



i dont even care.

Why did you post your lecture notes in a publicly-shared document?

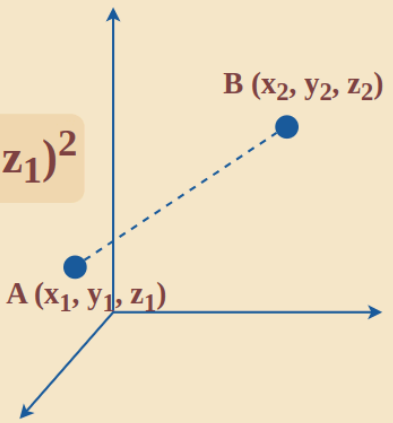
I'm not even in this class where did this come from

MODULE 1

LECTURE 1 3-Dimensional Rectangular System (8/23/24)

- 3d plane
- distance from point to xy plane = $|z|$
- distance from point to axis = find point on axis closest to it and use distance formula

Distance Between Two Points in 3D

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$


LECTURE 2 Vectors (8/26/24)

- graphing plane through point
- Outline
- MODULE 1
- [LECTURE 1 3-Dimensional Rectangular System \(8/23/24\)](#)
- LECTURE 2 Vectors (8/26/24)
- LECTURE 3 Dot Product (8/26/24)
- LECTURE 4 Cross Product (8/29/24)
- LECTURE 5 Lines and Planes (9/3/24)
- LECTURE 6 Cylinders and Quadric Surfaces (9/6/24)
- LECTURE 7 Vector-Valued Functions (9/9/24)
- LECTURE 8 Arc Length and Curvature (9/11/24)
- LECTURE 9: Motion in Space (9/14/24)
- MODULE 2
- LECTURE 10: Functions of Several Variables (9/19/24)
- LECTURE 11: Limits and Continuity (9/23/24)

- LECTURE 12: Partial Derivatives (9/25/24)
-
- - yz-plane through (x,y,z) , just look at x
- circle in \mathbb{R}^3 is a cylinder
- inequalities
- Vectors
 - represented by directed line segment (initial A terminal B) and length/magnitude $|AB|$
 - equal if same magnitude & direction, location doesn't matter
- Vector addition
 - tip to tail or parallelogram
- position vector has initial point at origin
- vector addition easy
- scalar multiplication easy
- properties of operations (vectors u,v,w scalars c,d)
 - $u+v=v+u$
 - $(u+v)+w = u+(v+w)$
 - $u+0=u$
 - $u+(-u)=0$
 - $c(u+v)=cu+cv$
 - $(c+d)u=cu+du$
 - $0u=0$
 - $c0=0$
 - $1u=u$
 - $c(du)=(cd)u=d(cu)$
- vector magnitude = $\sqrt{x^2+y^2+z^2}$
- unit vector has length 1

- to find unit vector, divide vector by magnitude
- standard basis vector notation
 - $\mathbf{i} = \langle 1, 0, 0 \rangle$
 - $\mathbf{j} = \langle 0, 1, 0 \rangle$
 - $\mathbf{k} = \langle 0, 0, 1 \rangle$
 - $487932\mathbf{i} + 6824\mathbf{j} - 2993\mathbf{k}$

LECTURE 3 Dot Product (8/

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \cdot \|\vec{v}\|}$$

26/24)

- dot product - multiply the components then add it all up

Properties of the Dot Product

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$$|\vec{u}| = \sqrt{u_1^2 + u_2^2 + u_3^2}$$

If \vec{u} , \vec{v} and \vec{w} are vectors and c is a scalar, then

1. $\vec{u} \cdot \vec{u} = \langle u_1, u_2, u_3 \rangle \cdot \langle u_1, u_2, u_3 \rangle = u_1^2 + u_2^2 + u_3^2 = |\vec{u}|^2$
2. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u} \leftarrow$
3. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w} \leftarrow$
4. $(\vec{v} + \vec{w}) \cdot \vec{u} = \vec{v} \cdot \vec{u} + \vec{w} \cdot \vec{u} \leftarrow$
5. $c(\vec{u} \cdot \vec{v}) = (c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v})$
6. $\vec{0} \cdot \vec{u} = 0 \quad \langle 0, 0, 0 \rangle \cdot \langle u_1, u_2, u_3 \rangle = 0 + 0 + 0 = 0$



Two nonzero vectors are **orthogonal** if the angle between them is $\theta = \frac{\pi}{2}$



Two nonzero vectors \vec{u} and \vec{v} are orthogonal if and only if $\vec{u} \cdot \vec{v} = 0$

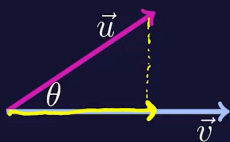


The zero vector $\vec{0}$ is orthogonal to all other vectors

- vector projection (yellow arrow)

Orthogonal Projections

The vector projection of \vec{u} onto \vec{v} : $\text{proj}_{\vec{v}} \vec{u}$



$$\begin{aligned} \text{length} &= |\vec{u}| \cos \theta \\ \text{unit vector} &= \hat{v} = \frac{\vec{v}}{|\vec{v}|} \\ \text{proj}_{\vec{v}} \vec{u} &= |\vec{u}| \cos \theta \frac{\vec{v}}{|\vec{v}|} \end{aligned}$$

Therefore, $\text{proj}_{\vec{v}} \vec{u} = (|\vec{u}| \cos \theta) \frac{\vec{v}}{|\vec{v}|} = \left(|\vec{u}| \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| |\vec{v}|} \right) \frac{\vec{v}}{|\vec{v}|}$

$$= \left(\frac{\vec{u} \cdot \vec{v}}{|\vec{v}|} \right) \frac{\vec{v}}{|\vec{v}|} = \underbrace{\frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2}}_{\text{scalar}} \vec{v} \text{ vector}$$

$$\cos \theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| |\vec{v}|}$$

- WORK =
 - $F \cdot D$
 - $|F| |D| \cos \theta$

LECTURE 4 Cross Product (8/29/24)

- cross product - easy

Algebraic Properties of the Cross Product

If \vec{u} , \vec{v} and \vec{w} are vectors and c is a scalar, then

1. $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$
2. $\vec{u} \times \vec{u} = \vec{0}$ parallel
3. $(c\vec{u}) \times \vec{v} = c(\vec{u} \times \vec{v}) = \vec{u} \times (c\vec{v})$
4. $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$
5. $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$
6. $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$

