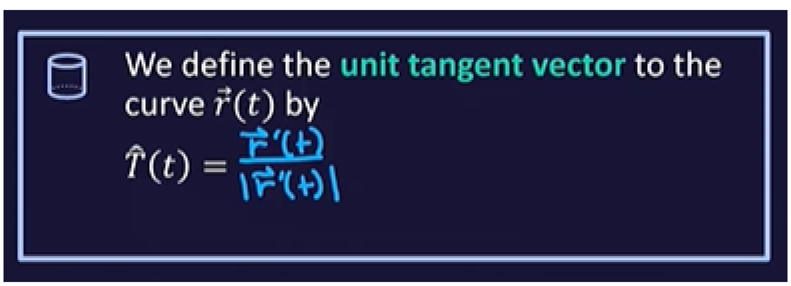
$y = 0 \quad z = -\frac{x^2}{a^2}$ parabola

$x = 0 \quad z = \frac{y^2}{b^2}$ parabola

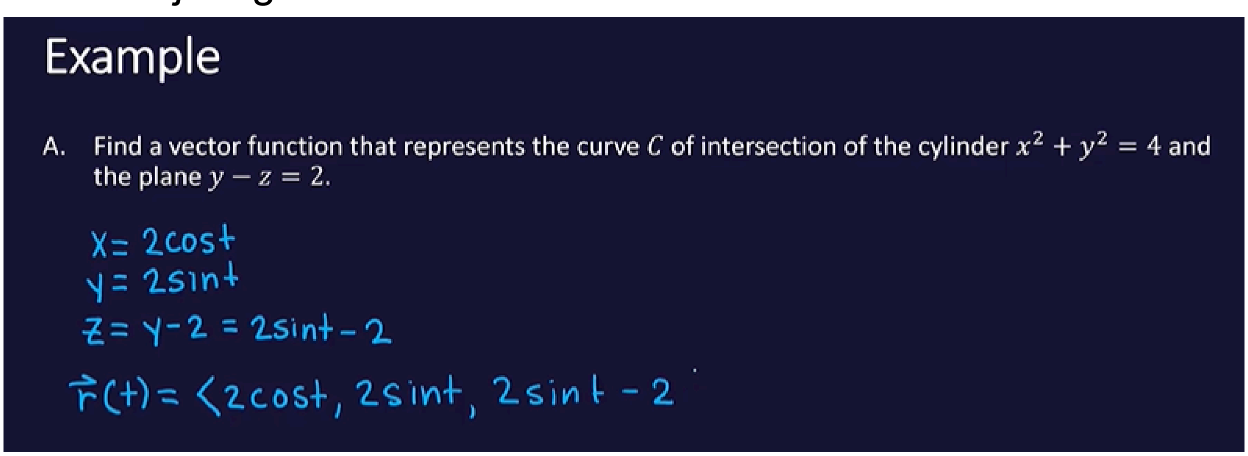


LECTURE 7 Vector-Valued Functions (9/9/24)

- domain of vector-valued function is intersection of domains of each component
- curve sketching
- helix
- the limit of vector-valued function is just taking limit of each component
  - if ANY component limit doesn't exist, the functions limit doesn't exist
- vector-valued function is continuous at a if
  - defined at each component
  - limit exists at each component
  - and they're equal (the point at a and the limit)
- first principle derivative XD
  - fuck that just derive each component



- this just gives direction ^



- there's a circle so sin and cos are used ^

Vector-Valued Functions

Definition

A curve $\vec{r}(t)$ is **smooth** on an interval I if

$\vec{r}'(t)$ is continuous and $\vec{r}'(t) \neq \vec{0}$

On I. A curve that is made up of a finite number of smooth curves pieced together in a continuous fashion is called **piece-wise smooth**.

Differentiation Rules

Let $\vec{u}$ and $\vec{v}$ be differentiable vector-valued functions, $f(t)$ a differentiable real-valued function, $\vec{C}$ a constant vector, and c a constant. Then

1. $\frac{d}{dt}\left[\vec{C}(t)\right] = \vec{0}$
2. $\frac{d}{dt}[\vec{u}(t) + \vec{v}(t)] = \vec{u}'(t) + \vec{v}'(t)$
3. $\frac{d}{dt}[c\,\vec{u}(t)] = c\,\vec{u}'(t)$
4. $\frac{d}{dt}[f(t)\vec{u}(t)] = f'(t)\vec{u}(t) + f(t)\vec{u}'(t)$
5. $\frac{d}{dt}[\vec{u}(t) \cdot \vec{v}(t)] = \vec{u}'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}'(t)$
6. $\frac{d}{dt}[\vec{u}(t) \times \vec{v}(t)] = \vec{u}'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}'(t)$
7. $\frac{d}{dt}[\vec{u}(f(t))] = \vec{u}'(f(t))\, f'(t)$

LECTURE 8 Arc Length and Curvature (9/11/24)

Arc Length

Extension to 3D

There is a natural extension of this to 3D. Consider a curve $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle, a \le t \le b$, where f', g', and h' are continuous. If the curve is traversed exactly once as t increases from a to b, then its length is

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}.$$

Arc Length Function

Definition

Suppose that C is a piecewise smooth curve given by a vector-valued function $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle, a \le t \le b$. The **arc length function** is defined

how far we've gone after some time t

$$s(t) = \int_a^t \sqrt{[f'(u)]^2 + [g'(u)]^2 + [h'(u)]^2}\, du$$

The Fundamental Theorem of Calculus tells us that s is differentiable function of t and

$$\frac{ds}{dt} = \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}$$

Reparameterization of a Curve

Arc Length

know the position

It is often useful to parameterize a curve with respect to arc length since arc length arises naturally from the shape of the curve and does not depend on a particular coordinate system. If a curve $\vec{r}(t)$ is given in terms of a parameter t and $s(t)$ is the arc length function, then

1. Solve for t as a function of s: $t = t(s)$
2. Reparametrize the curve in terms of s by substituting for t: $\vec{r} = \vec{r}\big(t(s)\big)$

