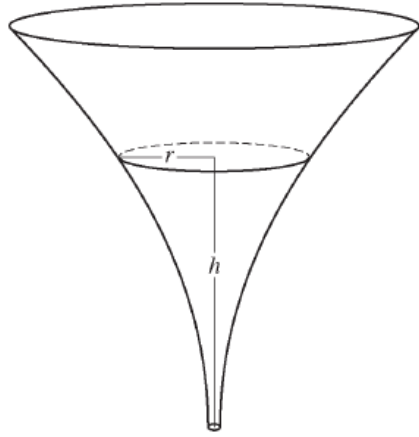


AB Free Response 7.5



1. The inside of a funnel of height 10 inches has circular cross sections, as shown in the figure above. At height h , the radius of the funnel is given by $r = \frac{1}{12}(3 + h^2)$, where $0 \leq h \leq 10$. The units of r and h are in inches.

(a) Find the average value of the radius of the funnel.

l: integral

l: antiderivative

l: answer

(b) Find the volume of the funnel.

l: integrand

l: antiderivative

l: answer

(c) The funnel contains liquid that is draining from the bottom. At the instant when the height of the liquid is $h = 3$ inches, the radius of the surface of the liquid is decreasing at a rate of $\frac{1}{5}$ inch per second. At this instant, what is the rate of change of the height of the liquid with respect to time?

2: *chain rule*

1: *answer*

Solutions 7.5 - 1

$$(a) \text{ Average radius} = \frac{1}{10} \int_0^{10} \frac{1}{12} (3 + h^2) dh$$

1: integral

$$= \frac{1}{120} \left[3h + \frac{h^3}{3} \right]_0^{10}$$

1: antiderivative

$$= \frac{1}{120} \left[\left(30 + \frac{1000}{3} \right) - 0 \right] = \frac{109}{12} \text{ in}$$

1: answer

$$(b) \text{ Volume} = \pi \int_0^{10} \left(\frac{1}{12} (3 + h^2) \right)^2 dh = \frac{\pi}{144} \int_0^{10} (9 + 6h^2 + h^4) dh$$

1: integrand

$$= \frac{\pi}{144} \left[9h + 2h^3 + \frac{h^5}{5} \right]_0^{10}$$

1: antiderivative

$$= \frac{\pi}{144} ((90 + 2000 + 20,000) - 0) = \frac{11,045\pi}{72}$$

1: answer

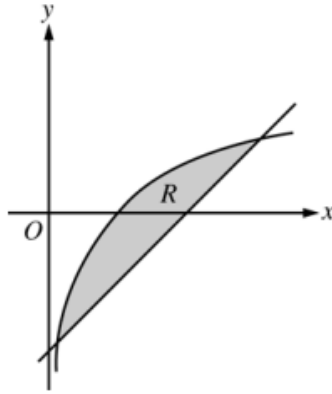
$$(c) \frac{dr}{dt} = \frac{1}{12} (2h) \frac{dh}{dt}$$

2: chain rule

$$-\frac{1}{5} = \frac{1}{2} \frac{dh}{dt}$$

$$\frac{dh}{dt} = -\frac{1}{5} \cdot 2 = -\frac{2}{5} \text{ in/sec}$$

1: answer



2. Let R be the shaded region bounded by the graph of $y = \ln x$ and the line $y = x - 3$, as shown above.

(a) Find the area of R .

1: integrand

1: limits

1: answer

(b) Find the volume of the solid generated when R is rotated about the horizontal line $y = -4$.

2: integrand

1: limits, constant, and answer

(c) Write, but do not evaluate, an integral expression that can be used to find the volume of the solid generated when R is rotated about the y -axis.

2: integrand

1: limits and constant

Solutions 7.5 - 2

$\ln(x) = x - 3$ when $x = 0.052$ and 4.505 .