- triple product - not easy

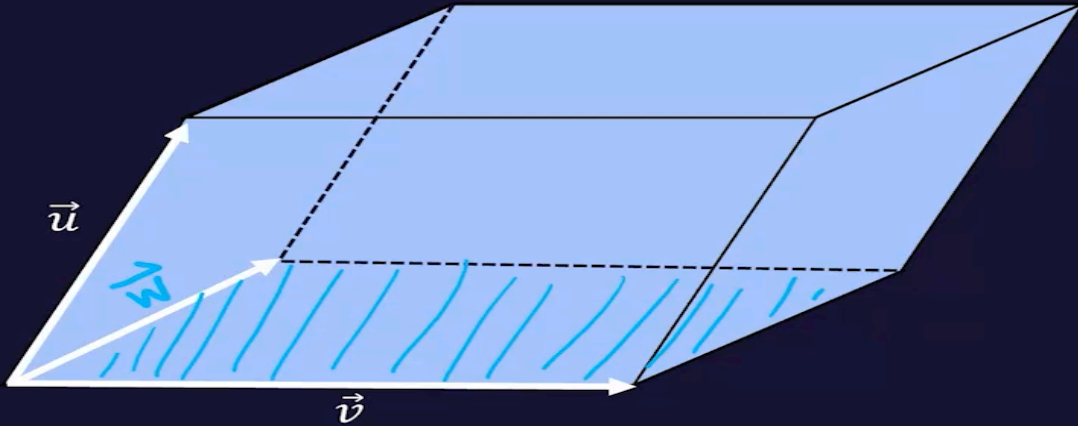
The triple product $\vec{u} \cdot (\vec{v} \times \vec{w})$ can be calculated using a determinant:

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

- $V = \text{height} \times \text{area of base}$

Example

Find the volume of a parallelepiped determined by the vectors \vec{u} , \vec{v} , and \vec{w} .



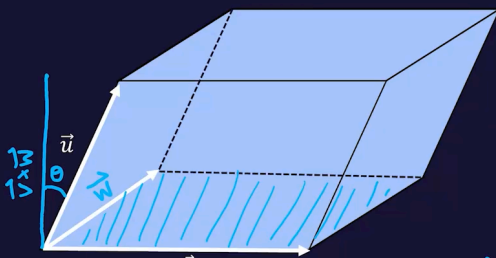
- the area of base is just area of parallelogram or magnitude of cross product $|\vec{v} \times \vec{w}|$
- the height is perpendicular to \vec{v} and \vec{w} so it is $|\vec{u} \cdot \frac{\vec{v} \times \vec{w}}{|\vec{v} \times \vec{w}|}|$

Triple Product

$$\vec{u} \cdot (\vec{v} \times \vec{w})$$

Example

Find the volume of a parallelepiped determined by the vectors \vec{u} , \vec{v} , and \vec{w} .



θ is the angle between $\vec{v} \times \vec{w}$ and \vec{u}
height = $|\vec{u}| \cos \theta$

$$\begin{aligned} V &= \text{height} \cdot \text{Area of the base} \\ &= \text{height} |\vec{v} \times \vec{w}| \\ &= |\vec{u}| \cos \theta |\vec{v} \times \vec{w}| \\ &= |\vec{u}| |\vec{v} \times \vec{w}| \cos \theta \\ &= \vec{u} \cdot (\vec{v} \times \vec{w}) \end{aligned}$$

The triple product $\vec{u} \cdot (\vec{v} \times \vec{w})$ can be calculated using a determinant:

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$= \downarrow u_1 \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - \downarrow u_2 \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + \downarrow u_3 \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}$$

LECTURE 5 Lines and Planes (9/3/24)

- to define line, we need point $P_0 \langle x_0, y_0, z_0 \rangle$ and vector $v = \langle a, b, c \rangle$ parallel to the line
 - $r = r_0 + vt$ (t is a scalar multiple)
 - $\langle x_0, y_0, z_0 \rangle + t \langle a, b, c \rangle$ (called vector equation)
- Line segment
 - $r(t) = (1-t)\langle x_0, y_0, z_0 \rangle + t\langle x_1, y_1, z_1 \rangle$
 - initial point at $t=0$, terminal point at $t=1$
 - will not be used until line integrals (module 4)
- Parametric equation - component by component

$$\begin{aligned}x &= x_0 + at \\y &= y_0 + bt \\z &= z_0 + ct\end{aligned}$$

- Symmetric equation - solve for t

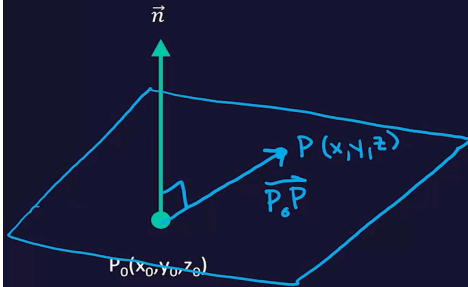
$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

- to define a plane, we need a point on the plane and a normal vector

Planes

1- point on the plane
2- normal

A plane in space is determined by a point $P_0(x_0, y_0, z_0)$ in the plane and a vector $\vec{n} = \langle a, b, c \rangle$ that is **orthogonal** to the plane.



Let $P(x, y, z)$ be any point on the plane.

Vector equation of the plane:

$$\vec{P_0P} \cdot \vec{n} = 0$$

$$\langle x - x_0, y - y_0, z - z_0 \rangle \cdot \langle a, b, c \rangle = 0$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$ax - ax_0 + by - by_0 + cz - cz_0 = 0$$

$$ax + by + cz + d = 0$$

- PRO STRAT FOR FIND PLANE WITH NORMAL & POINT
 - to find d take each component of normal and multiply it by each component of the point *-1

Equations of a Plane

Scalar Equation of the Plane

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Linear Equation of the Plane

$$ax + by + cz + d = 0$$



NOTE: The coefficients a , b , and c are the components of the normal vector to the plane.

Parallel and Orthogonal Planes

Two planes are said to be **parallel** if their normal vectors are parallel ($\vec{n}_1 = c\vec{n}_2$).

Two planes are said to be **orthogonal** if their normal vectors are orthogonal ($\vec{n}_1 \cdot \vec{n}_2 = 0$).



- literally just look at the normal vector to determine parallel/orthogonal/neither
 - if it's line vs plane it'll be reversed if using the vector of the line, so if the dot product is 0 they're parallel and if it looks parallel it's orthogonal. kinda stupid lol
- find the angle between two planes
 - just find angle between normal vectors using:

$$\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{\|\vec{n}_1\| \|\vec{n}_2\|}$$

- find parametric equations for line of intersection of two planes



- v is perpendicular to both normal vectors so cross both normal vectors to find v
- to find a point on the line, fix a value ($x/y/z$) and solve for the other two ($x/y/z$) (this might not work maybe)
- find distance from point to plane

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

- Proof:

$$D = |\vec{u}| |\cos \theta|$$

$$= \frac{|\vec{u}| |\vec{u} \cdot \vec{n}|}{|\vec{u}| |\vec{n}|}$$

$$= \frac{|\langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle \cdot \langle a, b, c \rangle|}{\sqrt{a^2 + b^2 + c^2}}$$

$$= \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}}$$

$$= \frac{|ax_1 - ax_0 + by_1 - by_0 + cz_1 - cz_0|}{\sqrt{a^2 + b^2 + c^2}}$$

- x_1, y_1, z_1 is the point and a, b, c, d is the plane
- find distance from two parallel planes
 - find a point on one of the planes then the problem is the same as point to plane

LECTURE 6 Cylinders and Quadric Surfaces (9/6/24)


- conic sections because i forgot

Review of Conic Sections


2. Ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$


$a > b$





$a = b$



$b > a$



3. Hyperbola:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$$


- whatever the - is on is the axis of the vertical/horizontal asymptote
 - technically they have diagonal asymptotes or something
- in this world a cylinder doesn't have to have a circle as a base, it can be a triangle or whatever tf
 - this makes any 2d graph in \mathbb{R}^3 a cylinder i think
- NO DRAWING ON EXAMS!!!