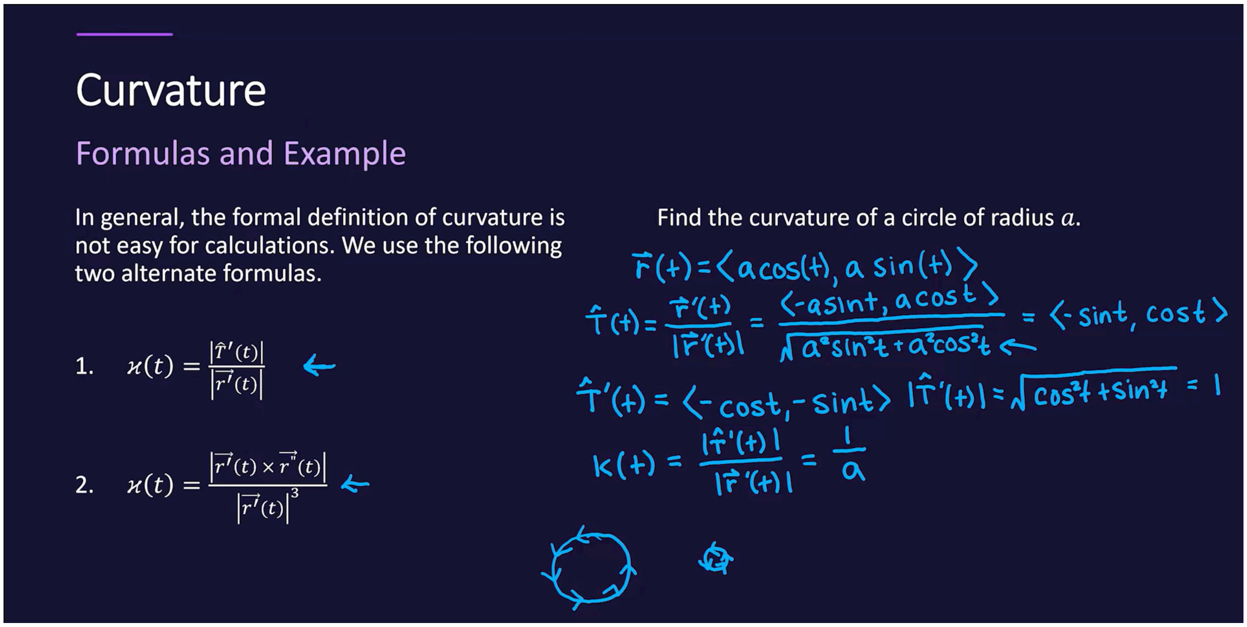
Example

$\cos t = 1$, $\sin t = 0$, $t = 0$ } $t = 0$

Reparametrize the helix $\vec{r}(t) = \cos t\,\hat{\imath} + \sin t\,\hat{\jmath} + t\,\hat{k}$ by its arc length starting from $(1,0,0)$ in the direction of increasing t. Use this information to determine the position after traveling $\sqrt{2}\,\pi$ units on the helix.

$$s(t) = \int_0^t \sqrt{(-\sin u)^2 + (\cos u)^2 + 1^2}\, du$$
$$= \int_0^t \sqrt{\sin^2 u + \cos^2 u + 1}\, du$$
$$= \int_0^t \sqrt{2}\, du$$
$$= \sqrt{2}\, u \Big|_0^t$$
$$s = \sqrt{2}\, t$$
$$t = \frac{s}{\sqrt{2}}$$ want $\vec{r}$ dependent on s

$$\vec{r}(s) = \left\langle \cos\left(\frac{s}{\sqrt{2}}\right), \sin\left(\frac{s}{\sqrt{2}}\right), \frac{s}{\sqrt{2}} \right\rangle$$
$$\vec{r}(\sqrt{2}\pi) = \langle \cos(\pi), \sin(\pi), \pi \rangle$$



Curvature

Formulas and Example

In general, the formal definition of curvature is not easy for calculations. We use the following two alternate formulas.

1. $\varkappa(t) = \frac{|\hat{T}'(t)|}{|\vec{r}'(t)|}$ ←

2. $\varkappa(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}$ ←

Find the curvature of a circle of radius a.

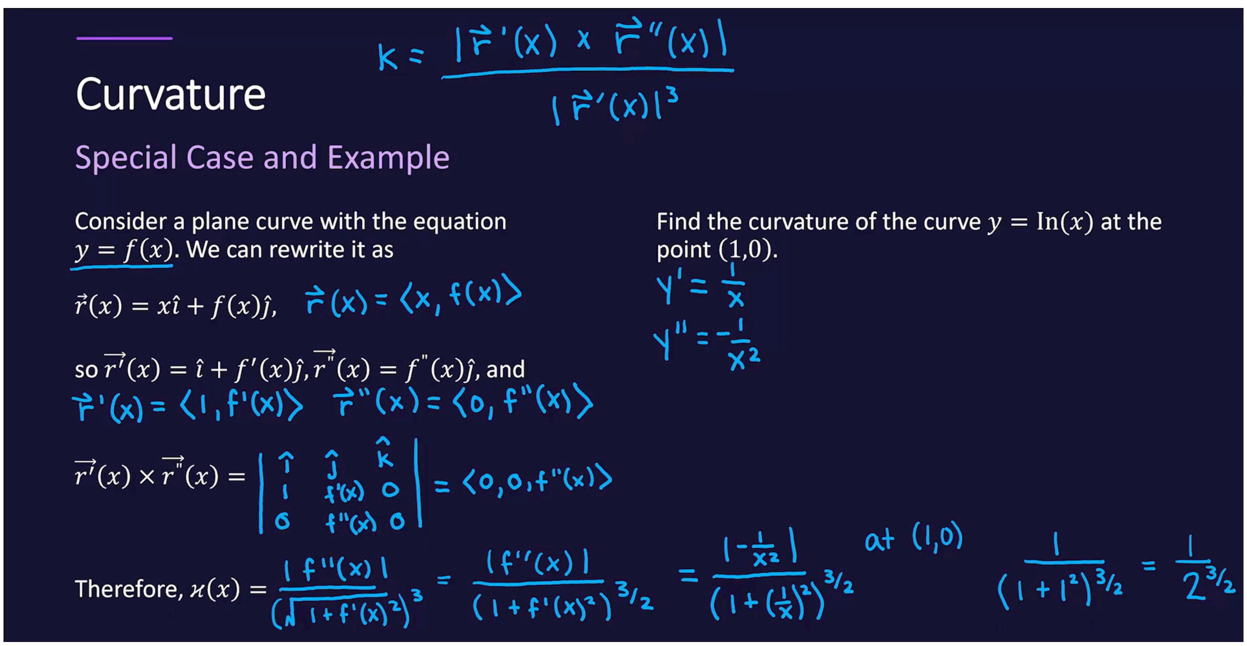
$\vec{r}(t) = \langle a\cos(t), a\sin(t) \rangle$

$\hat{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \frac{\langle -a\sin t, a\cos t \rangle}{\sqrt{a^2\sin^2 t + a^2\cos^2 t}} = \langle -\sin t, \cos t \rangle$

$\hat{T}'(t) = \langle -\cos t, -\sin t \rangle \quad |\hat{T}'(t)| = \sqrt{\cos^2 t + \sin^2 t} = 1$

$k(t) = \frac{|\hat{T}'(t)|}{|\vec{r}'(t)|} = \frac{1}{a}$

- curvature formula ^



$k = \frac{|\vec{r}'(x) \times \vec{r}''(x)|}{|\vec{r}'(x)|^3}$

Curvature

Special Case and Example

Consider a plane curve with the equation $y = f(x)$. We can rewrite it as

$\vec{r}(x) = x\hat{\imath} + f(x)\hat{\jmath}$, $\quad \vec{r}(x) = \langle x, f(x) \rangle$

so $\vec{r}'(x) = \hat{\imath} + f'(x)\hat{\jmath}$, $\vec{r}''(x) = f''(x)\hat{\jmath}$, and