Let $a = 0.052$ and $b = 4.505$.

$$\begin{aligned} \text{(a) Area of } R &= \int_a^b (\ln(x) - (x - 3)) \, dx \\ &= 5.694 \end{aligned}$$

1: integrand

1: limits

1: answer

$$\begin{aligned} \text{(b) Volume} &= \pi \int_a^b (\ln(x) + 4)^2 - (x - 3 + 4)^2 \, dx \\ &= 127.762 \end{aligned}$$

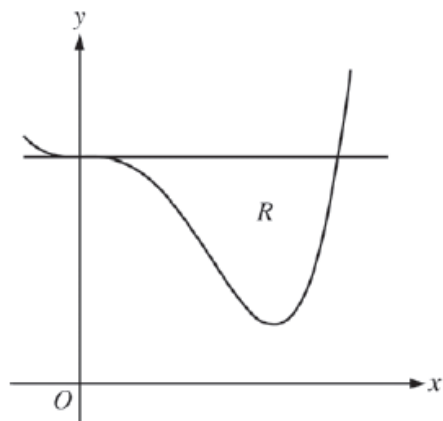
2: integrand

1: limits, constant, and answer

$$\text{(c) Volume} = \pi \int_{a-3}^{b-3} ((y + 3)^2 - (e^y)^2) \, dy$$

2: integrand

1: limits and constant



3. Let R be the region enclosed by the graph of $f(x) = x^4 - 2.6x^3 + 4$ and the horizontal line $y = 4$, as shown in the figure above.

(a) Find the volume of the solid generated when R is rotated about the horizontal line $y = -2$.

2: *integrand*

1: *limits*

1: *answer*

(b) Region R is the base of a solid. For this solid, each cross section perpendicular to the x -axis is an isosceles right triangle with a leg in R . Find the volume of the solid.

2: *integrand*

1: *answer*

(c) The vertical line $x = k$ divides R into two regions with equal areas. Write, but do not solve, an equation involving integral expressions whose solution gives the value k .

1: *area of one region*

1: *equation*

Solutions 7.5 - 3

$$(a) f(x) = 4 \rightarrow x = 0, 2.6$$

$$\begin{aligned} \text{Volume} &= \pi \int_0^{2.6} [(4 + 2)^2 - (f(x) + 2)^2] dx \\ &= 156.271 \end{aligned}$$

2: integrand

1: limits

1: answer

$$\begin{aligned} (b) \text{ Volume} &= \int_0^{2.6} \frac{1}{2} (4 - f(x))^2 dx \\ &= 10.773 \end{aligned}$$

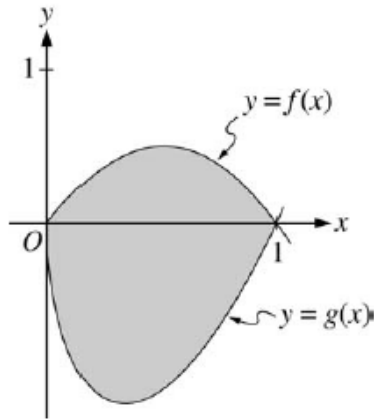
2: integrand

1: answer

$$(c) \int_0^k (4 - f(x)) dx = \int_k^{2.6} (4 - f(x)) dx$$

1: area of one region

1: equation



4. Let f and g be the functions given by $f(x) = 2x(1 - x)$ and $g(x) = 3(x - 1)\sqrt{x}$ for $0 \leq x \leq 1$. The graphs of f and g are shown in the figure above.

(a) Find the area of the shaded region enclosed by the graphs of f and g .

1: *integral*

1: *answer*

(b) Find the volume of the solid generated when the shaded region enclosed by the graphs of f and g is revolved about the horizontal line $y = 3$.

1: *limits and constant*

2: *integrand*

1: *answer*

(c) Let h be the function given by $h(x) = kx(4 - x)$ for $0 \leq x \leq 1$. For each $k > 0$, the region (not shown) enclosed by the graphs of h and g is the base of a solid with square cross sections perpendicular to the x -axis. There is a value of k for which the volume of this solid is equal to 11. Write, but do not solve, an equation involving an integral expression that could be used to find the value of k .

2: *integrand*

1: *answer*

Solutions 7.5 - 4

$$(a) \text{ Area} = \int_0^1 (f(x) - g(x)) \, dx$$

1: integral

$$= \int_0^1 (2x(1-x) - 3(x-1)\sqrt{x}) \, dx = 1.133$$

1: answer

$$(b) \text{ Volume} = \pi \int_0^1 ((3 - g(x))^2 - (3 - f(x))^2) \, dx$$

1: limits and constant

$$= \pi \int_0^1 ((3 - 3(x-1)\sqrt{x})^2 - (3 - 2x(1-x))^2) \, dx$$

2: integrand

$$= 23.300$$

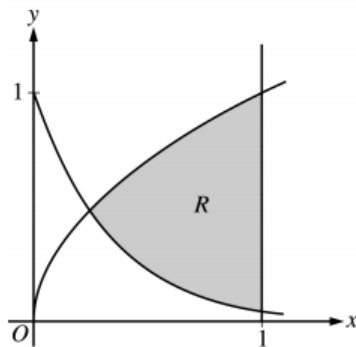
1: answer

$$(c) \text{ Volume} = \int_0^1 (h(x) - g(x))^2 \, dx$$

2: integrand

$$\int_0^1 (kx(4-x) - 3(x-1)\sqrt{x})^2 \, dx = 11$$

1: answer



5. Let R be the shaded region bounded by the graphs of $y = \sqrt{x}$ and $y = e^{-3x}$ and the vertical line $x = 1$, as shown in the figure above.