A **quadric surface** is a 3D surface whose equation is of the second degree.

The general equation is $Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$, given that $A^2 + B^2 + C^2 + D^2 + E^2 + F^2 \neq 0$.

With rotation and translation, these possibilities can be reduced to two distinct types.

1. $Ax^2 + By^2 + Cz^2 + J = 0$ (All three quadratic terms; no linear terms)
2. $Ax^2 + By^2 + Iz = 0$ (Only two quadratic terms plus one linear term)

type 1 surfaces

Basic Quadric Surface

0 negative signs

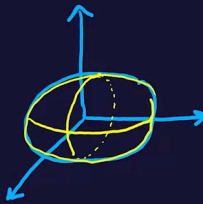
Ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$$z=0 \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{ellipse}$$

$$y=0 \quad \frac{x^2}{a^2} + \frac{z^2}{c^2} = 1 \quad \text{ellipse}$$

$$x=0 \quad \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \text{ellipse}$$



1 negative sign

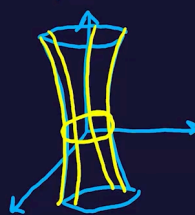
Hyperboloid of 1 sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

$$z=0 \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{ellipse}$$

$$y=0 \quad \frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 \quad \text{hyperbola}$$


$$x=0 \quad \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \quad \text{hyperbola}$$



Basic Quadric Surface

2 negative signs

Hyperboloid of 2 sheets

$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$


$$z=0 \quad -\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1 \quad \text{X}$$

$$z=2c \quad -\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{4c^2}{c^2} = 1$$

$$3 = \frac{x^2}{a^2} + \frac{y^2}{b^2} \quad \text{ellipse}$$

$$y=0 \quad -\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1 \quad \text{hyperbola}$$

$$x=0 \quad -\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \text{hyperbola}$$



1 negative sign

Elliptic Cone

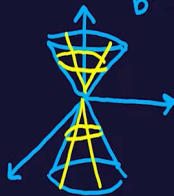
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$

$$z=0 \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$$

$$z=c \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{c^2}{c^2} = 0 \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{ellipses}$$

$$y=0 \quad \frac{x^2}{a^2} - \frac{z^2}{c^2} = 0 \quad \frac{x^2}{a^2} = \frac{z^2}{c^2} \quad \frac{x}{a} = \pm \frac{z}{c} \quad \text{lines}$$

$$x=0 \quad \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0 \quad \frac{y^2}{b^2} = \frac{z^2}{c^2} \quad \frac{y}{b} = \pm \frac{z}{c}$$

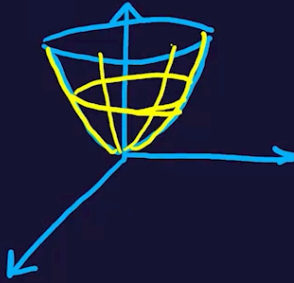


type 2 surfaces

Basic Quadric Surface

Elliptic Paraboloid

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$



Hyperbolic Paraboloid

$$z = \frac{y^2}{b^2} - \frac{x^2}{a^2} \quad \bar{z} = c$$

$$c = \frac{y^2}{b^2} - \frac{x^2}{a^2} \quad \text{hyperbola}$$

$$y=0 \quad z = -\frac{x^2}{a^2} \quad \text{parabola}$$

$$x=0 \quad z = \frac{y^2}{b^2} \quad \text{parabola}$$



LECTURE 7 Vector-Valued Functions (9/9/24)

- domain of vector-valued function is intersection of domains of each component
- curve sketching
- helix
- the limit of vector-valued function is just taking limit of each component
 - if ANY component limit doesn't exist, the functions limit doesn't exist
- vector-valued function is continuous at a if
 - defined at each component
 - limit exists at each component
 - and they're equal (the point at a and the limit)
- first principle derivative XD
 - fuck that just derive each component



We define the **unit tangent vector** to the curve $\vec{r}(t)$ by

$$\hat{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

- this just gives direction ^

Example

- A. Find a vector function that represents the curve C of intersection of the cylinder $x^2 + y^2 = 4$ and the plane $y - z = 2$.

$$x = 2\cos t$$

$$y = 2\sin t$$

$$z = y - 2 = 2\sin t - 2$$

$$\vec{r}(t) = \langle 2\cos t, 2\sin t, 2\sin t - 2 \rangle$$

- there's a circle so sin and cos are used ^

Vector-Valued Functions

Definition



A curve $\vec{r}(t)$ is **smooth** on an interval I if

$$\vec{r}'(t) \text{ is continuous and } \vec{r}'(t) \neq \vec{0}$$

On I . A curve that is made up of a finite number of smooth curves pieced together in a continuous fashion is called **piece-wise smooth**.

Differentiation Rules

Let \vec{u} and \vec{v} be differentiable vector-valued functions, $f(t)$ a differentiable real-valued function, \vec{C} a constant vector, and c a constant. Then

$$1. \quad \frac{d}{dt} [\vec{C}(t)] = \vec{0}$$

$$5. \quad \frac{d}{dt} [\vec{u}(t) \cdot \vec{v}(t)] = \vec{u}'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}'(t)$$

$$2. \quad \frac{d}{dt} [\vec{u}(t) + \vec{v}(t)] = \vec{u}'(t) + \vec{v}'(t)$$

$$6. \quad \frac{d}{dt} [\vec{u}(t) \times \vec{v}(t)] = \vec{u}'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}'(t)$$

$$3. \quad \frac{d}{dt} [c \vec{u}(t)] = c \vec{u}'(t)$$

$$7. \quad \frac{d}{dt} [\vec{u}(f(t))] = \vec{u}'(f(t)) f'(t)$$

$$4. \quad \frac{d}{dt} [f(t)\vec{u}(t)] = f'(t)\vec{u}(t) + f(t)\vec{u}'(t)$$

LECTURE 8 Arc Length and Curvature (9/11/24)

Arc Length

Extension to 3D

There is a natural extension of this to 3D. Consider a curve $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, $a \leq t \leq b$, where f' , g' , and h' are continuous. If the curve is traversed exactly once as t increases from a to b , then its length is

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2} dt$$

Arc Length Function

Definition

Suppose that C is a piecewise smooth curve given by a vector-valued function $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, $a \leq t \leq b$. The **arc length function** is defined *how far we've gone after some time t*

$$s(t) = \int_a^t \sqrt{[f'(u)]^2 + [g'(u)]^2 + [h'(u)]^2} du$$

The Fundamental Theorem of Calculus tells us that s is differentiable function of t and

$$\frac{ds}{dt} = \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}$$

Reparameterization of a Curve

Arc Length

know the position

It is often useful to parameterize a curve with respect to arc length since arc length arises naturally from the shape of the curve and does not depend on a particular coordinate system. If a curve $\vec{r}(t)$ is given in terms of a parameter t and $s(t)$ is the arc length function, then