$\vec{r}'(x) = \langle 1, f'(x) \rangle \quad \vec{r}''(x) = \langle 0, f''(x) \rangle$

$\vec{r}'(x) \times \vec{r}''(x) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & f'(x) & 0 \\ 0 & f''(x) & 0 \end{vmatrix} = \langle 0, 0, f''(x) \rangle$

Therefore, $\varkappa(x) = \frac{|f''(x)|}{\left(\sqrt{1+f'(x)^2}\right)^3} = \frac{|f''(x)|}{(1+f'(x)^2)^{3/2}} = \frac{\left|-\frac{1}{x^2}\right|}{\left(1+\left(\frac{1}{x}\right)^2\right)^{3/2}}$

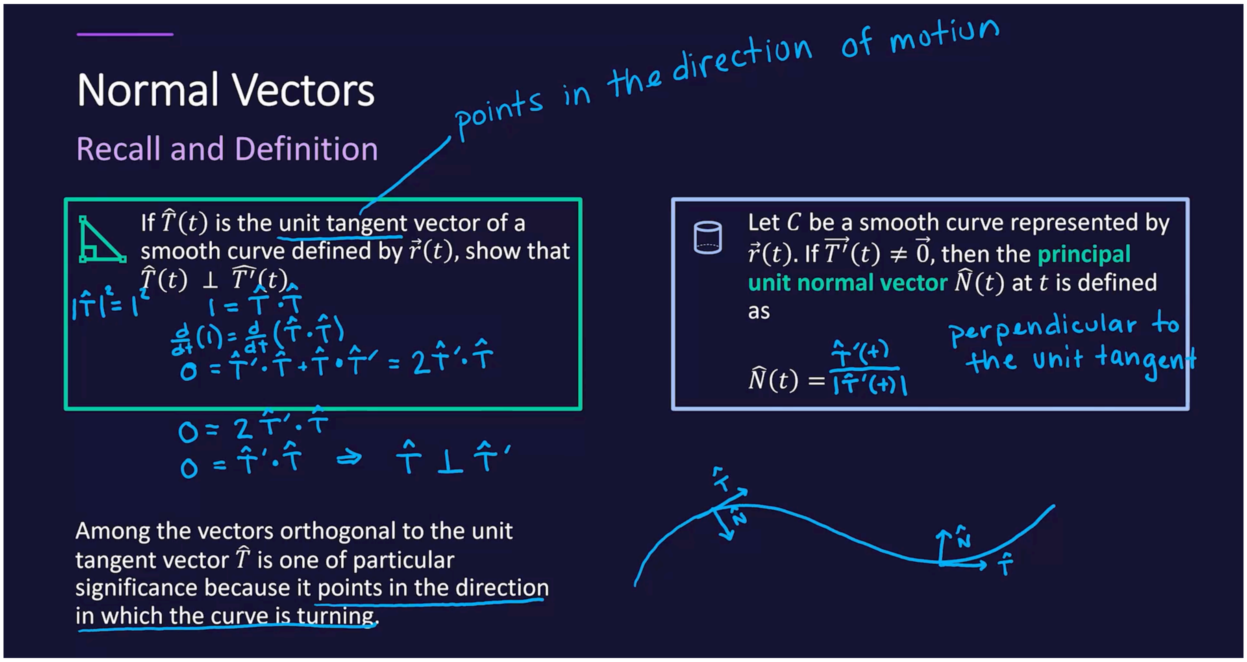
at $(1,0)$: $\frac{1}{(1+1^2)^{3/2}} = \frac{1}{2^{3/2}}$

Find the curvature of the curve $y = \text{In}(x)$ at the point $(1,0)$.

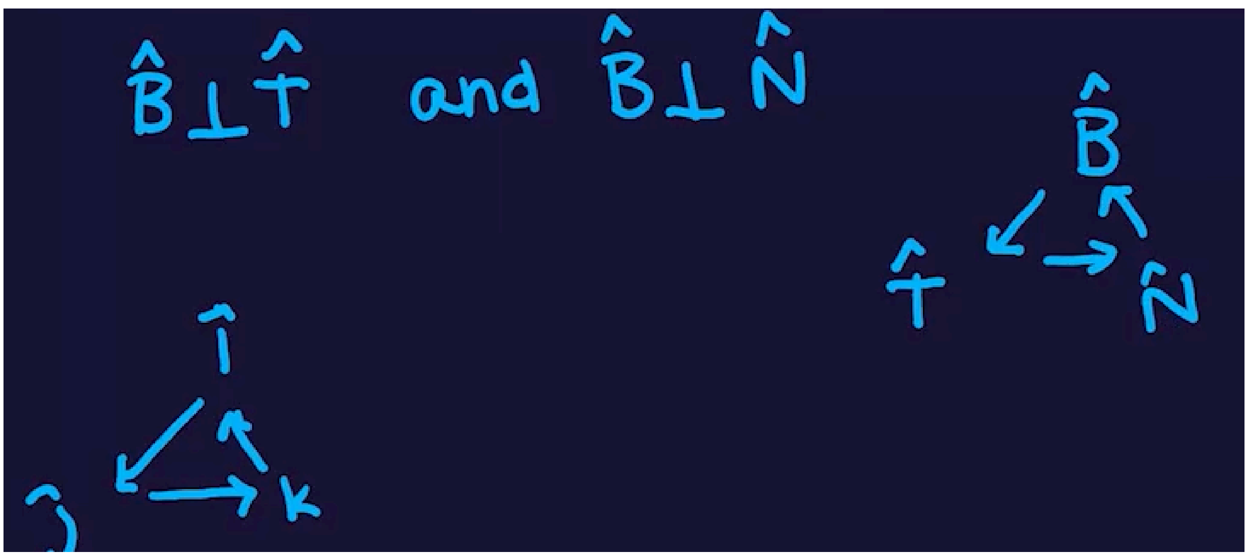
$y' = \frac{1}{x}$

$y'' = -\frac{1}{x^2}$

LECTURE 9: Motion in Space (9/14/24)



- bruh i needed this for HOMEWORK 8



- yo this triangle thing is cracked for cross product instead of right hand rule
- B is binormal vector its T x N

Find the principal unit normal vector of the helix

$$\vec{r}(t) = 2\cos(t)\hat{\imath} + 2\sin(t)\hat{\jmath} + t\hat{k}.$$

$$\hat{N} = \frac{T'(t)}{|T'(t)|} \qquad \hat{T} = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

$$\vec{r}'(t) = \langle -2\sin t, 2\cos t, 1 \rangle$$

$$|\vec{r}'(t)| = \sqrt{4\sin^2 t + 4\cos^2 t + 1} = \sqrt{5}$$

$$\hat{T} = \frac{1}{\sqrt{5}} \langle -2\sin t, 2\cos t, 1 \rangle$$

$$\hat{T}' = \frac{1}{\sqrt{5}} \langle -2\cos t, -2\sin t, 0 \rangle$$

$$|\hat{T}'| = \frac{1}{\sqrt{5}} \sqrt{4\cos^2 t + 4\sin^2 t} = \frac{2}{\sqrt{5}}$$

$$\hat{N} = \frac{\frac{1}{\sqrt{5}} \langle -2\cos t, -2\sin t, 0 \rangle}{\frac{2}{\sqrt{5}}}$$

$$= \langle -\cos t, -\sin t, 0 \rangle$$

Motion Along a Curve

Definition and Example

Let C be a smooth curve represented by $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$. The velocity vector, speed, and acceleration vector at time t are defined as follows.