(a) Find the area of R .

1: integrand

1: bounds

1: answer

(b) Find the volume of the solid generated when R is revolved about the horizontal line $y = 1$.

2: integrand

1: answer

(c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x -axis is a rectangle whose height is 3 times the length of its base in region R . Find the volume of this solid.

2: *integrand*

1: *answer*

Solutions 7.5 - 5

Point of intersection

$$e^{-3x} = \sqrt{x} \text{ at } (T, S) = (0.239, 0.489)$$

$$\begin{aligned} \text{(a) Area} &= \int_T^1 \sqrt{x} - e^{-3x} dx \\ &= 0.443 \end{aligned}$$

1: integrand

1: answer

$$\begin{aligned} \text{(b) Volume} &= \pi \int_T^1 ((1 - e^{-3x})^2 - (1 - \sqrt{x})^2) dx \\ &= 0.453\pi \text{ or } 1.424 \end{aligned}$$

2: integrand

1: answer

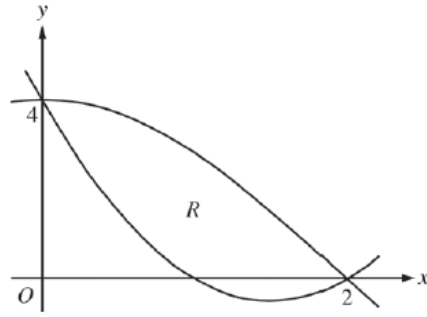
$$\text{(c) Length} = \sqrt{x} - e^{-3x}$$

2: integrand

$$\text{Height} = 3(\sqrt{x} - e^{-3x})$$

1: answer

$$\text{Volume} = \int_T^1 3(\sqrt{x} - e^{-3x})^2 dx = 0.933$$



6. Let $f(x) = 2x^2 - 6x + 4$ and $g(x) = 4\cos(\frac{1}{4}\pi x)$. Let R be the region bounded by the graphs of f and g , as shown in the figure above.

(a) Find the area of R .

1: *integrand*

2: *antiderivative*

1: *answer*

(b) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = 4$.

2: integrand

1: limits and constant

(c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x -axis is a semicircle. Write, but do not evaluate, an integral expression that gives the volume of the solid.

1: integrand

1: limits and constant

Solutions 7.5 - 6

$$(a) \text{ Area} = \int_0^2 [g(x) - f(x)] \, dx \quad 1: \text{ integrand}$$

$$= \int_0^2 [4\cos(\frac{\pi}{4}x) - (2x^2 - 6x + 4)] \, dx$$

$$= \left[4 \frac{4}{\pi} \sin\left(\frac{\pi}{4}x\right) - \left(\frac{2x^3}{3} - 3x^2 + 4x\right) \right]_0^2 \quad 2: \text{ antiderivative}$$

$$= \frac{16}{\pi} - \left(\frac{16}{3} - 12 + 8\right)$$

$$= \frac{16}{\pi} - \frac{4}{3} \quad 1: \text{ answer}$$

$$(b) \text{ Volume} = \pi \int_0^2 [(4 - f(x))^2 - (4 - g(x))^2] \, dx \quad 2: \text{ integrand}$$

$$= \pi \int_0^2 [(4 - (2x^2 - 6x + 4))^2 - (4 - 4\cos(\frac{\pi}{4}x))^2] \, dx \quad 1: \text{ limits and constant}$$

$$(c) \text{ Volume} = \int_0^2 \pi \left(\frac{g(x) - f(x)}{2} \right)^2 \, dx \quad 1: \text{ integrand}$$

$$= \pi \int_0^2 \frac{\left(4\cos(\frac{\pi}{4}x) - (2x^2 - 6x + 4) \right)^2}{4} \, dx \quad 1: \text{ limits and constant}$$