1. Solve for t as a function of s : $t = t(s)$
2. Reparametrize the curve in terms of s by substituting for t : $\vec{r} = \vec{r}(t(s))$

Example

$$\left. \begin{array}{l} \cos t = 1 \\ \sin t = 0 \\ t = 0 \end{array} \right\} t = 0$$

Reparameterize the helix $\vec{r}(t) = \cos t \hat{i} + \sin t \hat{j} + t \hat{k}$ by its arc length starting from $(1,0,0)$ in the direction of increasing t . Use this information to determine the position after traveling $\sqrt{2}\pi$ units on the helix.

$$\begin{aligned} s(t) &= \int_0^t \sqrt{(-\sin u)^2 + (\cos u)^2 + 1^2} \, du \\ &= \int_0^t \sqrt{\sin^2 u + \cos^2 u + 1} \, du \\ &= \int_0^t \sqrt{2} \, du \\ &= \sqrt{2} u \Big|_0^t \\ s &= \sqrt{2} t \end{aligned}$$

$$\begin{aligned} \vec{r}(s) &= \left\langle \cos\left(\frac{s}{\sqrt{2}}\right), \sin\left(\frac{s}{\sqrt{2}}\right), \frac{s}{\sqrt{2}} \right\rangle \\ \vec{r}(\sqrt{2}\pi) &= \langle \cos(\pi), \sin(\pi), \pi \rangle \end{aligned}$$

∴

$t = \frac{s}{\sqrt{2}}$ want \vec{r} dependent on s

Curvature

Formulas and Example

In general, the formal definition of curvature is not easy for calculations. We use the following two alternate formulas.

$$1. \quad \kappa(t) = \frac{|\hat{r}'(t)|}{|\vec{r}'(t)|} \leftarrow$$

$$2. \quad \kappa(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3} \leftarrow$$

Find the curvature of a circle of radius a .

$$\begin{aligned} \vec{r}(t) &= \langle a \cos(t), a \sin(t) \rangle \\ \vec{r}'(t) &= \langle -a \sin t, a \cos t \rangle \\ |\vec{r}'(t)| &= \sqrt{a^2 \sin^2 t + a^2 \cos^2 t} = a \\ \hat{r}(t) &= \langle -\sin t, \cos t \rangle \\ \hat{r}'(t) &= \langle -\cos t, -\sin t \rangle \quad |\hat{r}'(t)| = \sqrt{\cos^2 t + \sin^2 t} = 1 \\ \kappa(t) &= \frac{|\hat{r}'(t)|}{|\vec{r}'(t)|} = \frac{1}{a} \end{aligned}$$



• curvature formula ^

Curvature

$$\kappa = \frac{|\vec{r}'(x) \times \vec{r}''(x)|}{|\vec{r}'(x)|^3}$$

Special Case and Example

Consider a plane curve with the equation $y = f(x)$. We can rewrite it as

$$\vec{r}(x) = x\hat{i} + f(x)\hat{j}, \quad \vec{r}'(x) = \langle x, f(x) \rangle$$

$$\text{so } \vec{r}'(x) = \hat{i} + f'(x)\hat{j}, \quad \vec{r}''(x) = f''(x)\hat{j}, \text{ and}$$

$$\vec{r}'(x) = \langle 1, f'(x) \rangle \quad \vec{r}''(x) = \langle 0, f''(x) \rangle$$

$$\vec{r}'(x) \times \vec{r}''(x) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & f'(x) & 0 \\ 0 & f''(x) & 0 \end{vmatrix} = \langle 0, 0, f''(x) \rangle$$

$$\text{Therefore, } \kappa(x) = \frac{|f''(x)|}{(\sqrt{1+f'(x)^2})^3} = \frac{|f''(x)|}{(1+f'(x)^2)^{3/2}} = \frac{|-\frac{1}{x^2}|}{(1+(\frac{1}{x})^2)^{3/2}} \text{ at } (1,0) = \frac{1}{(1+1^2)^{3/2}} = \frac{1}{2^{3/2}}$$

Find the curvature of the curve $y = \ln(x)$ at the point $(1,0)$.

$$y' = \frac{1}{x}$$

$$y'' = -\frac{1}{x^2}$$

LECTURE 9: Motion in Space (9/14/24)

Normal Vectors

Recall and Definition

points in the direction of motion

If $\hat{T}(t)$ is the unit tangent vector of a smooth curve defined by $\vec{r}(t)$, show that $\hat{T}(t) \perp \hat{T}'(t)$.


$|\hat{T}|^2 = 1$
 $1 = \hat{T} \cdot \hat{T}$
 $\frac{d}{dt}(1) = \frac{d}{dt}(\hat{T} \cdot \hat{T})$
 $0 = \hat{T}' \cdot \hat{T} + \hat{T} \cdot \hat{T}' = 2\hat{T}' \cdot \hat{T}$
 $0 = 2\hat{T}' \cdot \hat{T}$
 $0 = \hat{T}' \cdot \hat{T} \Rightarrow \hat{T} \perp \hat{T}'$

Let C be a smooth curve represented by $\vec{r}(t)$. If $\hat{T}'(t) \neq \vec{0}$, then the **principal unit normal vector** $\hat{N}(t)$ at t is defined as

perpendicular to the unit tangent


$$\hat{N}(t) = \frac{\hat{T}'(t)}{|\hat{T}'(t)|}$$

Among the vectors orthogonal to the unit tangent vector \hat{T} is one of particular significance because it points in the direction in which the curve is turning.



- bruh i needed this for HOMEWORK 8

$\hat{B} \perp \hat{T}$ and $\hat{B} \perp \hat{N}$



- yo this triangle thing is cracked for cross product instead of right hand rule
- B is binormal vector its $\hat{T} \times \hat{N}$

Find the principal unit normal vector of the helix

$$\vec{r}(t) = 2\cos(t)\hat{i} + 2\sin(t)\hat{j} + t\hat{k}.$$

$$\hat{N} = \frac{\hat{T}'(t)}{|\hat{T}'(t)|}$$

$$\hat{T} = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

$$\vec{r}'(t) = \langle -2\sin t, 2\cos t, 1 \rangle$$

$$|\vec{r}'(t)| = \sqrt{4\sin^2 t + 4\cos^2 t + 1} = \sqrt{5}$$

$$\hat{T} = \frac{1}{\sqrt{5}} \langle -2\sin t, 2\cos t, 1 \rangle$$

$$\hat{T}' = \frac{1}{\sqrt{5}} \langle -2\cos t, -2\sin t, 0 \rangle$$

$$|\hat{T}'| = \frac{1}{\sqrt{5}} \sqrt{4\cos^2 t + 4\sin^2 t} = \frac{2}{\sqrt{5}}$$

$$\hat{N} = \frac{1}{\frac{2}{\sqrt{5}}} \langle -2\cos t, -2\sin t, 0 \rangle$$

$$= \langle -\cos t, -\sin t, 0 \rangle$$

Motion Along a Curve

Definition and Example

Let C be a smooth curve represented by $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$. The velocity vector, speed, and acceleration vector at time t are defined as follows.