Velocity $= \vec{v}(t) = \vec{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$

Speed $= |\vec{v}(t)| = |\vec{r}'(t)| = \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}$

Acceleration $= \vec{a}(t) = \vec{r}''(t) = \vec{v}'(t) = \langle f''(t), g''(t), h''(t) \rangle$

Find the velocity, speed, and acceleration of a particle with position vector

$$\vec{r}(t) = \langle 2\cos(t), 2\sin(t), t \rangle.$$

$$\vec{r}'(t) = \langle -2\sin t, 2\cos t, 1 \rangle$$

$$\text{Speed} = |\vec{r}'(t)| = \sqrt{4\sin^2 t + 4\cos^2 t + 1} = \sqrt{5}$$

$$\vec{a} = \vec{r}''(t) = \langle -2\cos t, -2\sin t, 0 \rangle$$

Components of Acceleration

Speed and Direction

We now use the vectors $\hat{T}$ and $\hat{N}$ to gain insight into how moving objects accelerate. The two ways to change the velocity of an object are to change its **speed** and change its **direction** of motion. We now show that changing the speed produces acceleration **in the direction of $\hat{T}$** and changing the direction produces acceleration **in the direction of $\hat{N}$**.

tangential — normal — centripetal force

Components of Accleration

Recall

$$\hat{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \frac{\vec{v}(t)}{|\vec{v}(t)|}$$

then $\vec{v} = |\vec{v}|\hat{T}$

and $\vec{a} = \frac{d}{dt}(\vec{v}) = \frac{d}{dt}|\vec{v}|\hat{T}$

$$= |\vec{v}|'\hat{T} + |\vec{v}|\hat{T}'\frac{|\hat{T}'(t)|}{|T'(t)|}$$

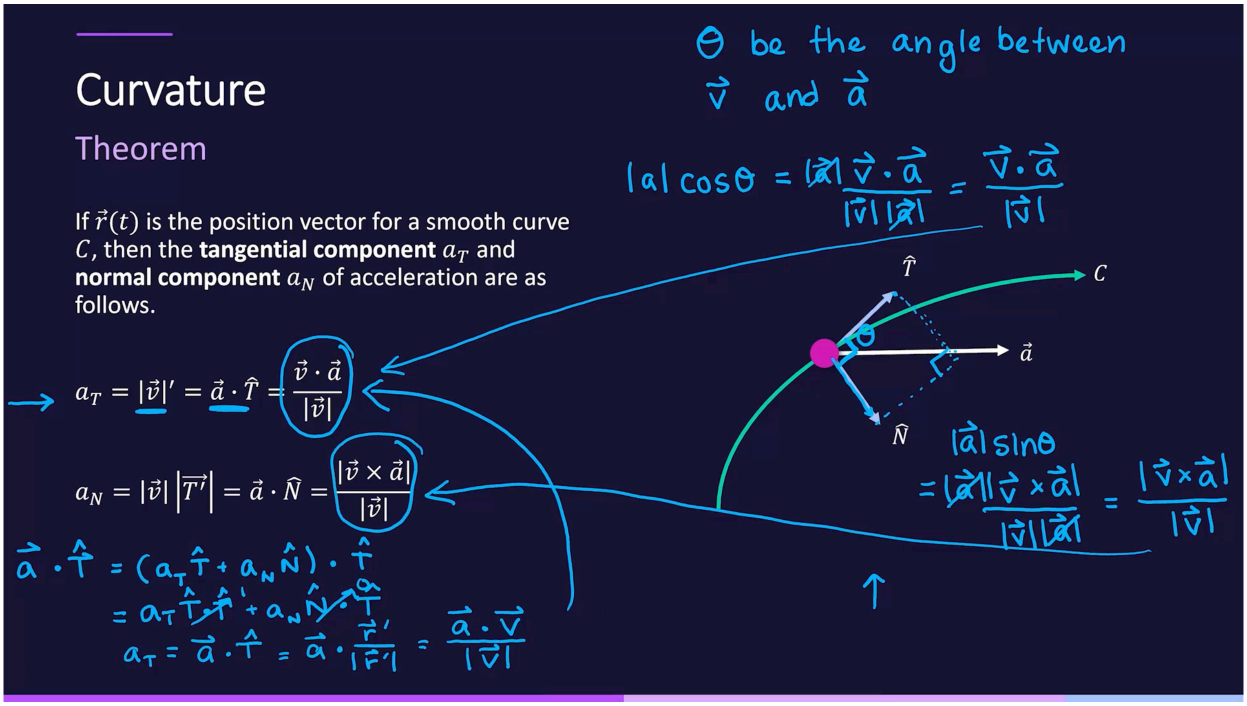
$$= |\vec{v}|'\hat{T} + |\vec{v}||\hat{T}'(t)|\frac{\hat{T}'}{|\hat{T}'(t)|}$$

$$= \underbrace{|\vec{v}|'}_{a_T}\hat{T} + \underbrace{|\vec{v}||\hat{T}'|}_{a_N}\hat{N}$$