$$\text{Velocity} = \vec{v}(t) = \vec{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$$

$$\text{Speed} = |\vec{v}(t)| = |\vec{r}'(t)| = \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}$$

$$\text{Acceleration} = \vec{a}(t) = \vec{r}''(t) = \vec{v}'(t) = \langle f''(t), g''(t), h''(t) \rangle$$

Find the velocity, speed, and acceleration of a particle with position vector

$$\vec{r}(t) = \langle 2\cos(t), 2\sin(t), t \rangle$$

$$\vec{r}'(t) = \langle -2\sin t, 2\cos t, 1 \rangle$$

$$\text{Speed} = |\vec{r}'(t)| = \sqrt{4\sin^2 t + 4\cos^2 t + 1}$$

$$= \sqrt{5}$$

$$\vec{a} = \vec{r}''(t) = \langle -2\cos t, -2\sin t, 0 \rangle$$

Components of Acceleration

Speed and Direction

We now use the vectors \hat{T} and \hat{N} to gain insight into how moving objects accelerate. The two ways to change the velocity of an object are to change its speed and change its direction of motion. We now show that changing the speed produces acceleration in the direction of \hat{T} and changing the direction produces acceleration in the direction of \hat{N} .

tangential

normal
centripetal force

Components of Acceleration

Recall

$$\hat{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \frac{\vec{v}(t)}{|\vec{v}(t)|}$$

then $\vec{v} = |\vec{v}| \hat{T}$

and $\vec{a} = \frac{d}{dt}(\vec{v}) = \frac{d}{dt}(|\vec{v}| \hat{T})$

Because \vec{a} is written as a linear combination of \hat{T} and \hat{N} , it follows that the acceleration vector \vec{a} lies in the plane determined by \hat{T} and \hat{N} .

$$= |\vec{v}|' \hat{T} + |\vec{v}| \hat{T}' \frac{|\hat{T}'(t)|}{|\hat{T}'(t)|}$$

$$= |\vec{v}|' \hat{T} + |\vec{v}| |\hat{T}'(t)| \frac{\hat{T}'}{|\hat{T}'(t)|}$$

→ Slide tangential \hat{T} \hat{N} ← normal spins

- “this isn’t how we compute these parts”

Curvature

Theorem

If $\vec{r}(t)$ is the position vector for a smooth curve C , then the **tangential component** a_T and **normal component** a_N of acceleration are as follows.

θ be the angle between \vec{v} and \vec{a}

$$|\vec{a}| \cos \theta = |\vec{a}| \frac{\vec{v} \cdot \vec{a}}{|\vec{v}| |\vec{a}|} = \frac{\vec{v} \cdot \vec{a}}{|\vec{v}|}$$

$a_T = |\vec{v}'| = \vec{a} \cdot \hat{T} = \frac{\vec{v} \cdot \vec{a}}{|\vec{v}|}$

$a_N = |\vec{v}| |\vec{T}'| = \vec{a} \cdot \hat{N} = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|}$

$\vec{a} \cdot \hat{T} = (a_T \hat{T} + a_N \hat{N}) \cdot \hat{T} = a_T \hat{T} \cdot \hat{T} + a_N \hat{N} \cdot \hat{T} = a_T$

$a_T = \vec{a} \cdot \hat{T} = \vec{a} \cdot \frac{\vec{v}}{|\vec{v}|} = \frac{\vec{a} \cdot \vec{v}}{|\vec{v}|}$

$|\vec{a}| \sin \theta = |\vec{a}| \frac{|\vec{v} \times \vec{a}|}{|\vec{v}| |\vec{a}|} = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|}$

- the circled equations are how you compute them

Example



$$a_N = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|}$$

A particle moves with position function $\vec{r}(t) = \langle 2\cos(t), 2\sin(t), t \rangle$. Find the tangential and normal components of acceleration.

$$a_T = \frac{\vec{v} \cdot \vec{a}}{|\vec{v}|} \quad \vec{r}'(t) = \langle -2\sin t, 2\cos t, 1 \rangle = \vec{v}$$

$$\vec{a} = \langle -2\cos t, -2\sin t, 0 \rangle$$

$$a_T = \frac{\langle -2\sin t, 2\cos t, 1 \rangle \cdot \langle -2\cos t, -2\sin t, 0 \rangle}{\sqrt{4\sin^2 t + 4\cos^2 t + 1}} = \frac{4\sin t \cos t - 4\sin t \cos t}{\sqrt{5}} = 0$$

$$\vec{v} \times \vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2\sin t & 2\cos t & 1 \\ -2\cos t & -2\sin t & 0 \end{vmatrix} = \langle 2\sin t, -2\cos t, 4 \rangle \quad |\vec{v} \times \vec{a}| = \sqrt{4\sin^2 t + 4\cos^2 t + 16} = \sqrt{20}$$

$$a_N = \frac{\sqrt{20}}{\sqrt{5}} = \sqrt{4} = 2 \quad \vec{a} = 0 \cdot \hat{T} + 2 \hat{N}$$

- WTF EQUATION SUMMARY

Formulas for Curves in Space

Summary

Position function: $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle \leftarrow$

Unit binormal vector: $\hat{B} = \hat{T} \times \hat{N}$

Velocity: $\vec{v} = \vec{r}'$

Curvature: $\kappa = \left| \frac{d\hat{T}}{ds} \right| = \frac{|\hat{T}'|}{|\vec{v}|} = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3}$ *how fast a curve is changing along a distance*

Acceleration: $\vec{a} = \vec{v}' = \vec{r}'' \leftarrow$

Components of acceleration: $\vec{a} = a_T \hat{T} + a_N \hat{N}$

Unit tangent vector: $\hat{T} = \frac{\vec{v}}{|\vec{v}|}$ *direction particle is moving in*

with $a_T = \vec{a} \cdot \hat{T} = |\vec{v}'| = \frac{\vec{v} \cdot \vec{a}}{|\vec{v}|}$

Principal unit normal vector: $\hat{N} = \frac{\hat{T}'}{|\hat{T}'|}$ *how the curve changes direction*

and $a_N = \vec{a} \cdot \hat{N} = |\vec{v}| |\hat{T}'| = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|}$

- NVM THESE ARE GIVEN ON EXAM BRUHH

MODULE 2

LECTURE 10: Functions of Several Variables (9/19/24)

- domain = number we can't put in (log of negative, sqrt of negative, etc.)
- range = range of possible outputs

Function of Two Variables

Definition and Examples

$y = f(x)$ independent

two independent



A **function of two variables** $z = f(x, y)$ assigns each ordered pair (x, y) in a subset D of \mathbb{R}^2 a unique real number z . The set D is called the **domain** of f that contains allowable inputs, and its **range** is the set of outputs, the z -values taken on by f .

Find the domain and range of the functions:

$$f(x, y) = x^2 + y^2$$

Domain $\mathbb{R}^2 \{ (x, y) \mid -\infty < x < \infty, -\infty < y < \infty \}$

Range $[0, \infty)$

$$f(x, y) = x^2 \sin(y)$$

Domain $\mathbb{R}^2 \{ (x, y) \mid -\infty < x < \infty, -\infty < y < \infty \}$

$$\text{fix } y = \frac{\pi}{2} \quad f(x, \frac{\pi}{2}) = x^2 \sin(\frac{\pi}{2}) = x^2 \geq 0$$

$$y = \frac{3\pi}{2} \quad f(x, \frac{3\pi}{2}) = x^2 \sin(\frac{3\pi}{2}) = -x^2 \leq 0$$

$(-\infty, \infty)$

Examples

Find the domain and range of the function:

$$f(x, y) = \frac{x}{y-3}$$

$$\text{Domain } \{(x, y) \mid -\infty < x < \infty, y \neq 3\}$$

$$\text{Fix } y=4 \quad f(x, 4) = \frac{x}{4-3} = x$$

$$\text{Range } (-\infty, \infty)$$

Find the domain and range of the function:

$$f(x, y) = \sqrt{1-x^2-y^2}$$

$$1-x^2-y^2 \geq 0$$

$$x^2+y^2 \leq 1$$



$$\text{Domain } \{(x, y) \mid x^2+y^2 \leq 1\}$$

$$\text{Range } [0, 1]$$

- Shadow underneath surface is the domain

Graphs

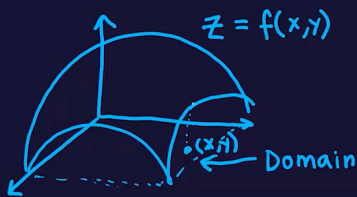
Definition and Note



The **graph** of a function f of two variables is the set of all points (x, y, z) for which $z = f(x, y)$ and (x, y) is in the domain of f . This graph can be interpreted geometrically as a **surface** in space, as discussed in Lecture 6.



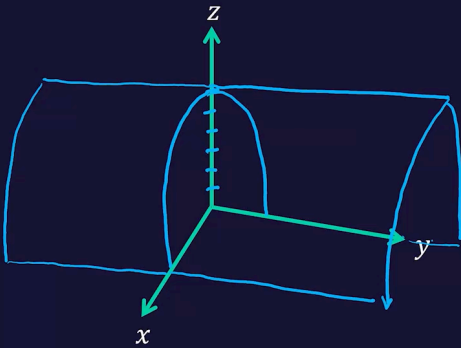
NOTE: The graph of $z = f(x, y)$ is a surface whose projection onto the xy -plane is D , the domain of f .



Examples

Find the domain and range of the function:

$z = f(x, y) = 6 - x^2$. Describe the graph of f .
Domain is \mathbb{R}^2 Range $(-\infty, 6]$



$$16 - 4x^2 - y^2 \geq 0$$

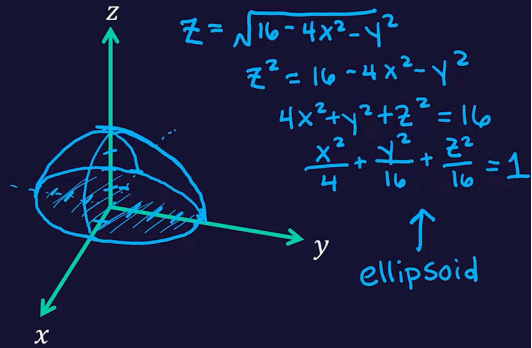
$$\text{Domain } \{ (x, y) \mid 4x^2 + y^2 \leq 16 \}$$

$$\frac{x^2}{4} + \frac{y^2}{16} \leq 1 \text{ ellipse}$$

Find the domain and range of the function:

$f(x, y) = \sqrt{16 - 4x^2 - y^2}$. Range $[0, 4]$

Describe the graph of f .



Level Curves

Definition

We noted that traces could be used to help visualize the shape of surfaces and now we introduce the related concept of a level curve (or contour curve).



Given a function $f(x, y)$ and a number k in the range of f , a level curve of f for the value k is defined to be the set of points satisfying the equation.

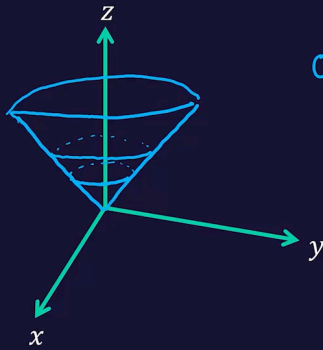
$$f(x, y) = k$$

A collection of level curves of f is called a contour map.

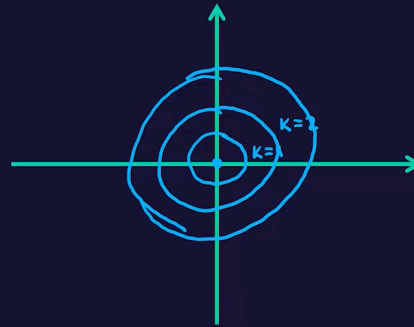
Level Curves

Example

Describe the surface and the level curves for $z = f(x, y) = \sqrt{x^2 + y^2}$.



$z = \sqrt{x^2 + y^2}$ $[0, \infty)$
 $z^2 = x^2 + y^2$
 $0 = x^2 + y^2 - z^2$



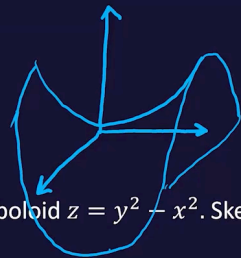
k values have to be in the range

$k=0 \quad 0 = \sqrt{x^2 + y^2} \quad 0 = x^2 + y^2$
 $k=1 \quad 1 = \sqrt{x^2 + y^2} \quad 1 = x^2 + y^2$
 $k=2 \quad 2 = \sqrt{x^2 + y^2} \quad 4 = x^2 + y^2$

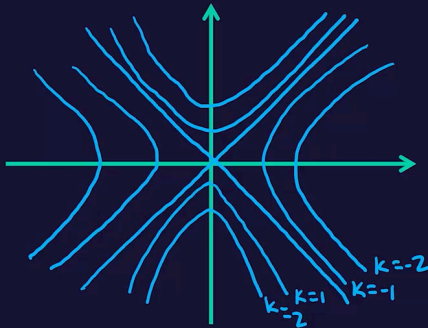
Level Curves

Example

Consider the hyperbolic paraboloid $z = y^2 - x^2$. Sketch a contour map of this surface.



Range \mathbb{R}



$k=-2 \quad y^2 - x^2 = -2 \quad x^2 - y^2 = 2$
 $k=-1 \quad y^2 - x^2 = -1 \quad x^2 - y^2 = 1 \text{ hyperbola}$
 $k=0 \quad y^2 - x^2 = 0 \quad y^2 = x^2 \quad y = \pm x$
 $k=1 \quad y^2 - x^2 = 1 \text{ hyperbola}$
 $k=2 \quad y^2 - x^2 = 2$

Level Surfaces

Definition



We have examined functions of two variables and now we can extend the concepts naturally to functions of three or more variables. With the function $w = f(x, y, z)$, level curves become **level surfaces** $w = f(x, y, z) = k$, where k is in the range of f .