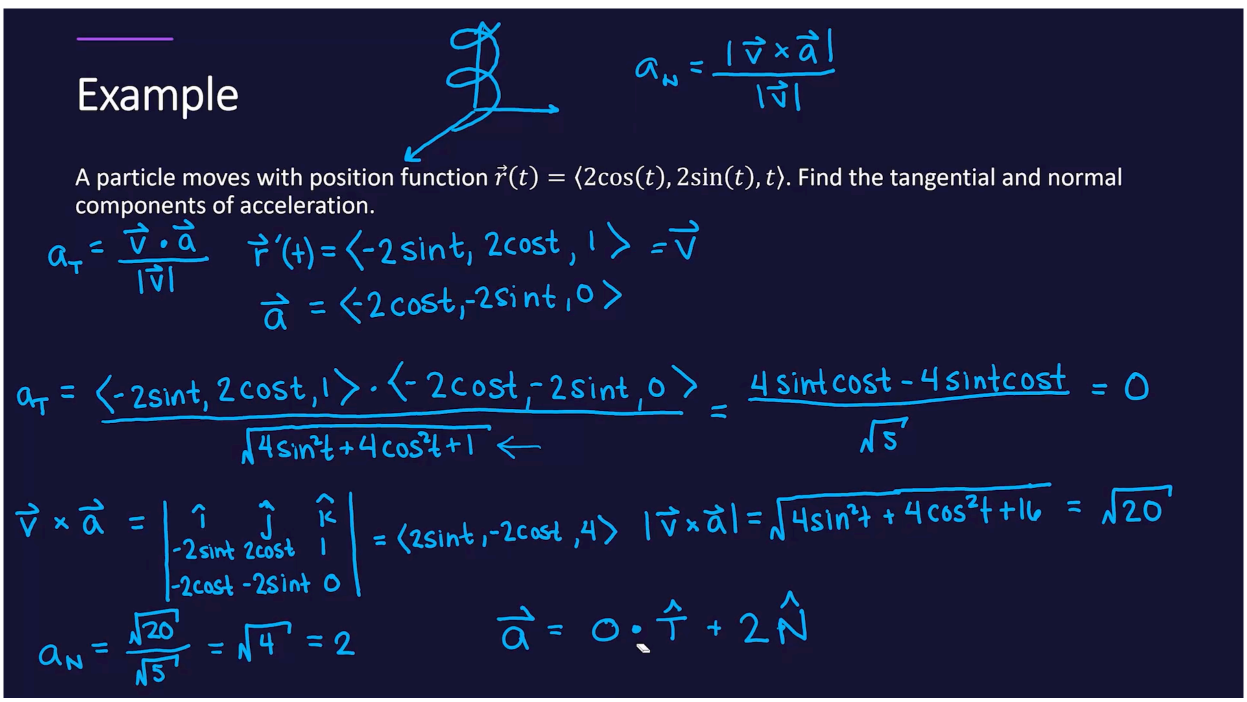
tangential → slide; ← normal spins

Because $\vec{a}$ is written as a linear combination of $\hat{T}$ and $\hat{N}$, it follows that the acceleration vector $\vec{a}$ lies in the plane determined by $\hat{T}$ and $\hat{N}$.

- "this isn't how we compute these parts"



- the circled equations are how you compute them



- WTF EQUATION SUMMARY

Formulas for Curves in Space

Summary

Position function: $\overrightarrow{r(t)} = \langle f(t), g(t), h(t)\rangle$ ←

Velocity: $\vec{v} = \vec{r}'$

Acceleration: $\vec{a} = \vec{v}' = \vec{r}''$ ←

Unit tangent vector: $\hat{T} = \frac{\vec{v}}{|\vec{v}|}$ direction particle is moving in

Principal unit normal vector: $\hat{N} = \frac{\hat{T}'}{|\hat{T}'|}$

how the curve changes direction

Unit binormal vector: $\hat{B} = \hat{T} \times \hat{N}$

Curvature: $\varkappa = \left|\frac{d\hat{T}}{ds}\right| = \frac{|\hat{T}'|}{|\vec{v}|} = \frac{|\vec{v}\times\vec{a}|}{|\vec{v}|^3}$ how fast a curve is changing along a distance

Components of acceleration: $\vec{a} = a_T\hat{T} + a_N\hat{N}$

with $a_T = \vec{a}\cdot\hat{T} = |\vec{v}|' = \frac{\vec{v}\cdot\vec{a}}{|\vec{v}|}$

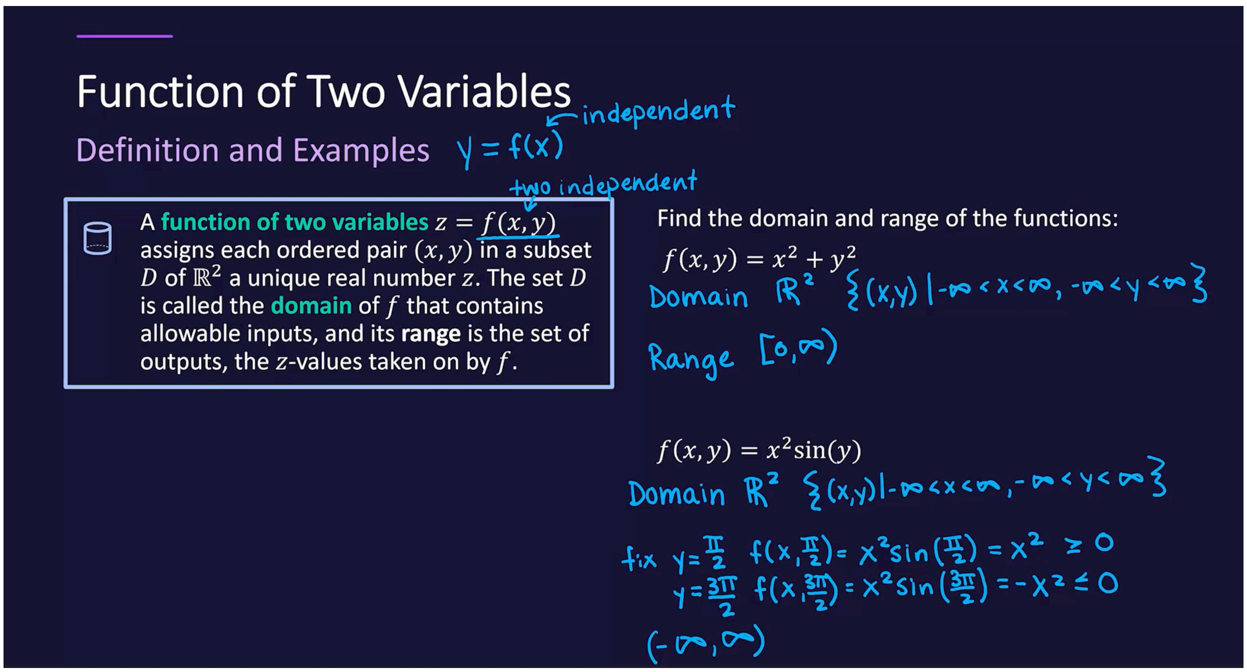
and $a_N = \vec{a}\cdot\hat{N} = |\vec{v}||\hat{T}'| = \frac{|\vec{v}\times\vec{a}|}{|\vec{v}|}$

- NVM THESE ARE GIVEN ON EXAM BRUHH

MODULE 2

LECTURE 10: Functions of Several Variables (9/19/24)

- domain = number we can't put in (log of negative, sqrt of negative, etc.)
- range = range of possible outputs



Examples

Find the domain and range of the function:

$$f(x,y) = \frac{x}{y-3}$$

Domain $\{(x,y) \mid -\infty < x < \infty, y \neq 3\}$

Fix $y=4$ $f(x,4) = \frac{x}{4-3} = x$

Range $(-\infty, \infty)$

Find the domain and range of the function:

$$f(x,y) = \sqrt{1 - x^2 - y^2}$$

$1 - x^2 - y^2 \geq 0$

$x^2 + y^2 \leq 1$

Domain $\{(x,y) \mid x^2 + y^2 \leq 1\}$

Range $[0,1]$

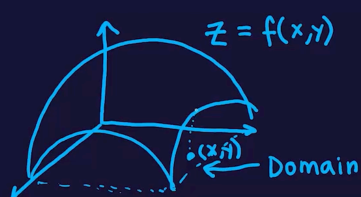
- Shadow underneath surface is the domain

Graphs

Definition and Note

The **graph** of a function f of two variables is the set of all points (x, y, z) for which $z = f(x, y)$ and (x, y) is in the domain of f. This graph can be interpreted geometrically as a **surface** in space, as discussed in Lecture 6.