# Independent Variables	Explicit Form	Implicit Form	Graph Resides In	
1	$y = f(x)$	$F(x, y) = 0$	\mathbb{R}^2	
2	$z = f(x, y)$	$F(x, y, z) = 0$	\mathbb{R}^3	level curve
3	$w = f(x, y, z)$	$F(x, y, z, w) = 0$	\mathbb{R}^4	level surface

Example

Find the domain and range of the function $f(x, y, z) = \ln(16 - 4x^2 - 4y^2 - z^2)$.

$$16 - 4x^2 - 4y^2 - z^2 > 0$$

Domain $\{(x, y, z) \mid 4x^2 - 4y^2 - z^2 < 16\}$

Range $(-\infty, \ln 16]$

$$\lim_{x \rightarrow 0^-} \ln x = -\infty$$



Characterize the level surfaces of the function $f(x, y, z) = \ln(16 - 4x^2 - 4y^2 - z^2)$.

$$k = \ln 16$$

$$\ln(16 - 4x^2 - 4y^2 - z^2) = \ln 16$$

$$16 - 4x^2 - 4y^2 - z^2 = 16$$

$$4x^2 + 4y^2 + z^2 = 0 \text{ point at the origin}$$

$$k < \ln 16$$

$$\ln(16 - 4x^2 - 4y^2 - z^2) = k$$

$$16 - 4x^2 - 4y^2 - z^2 = e^k$$

$$16 - e^k = 4x^2 + 4y^2 + z^2 \Rightarrow \text{ellipsoid}$$

$$> 0$$

↓
Summary

Curve $y = f(x)$

Surface $z = f(x, y)$

$\frac{df}{dx}$ ✓

$\frac{d^2f}{dx^2}$ ✓

$y - y_0 = \frac{df}{dx}(x - x_0)$ ←

$\Delta f \approx \frac{df}{dx} \Delta x$ ←

$\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt}$

$\frac{df}{dx} = 0$ critical point

partial derivatives

L12

$\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

$\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}, \frac{\partial^2 f}{\partial y^2}$

tangent planes

L13

$z - z_0 = \frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial y}(y - y_0)$

linear approximation and differentials

$\Delta f \approx \frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y$

Chain rule

L14

$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$

L17/L18 max/min

$\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

LECTURE 11: Limits and Continuity (9/23/24)

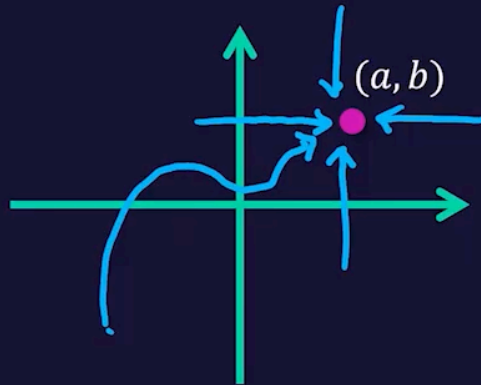
- continuity indicates
 - defined at $x = a$
 - limit exists at $x = a$
 - limit = value at $x = a$

Limits

Information

Intuitively, $\lim_{(x,y) \rightarrow (a,b)} f(x,y) = L$ means that as the point (x,y) gets very close to (a,b) , then $f(x,y)$ gets very close to L .

In this case, the point (a,b) lies in a plane and there exists an infinite many paths to approach this point. How can we test them all?




- if 2 paths to the point are equal, it does not mean the limit exists in 3D

Limits

Example and Note


Calculate the limit $\lim_{(x,y) \rightarrow (0,0)} \frac{x^3 - y^3}{x^3 + y^3}$ by

1. Approaching (0,0) along the line $y = 0$



$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^3 - y^3}{x^3 + y^3} = \lim_{x \rightarrow 0} \frac{x^3}{x^3} = 1$$

2. Approaching (0,0) along the line $x = 0$



$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^3 - y^3}{x^3 + y^3} = \lim_{y \rightarrow 0} \frac{-y^3}{y^3} = -1$$


-1 ≠ 1
limit DNE



We can conclude that the limit does not exist since we found two approaches that produced different limits. However, we **cannot** conclude that the limit exists when two approaches produce the same limit.


Example: Show that $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y}{x^4 + y^2}$ DNE.

path $y = x$



$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y}{x^4 + y^2} = \lim_{x \rightarrow 0} \frac{x^2 x}{x^4 + x^2} = \lim_{x \rightarrow 0} \frac{x}{x^2 + 1} = 0$$

path $y = x^2$



$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y}{x^4 + y^2} = \lim_{x \rightarrow 0} \frac{x^2 x^2}{x^4 + (x^2)^2} = \lim_{x \rightarrow 0} \frac{x^4}{x^4 + x^4} = \frac{1}{2}$$

$0 \neq \frac{1}{2}$ limit DNE.

Theorem (Limit Laws)

Assume that $\lim_{(x,y) \rightarrow (a,b)} f(x,y) = L$ and $\lim_{(x,y) \rightarrow (a,b)} g(x,y) = M$, and let c be a constant.
Then

1. $\lim_{(x,y) \rightarrow (a,b)} c = c$
2. $\lim_{(x,y) \rightarrow (a,b)} x = a$ and $\lim_{(x,y) \rightarrow (a,b)} y = b$
3. $\lim_{(x,y) \rightarrow (a,b)} [f(x,y) \pm g(x,y)] = L \pm M$
4. $\lim_{(x,y) \rightarrow (a,b)} [cf(x,y)] = cL$
5. $\lim_{(x,y) \rightarrow (a,b)} [f(x,y)g(x,y)] = LM$
6. $\lim_{(x,y) \rightarrow (a,b)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M}$ for $M \neq 0$
7. $\lim_{(x,y) \rightarrow (a,b)} [f(x,y)]^n = L^n$ for an integer $n > 0$
8. $\lim_{(x,y) \rightarrow (a,b)} \sqrt[n]{f(x,y)} = L^{\frac{1}{n}}$ for an integer $n > 0$
(assume $L > 0$ if n is even)

Limits

Note

$$\begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta \\ r^2 &= x^2 + y^2 \end{aligned}$$



take all the paths

Limits at (0,0) may be easier to evaluate by converting to **polar coordinates**.



$(x,y) \rightarrow (0,0)$ implies $r \rightarrow 0$.

Find the Limit

Examples

$$\begin{aligned} 1. \quad & \lim_{(x,y) \rightarrow (0,0)} \frac{\sin(2x^2+2y^2)}{x^2+y^2} \\ &= \lim_{r \rightarrow 0} \frac{\sin(2r^2)}{r^2} \\ &\stackrel{L'H}{=} \lim_{r \rightarrow 0} \frac{4r \cos(2r^2)}{2r} \\ &= \lim_{r \rightarrow 0} 2 \cos(2r^2) \\ &= 2 \end{aligned}$$

$$\begin{aligned} 2. \quad & \lim_{(x,y) \rightarrow (0,0)} \frac{x^2}{\sqrt{x^2+y^2}} \\ &= \lim_{r \rightarrow 0} \frac{r^2 \cos^2 \theta}{r} \\ &= \lim_{r \rightarrow 0} r \cos^2 \theta \end{aligned}$$

$$\begin{aligned} 0 &\leq \cos^2 \theta \leq 1 \\ 0 &\leq r \cos^2 \theta \leq r \\ \downarrow &\qquad \qquad \downarrow \qquad \qquad \downarrow \\ 0 &\qquad \qquad \qquad 0 \end{aligned}$$

= 0 by the squeeze thm

Continuity

Definition



A function $f(x, y)$ is **continuous** at (a, b) if

$$\lim_{(x,y) \rightarrow (a,b)} f(x,y) = f(a,b)$$

We say f is continuous on D if f is continuous at every point (a, b) in D .