NOTE: The graph of $z = f(x, y)$ is a surface whose projection onto the xy-plane is D, the domain of f.

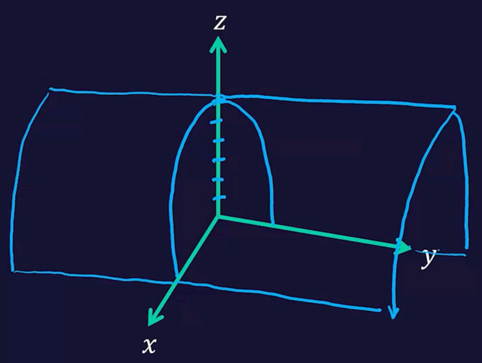


Examples

Find the domain and range of the function:

$z = f(x, y) = 6 - x^2$. Describe the graph of f.

Domain is $\mathbb{R}^2$ Range $(-\infty, 6]$



$16 - 4x^2 - y^2 \geq 0$

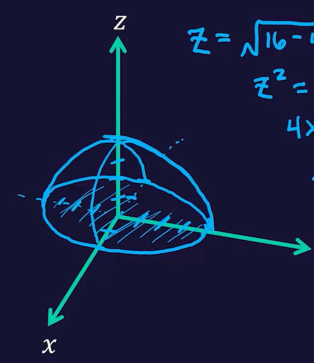
Domain $\{(x,y) \mid 4x^2 + y^2 \leq 16\}$

$\frac{x^2}{4} + \frac{y^2}{16} \leq 1$ ellipse

Find the domain and range of the function:

$f(x, y) = \sqrt{16 - 4x^2 - y^2}$. Range $[0, 4]$

Describe the graph of f.



$z = \sqrt{16 - 4x^2 - y^2}$

$z^2 = 16 - 4x^2 - y^2$

$4x^2 + y^2 + z^2 = 16$

$\frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{16} = 1$

↑ ellipsoid

Level Curves

Definition

We noted that traces could be used to help visualize the shape of surfaces and now we introduce the related concept of a level curve (or contour curve).

Given a function $f(x, y)$ and a number k in the range of f, a **level curve** of f for the value k is defined to be the set of points satisfying the equation.

$$f(x,y) = k$$

A collection of level curves of f is called a **contour map**.

Level Curves

Example

K values have to be in the range

$k=0 \quad 0=\sqrt{x^2+y^2} \quad 0=x^2+y^2$

$k=1 \quad 1=\sqrt{x^2+y^2} \quad 1=x^2+y^2$

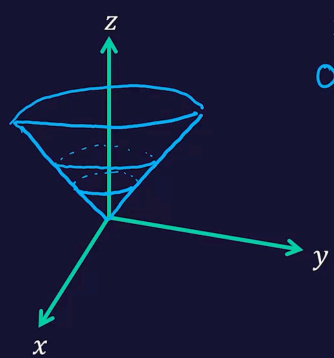
$k=2 \quad 2=\sqrt{x^2+y^2} \quad 4=x^2+y^2$

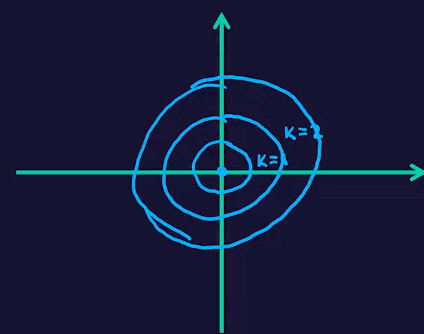
Describe the surface and the level curves for $z = f(x,y) = \sqrt{x^2+y^2}$.

$z=\sqrt{x^2+y^2}$ $[0,\infty)$

$z^2=x^2+y^2$

$0=x^2+y^2-z^2$

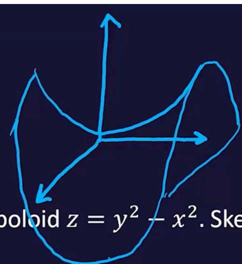




Level Curves

Example

Range $\mathbb{R}$



Consider the hyperbolic paraboloid $z = y^2 - x^2$. Sketch a contour map of this surface.

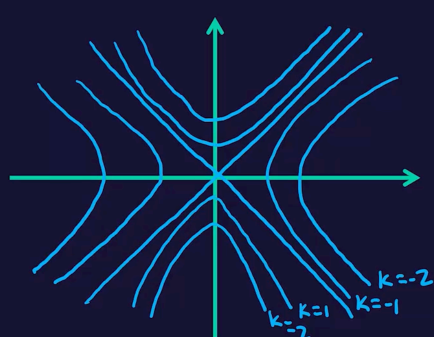
$k=-2 \quad y^2-x^2=-2 \quad x^2-y^2=2$

$k=-1 \quad y^2-x^2=-1 \quad x^2-y^2=1$ hyperbola

$k=0 \quad y^2-x^2=0 \quad y^2=x^2 \quad y=\pm x$

$k=1 \quad y^2-x^2=1$ hyperbola

$k=2 \quad y^2-x^2=2$



Level Surfaces

Definition

We have examined functions of two variables and now we can extend the concepts naturally to functions of three or more variables. With the function $w = f(x, y, z)$, level curves become **level surfaces** $w = f(x, y, z) = k$, where k is in the range of f.

# Independent Variables	Explicit Form	Implicit Form	Graph Resides In	
1	$y = f(x)$	$F(x, y) = 0$	$\mathbb{R}^2$	
2	$z = f(x, y)$	$F(x, y, z) = 0$	$\mathbb{R}^3$	level curve
3	$w = f(x, y, z)$	$F(x, y, z, w) = 0$	$\mathbb{R}^4$	level surface

Example

Find the domain and range of the function $f(x, y, z) = \ln(16 - 4x^2 - 4y^2 - z^2)$.

$16 - 4x^2 - 4y^2 - z^2 > 0$

Domain $\{(x,y,z) \mid 4x^2 - 4y^2 - z^2 < 16\}$

Range $(-\infty, \ln 16]$

$\lim_{x \to 0^-} \ln x = -\infty$

Characterize the level surfaces of the function $f(x, y, z) = \ln(16 - 4x^2 - 4y^2 - z^2)$.