1. Using the Limit Laws, we can prove that all polynomials are continuous on \mathbb{R}^2 and all rational functions are continuous on their domains. So, we can use **direct substitution** to find the limit:

$$\begin{aligned}\lim_{(x,y) \rightarrow (2,-1)} (x^2 - 4xy + 3y^2 - 1) &= \\ &= 2^2 - 4(2)(-1) + 3(-1)^2 - 1 \\ &= 14\end{aligned}$$

2. The basic properties of continuous functions of a single variable carry over to multivariable functions; that is the sum, difference, product, quotient, and composition of continuous functions are continuous.

For example, $\lim_{(x,y) \rightarrow (2,-1)} \frac{\sqrt{2x+3y}}{4x-y} =$

$$\begin{aligned}&= \frac{\sqrt{2(2)+3(-1)}}{4(2)-(-1)} \\ &= \frac{1}{9}\end{aligned}$$

Continuity

Notes



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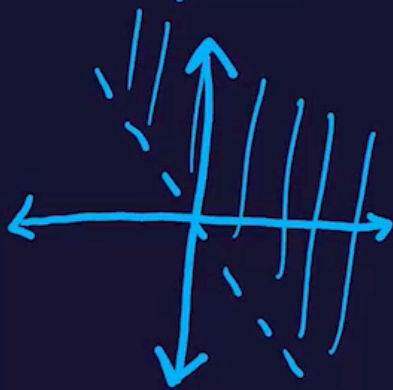
Continuity

Examples

Discuss the continuity of $f(x, y) = \ln(x + y)$.

$$x + y > 0$$

$$y > -x$$



Continuity

Examples

$$f(x, y) = \begin{cases} \frac{x^4 - y^4}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

$$\lim_{(x,y) \rightarrow (0,0)} f(x,y) = 0 = f(0,0) \quad \leftarrow$$

$$\begin{aligned} \lim_{(x,y) \rightarrow (0,0)} f(x,y) &= \lim_{(x,y) \rightarrow (0,0)} \frac{x^4 - y^4}{x^2 + y^2} \\ &= \lim_{(x,y) \rightarrow (0,0)} \frac{(x^2 - y^2)(x^2 + y^2)}{(x^2 + y^2)} \\ &= \lim_{(x,y) \rightarrow (0,0)} x^2 - y^2 \\ &= 0 \end{aligned}$$

continuous everywhere

$$g(x, y) = \begin{cases} \frac{x^4 - y^4}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 1 & (x, y) = (0, 0) \end{cases}$$

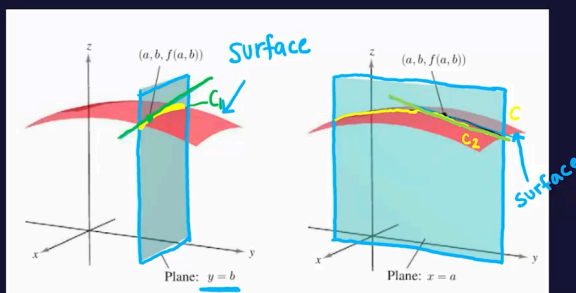
$$\lim_{(x,y) \rightarrow (0,0)} g(x,y) = 1 = g(0,0)$$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^4 - y^4}{x^2 + y^2} = 0 \neq 1$$

continuous everywhere except the origin

LECTURE 12: Partial Derivatives (9/25/24)

Geometric Interpretation



$f_x(a, b)$ is the slope of the tangent line of the curve C_1

$f_y(a, b)$ is the slope of the tangent line of the curve C_2

to what we did in Calc 1.

Let C_1 be the curve of intersection of the plane $y = b$ with the surface $z = f(x, y)$. Then $f_x(a, b)$ is the slope of the tangent line at the point $(a, b, f(a, b))$ to the curve C_1 . Similarly, $f_y(a, b)$ is the slope of the tangent line at the same point to the curve C_2 , where C_2 is the curve of intersection of the plane $x = a$ with the surface.

Calculation of a partial derivative of $z = f(x, y)$:

To find f_x , take the standard derivative of the function with respect to x and treat y as a constant.

Find the partial derivatives of f :

1. $f(x, y) = y \sin(x^2 + xy + y^2)$

$$f_x = y \cos(x^2 + xy + y^2)(2x + y)$$

$$f_y = 1 \sin(x^2 + xy + y^2) + y \cos(x^2 + xy + y^2)(x + 2y)$$

2. $f(x, y, z) = e^{2x} \cos(z^2) + e^{3y} \sin(2z)$

$$f_x = 2e^{2x} \cos(z^2)$$

$$f_y = 3e^{3y} \sin(2z)$$

$$f_z = -e^{2x} \sin(z^2)(2z) + e^{3y} \cos(2z)(2)$$

Partial Derivatives

Examples

Find the slope of the tangent line to the curve of intersection of the surface

$z = \sqrt{36 - 9x^2 - 4y^2}$ and the plane $x = 1$ at the point $(1, -2, \sqrt{11})$.

$$\frac{\partial z}{\partial y} = \frac{1}{2} (36 - 9x^2 - 4y^2)^{-\frac{1}{2}} (-8y)$$

$$= \frac{-4y}{\sqrt{36 - 9x^2 - 4y^2}}$$

$$\left. \frac{\partial z}{\partial y} \right|_{(1, -2)} = \frac{-4(-2)}{\sqrt{36 - 9(1)^2 - 4(2)^2}} = \frac{8}{\sqrt{11}}$$

$$z = f(x, y)$$

Find $\frac{\partial z}{\partial x}$ if $yz - \ln(z) = x^2 + y^2$.

$$y \frac{\partial z}{\partial x} - \frac{1}{z} \frac{\partial z}{\partial x} = 2x$$

$$\frac{\partial z}{\partial x} \left(y - \frac{1}{z} \right) = 2x \quad \cdot z$$

$$\frac{\partial z}{\partial x} = \frac{2xz}{y - \frac{1}{z}}$$

$$\frac{\partial z}{\partial x} = \frac{2xz}{yz - 1}$$

Higher Derivatives

$f(x, y)$ has the following **second partial derivatives**. *mixed partials*

1. Differentiate twice w.r.t. x :

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = (f_x)_x = f_{xx}$$

2. Differentiate twice w.r.t. y :

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = (f_y)_y = f_{yy}$$

3. Differentiate first w.r.t. x and then w.r.t. y :

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = (f_x)_y = f_{xy}$$

4. Differentiate first w.r.t. y and then w.r.t. x :

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = (f_y)_x = f_{yx}$$