$k = \ln 16$

$\ln(16 - 4x^2 - 4y^2 - z^2) = \ln 16$

$16 - 4x^2 - 4y^2 - z^2 = 16$

$4x^2 + 4y^2 + z^2 = 0$ point at the origin

$k < \ln 16$

$\ln(16 - 4x^2 - 4y^2 - z^2) = k$

$16 - 4x^2 - 4y^2 - z^2 = e^k$

$16 - e^k = 4x^2 + 4y^2 + z^2 \Rightarrow$ ellipsoid

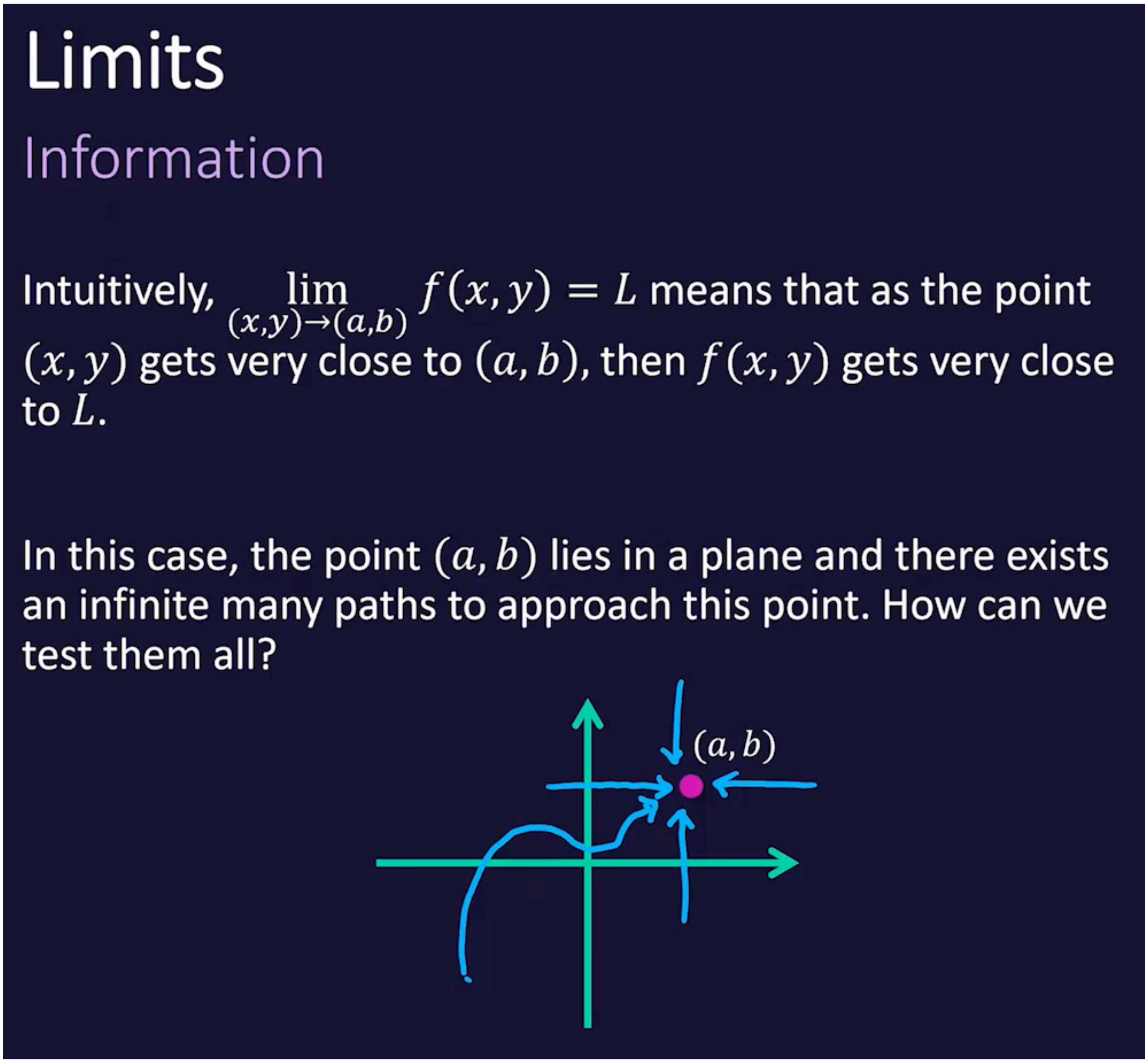
$16 - e^k > 0$

Summary

Curve $y = f(x)$		Surface $z = f(x, y)$
$\frac{df}{dx}$ ✓	partial derivatives, L12	$\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$
$\frac{d^2f}{dx^2}$ ✓		$\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}, \frac{\partial^2 f}{\partial y^2}$
$y - y_0 = \frac{df}{dx}(x - x_0)$	L13, tangent planes	$z - z_0 = \frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial y}(y - y_0)$
$\Delta f \approx \frac{df}{dx}\Delta x$	linear approximation and differentials	$\Delta f \approx \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y$
$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$	Chain rule, L14	$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{\partial f}{dt}$
$\frac{df}{dx} = 0$ critical point	L17/L18, max/min	$\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

LECTURE 11: Limits and Continuity (9/23/24)

- continuity indicates
  - defined at x = a
  - limit exists at x = a
  - limit = value at x = a



- if 2 paths to the point are equal, it does not mean the limit exists in 3D

Limits

Example and Note

Calculate the limit $\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^3+y^3}$ by

1. Approaching (0,0) along the line $y = 0$

$\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^3+y^3} = \lim_{x\to 0} \frac{x^3}{x^3} = 1$

2. Approaching (0,0) along the line $x = 0$

$\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^3+y^3} = \lim_{y\to 0} \frac{-y^3}{y^3} = -1$

$-1 \neq 1$

limit DNE

We can conclude that the limit does not exist since we found two approaches that produced different limits. However, we **cannot** conclude that the limit exists when two approaches produce the same limit.

Example: Show that $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$ DNE.

path $y = x$

$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} = \lim_{x\to 0} \frac{x^2x}{x^4+x^2} = \lim_{x\to 0} \frac{x}{x^2+1} = 0$

path $y = x^2$

$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} = \lim_{x\to 0} \frac{x^2x^2}{x^4+(x^2)^2} = \lim_{x\to 0} \frac{x^4}{x^4+x^4} = \frac{1}{2}$

$0 \neq \frac{1}{2}$ limit DNE.

Theorem (Limit Laws)

Assume that $\lim_{(x,y)\to(a,b)} f(x,y) = L$ and $\lim_{(x,y)\to(a,b)} g(x,y) = M$, and let c be a constant. Then

1. $\lim_{(x,y)\to(a,b)} c = C$
2. $\lim_{(x,y)\to(a,b)} x = a$ and $\lim_{(x,y)\to(a,b)} y = b$
3. $\lim_{(x,y)\to(a,b)} [f(x,y) \pm g(x,y)] = L + M$
4. $\lim_{(x,y)\to(a,b)} [cf(x,y)] = cL$
5. $\lim_{(x,y)\to(a,b)} [f(x,y)g(x,y)] = LM$
6. $\lim_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M}$ for $M \neq 0$
7. $\lim_{(x,y)\to(a,b)} [f(x,y)]^n = L^n$ for an integer $n > 0$
8. $\lim_{(x,y)\to(a,b)} \sqrt[n]{f(x,y)} = L^{\frac{1}{n}}$ for an integer n> 0

(assume $L > 0$ if n is even)

Limits

Note

$x = r\cos\theta$
$y = r\sin\theta$
$r^2 = x^2 + y^2$

take all the paths

Limits at (0,0) may be easier to evaluate by converting to **polar coordinates**.

$(x,y) \to (0,0)$ implies $r \to 0$.

Find the Limit

Examples

1. $\lim_{(x,y)\to(0,0)} \frac{\sin(2x^2+2y^2)}{x^2+y^2}$

$= \lim_{r\to 0} \frac{\sin(2r^2)}{r^2}$

$\overset{L'H}{=} \lim_{r\to 0} \frac{4r\cos(2r^2)}{2r}$

$= \lim_{r\to 0} 2\cos(2r^2)$

$= 2$

2. $\lim_{(x,y)\to(0,0)} \frac{x^2}{\sqrt{x^2+y^2}}$

$= \lim_{r\to 0} \frac{r^2\cos^2\theta}{r}$

$= \lim_{r\to 0} r\cos^2\theta$

$0 \le \cos^2\theta \le 1$

$0 \le r\cos^2\theta \le r$

$\downarrow \qquad \downarrow \qquad \downarrow$

$0 \qquad\qquad\quad 0$

$= 0$ by the squeeze thm

Continuity

Definition

A function $f(x, y)$ is **continuous** at (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

We say f is continuous on D if f is continuous at every point (a, b) in D.

1. Using the Limit Laws, we can prove that all polynomials are continuous on $\mathbb{R}^2$ and all rational functions are continuous on their domains. So, we can use direct substitution to find the limit:

$$\lim_{(x,y)\to(2,-1)} \left(x^2 - 4xy + 3y^2 - 1\right) =$$

$$= 2^2 - 4(2)(-1) + 3(-1)^2 - 1$$

$$= 14$$

2. The basic properties of continuous functions of a single variable carry over to multivariable functions; that is the sum, difference, product, quotient, and composition of continuous functions are continuous.

For example, $\lim_{(x,y)\to(2,-1)} \frac{\sqrt{2x+3y}}{4x-y} =$

$$= \frac{\sqrt{2(2)+3(-1)}}{4(2)-(-1)}$$

$$= \frac{1}{9}$$

Continuity

Notes

1. Using the Limit Laws, we can prove that all polynomials are continuous on $\mathbb{R}^2$ and all rational functions are continuous on their domains. So, we can use direct substitution to find the limit:

$$\lim_{(x,y)\to(2,-1)} \left(x^2 - 4xy + 3y^2 - 1\right) =$$

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$$= \frac{\sqrt{2(2)+3(-1)}}{4(2)-(-1)}$$

$$= \frac{1}{9}$$

Continuity

Examples

Discuss the continuity of $f(x, y) = \mathrm{In}(x + y)$.

$x + y > 0$

$y > -x$

Continuity

Examples

$$f(x,y) = \begin{cases} \frac{x^4-y^4}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

$\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$ ←

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^2+y^2}$$

$$= \lim_{(x,y)\to(0,0)} \frac{(x^2-y^2)(x^2+y^2)}{(x^2+y^2)}$$

$$= \lim_{(x,y)\to(0,0)} x^2-y^2$$

$$= 0$$

continuous everywhere

$$g(x,y) = \begin{cases} \frac{x^4-y^4}{x^2+y^2} & (x,y) \neq (0,0) \\ 1 & (x,y) = (0,0) \end{cases}$$

$\lim_{(x,y)\to(0,0)} g(x,y) = 1 = g(0,0)$

$$\lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^2+y^2} = 0 \neq 1$$

continuous everywhere except the origin

LECTURE 12: Partial Derivatives (9/25/24)

Geometric Interpretation

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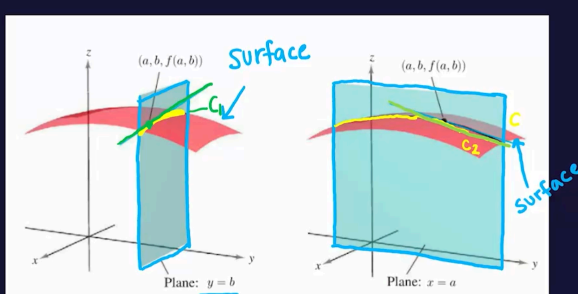

Let C_1 be the curve of intersection of the plane $y = b$ with the surface $z = f(x, y)$. Then $f_x(a, b)$ is the **slope** of the tangent line at the point $(a, b, f(a, b))$ to the curve C_1. Similarly, $f_y(a, b)$ is the slope of the tangent line at the same point to the curve C_2, where C_2 is the curve of intersection of the plane $x = a$ with the surface.

$f_x(a,b)$ is the slope of the ~~the~~ tangent line of the curve C_1

$f_y(a,b)$ is the slope of the tangent line of the curve C_2

Calculation of a partial derivative of $z = f(x, y)$:

To find f_x, take the standard derivative of the function with respect to x and treat y as a constant.

to what we did in Calc 1.

Find the partial derivatives of f:

1. $f(x,y) = y\sin(x^2+xy+y^2)$

$f_x = y\cos(x^2+xy+y^2)(2x+y)$

$f_y = 1\sin(x^2+xy+y^2) + y\cos(x^2+xy+y^2)(x+2y)$

2. $f(x,y,z) = e^{2x}\cos(z^2) + e^{3y}\sin(2z)$

$f_x = 2e^{2x}\cos(z^2)$

$f_y = 3e^{3y}\sin(2z)$

$f_z = -e^{2x}\sin(z^2)(2z) + e^{3y}\cos(2z)(2)$

Partial Derivatives

Examples

Find the slope of the tangent line to the curve of intersection of the surface $z = \sqrt{36-9x^2-4y^2}$ and the plane $x = 1$ (fix) at the point $(1,-2,\sqrt{11})$.

$$\frac{\partial z}{\partial y} = \frac{1}{2}(36-9x^2-4y^2)^{-\frac{1}{2}}(-8y)$$

$$= \frac{-4y}{\sqrt{36-9x^2-4y^2}}$$

$$\left.\frac{\partial z}{\partial y}\right|_{(1,-2)} = \frac{-4(-2)}{\sqrt{36-9(1)^2-4(-2)^2}} = \frac{8}{\sqrt{11}}$$

$z = f(x,y)$

Find $\frac{\partial z}{\partial x}$ if $yz - \text{In}(z) = x^2 + y^2$.

$$y\frac{\partial z}{\partial x} - \frac{1}{z}\frac{\partial z}{\partial x} = 2x$$

$$\frac{\partial z}{\partial x}\left(y - \frac{1}{z}\right) = 2x$$

$$\frac{\partial z}{\partial x} = \frac{2x}{y-\frac{1}{z}} \cdot \frac{z}{z}$$

$$\frac{\partial z}{\partial x} = \frac{2xz}{yz-1}$$

Higher Derivatives

$f(x,y)$ has the following **second partial derivatives**. mixed partials

1. Differentiate twice w.r.t. x:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = (f_x)_x = f_{xx}$$

2. Differentiate twice w.r.t. y:

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = (f_y)_y = f_{yy}$$

3. Differentiate first w.r.t. x and then w.r.t. y:

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = (f_x)_y = f_{xy}$$

4. Differentiate first w.r.t. y and then w.r.t. x:

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = (f_y)_x = f_{yx}$$

n