See this page in the course material.

Learning Outcomes

- Find and evaluate the inverse of a function in tabular form.
- Find and evaluate the inverse of a function given as an equation.
- Graph a function and its inverse.

Once we have a one-to-one function, we can evaluate its inverse at specific inverse function inputs or construct a complete representation of the inverse function in many cases.

Inverting Tabular Functions

Suppose we want to find the inverse of a function represented in table form. Remember that the domain of a function is the range of the inverse and the range of the function is the domain of the inverse. So we need to interchange the domain and range.

Each row (or column) of inputs becomes the row (or column) of outputs for the inverse function. Similarly, each row (or column) of outputs becomes the row (or column) of inputs for the inverse function.

Example: Interpreting the Inverse of a Tabular Function

A function [latex]f\left(t\right)[/latex] is given below, showing distance in miles that a car has traveled in [latex]t[/latex] minutes. Find and interpret [latex]{f}^{-1}\left(70\right)[/latex].

[latex]t\text{ (minutes)}[/latex]	30	50	70	90
[latex]f\left(t\right)\text{ (miles)}[/latex]	20	40	60	70

Show Solution

The inverse function takes an output of [latex]f[/latex] and returns an input for [latex]f[/latex]. So in the expression [latex]{f}^{-1}\left(70\right)[/latex], 70 is an output value of the original function, representing 70 miles. The inverse will return the corresponding input of the original function [latex]f[/latex], 90 minutes, so [latex]{f}^{-1}\left(70\right)=90[/latex]. The interpretation of this is that, to drive 70 miles, it took 90 minutes.

Alternatively, recall that the definition of the inverse was that if [latex]f\left(a\right)=b[/latex], then [latex]{f}^{-1}\left(b\right)=a[/latex]. By this definition, if we are given [latex]{f}^{-1}\left(70\right)=a[/latex], then we are looking for a value [latex]a[/latex] so that [latex]f\left(a\right)=70[/latex]. In this case, we are looking for a [latex]t[/latex] so that [latex]f\left(t\right)=70[/latex], which is when [latex]t=90[/latex].

Try It

Using the table below, find and interpret (a) [latex]\text{ }f\left(60\right)[/latex], and (b) [latex]\text{ }{f}^{-1}\left(60\right)[/latex].

[latex]t\text{ (minutes)}[/latex]	30	50	60	70	90
[latex]f\left(t\right)\text{ (miles)}[/latex]	20	40	50	60	70

Show Solution

a. [latex]f\left(60\right)=50[/latex]. In 60 minutes, 50 miles are traveled.

b. [latex]{f}^{-1}\left(60\right)=70[/latex]. To travel 60 miles, it will take 70 minutes.

Evaluating the Inverse of a Function, Given a Graph of the Original Function

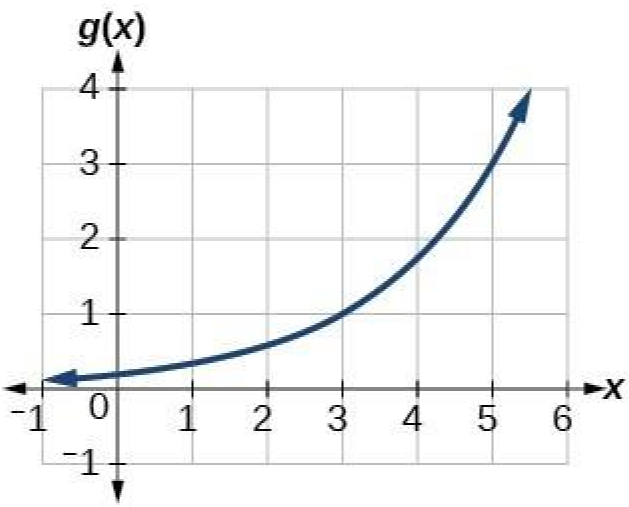
The domain of a function can be read by observing the horizontal extent of its graph. We find the domain of the inverse function by observing the *vertical* extent of the graph of the original function, because this corresponds to the horizontal extent of the inverse function. Similarly, we find the range of the inverse function by observing the *horizontal* extent of the graph of the original function, as this is the vertical extent of the inverse function. If we want to evaluate an inverse function, we find its input within its domain, which is all or part of the vertical axis of the original function’s graph.

How To: Given the graph of a function, evaluate its inverse at specific points.

1. Find the desired input of the inverse function on the [latex]y[/latex]-axis of the given graph.
2. Read the inverse function’s output from the [latex]x[/latex]-axis of the given graph.

Example: Evaluating a Function and Its Inverse from a Graph at Specific Points

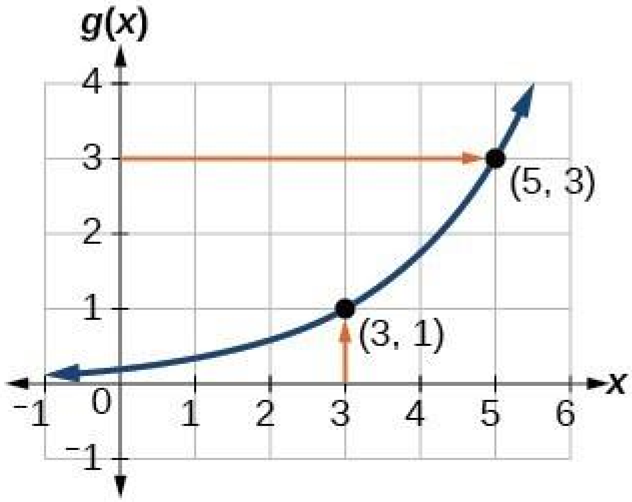
A function [latex]g\left(x\right)[/latex] is given below. Find [latex]g\left(3\right)[/latex] and [latex]{g}^{-1}\left(3\right)[/latex].



Show Solution

To evaluate [latex]g\left(3\right)[/latex], we find 3 on the *x*-axis and find the corresponding output value on the [latex]y[/latex]-axis. The point [latex]\left(3,1\right)[/latex] tells us that [latex]g\left(3\right)=1[/latex].

To evaluate [latex]{g}^{-1}\left(3\right)[/latex], recall that by definition [latex]{g}^{-1}\left(3\right)[/latex] means the value of *x* for which [latex]g\left(x\right)=3[/latex]. By looking for the output value 3 on the vertical axis, we find the point [latex]\left(5,3\right)[/latex] on the graph, which means [latex]g\left(5\right)=3[/latex], so by definition, [latex]{g}^{-1}\left(3\right)=5[/latex].



Try It

Using the graph in the previous example, (a) find [latex]{g}^{-1}\left(1\right)[/latex], and (b) estimate [latex]{g}^{-1}\left(4\right)[/latex].

Show Solution

a. 3; b. 5.6

Finding Inverses of Functions Represented by Formulas

Sometimes we will need to know an inverse function for all elements of its domain, not just a few. If the original function is given as a formula—for example, [latex]y[/latex] as a function of [latex]x-[/latex] we can often find the inverse function by solving to obtain [latex]x[/latex] as a function of [latex]y[/latex].

How To: Given a function represented by a formula, find the inverse.

1. Verify that [latex]f[/latex] is a one-to-one function.
2. Replace [latex]f\left(x\right)[/latex] with [latex]y[/latex].

3. Interchange [latex]x[/latex] and [latex]y[/latex].
4. Solve for [latex]y[/latex], and rename the function [latex]{f}^{-1}\left(x\right)[/latex].

Example: Inverting the Fahrenheit-to-Celsius Function

Find a formula for the inverse function that gives Fahrenheit temperature as a function of Celsius temperature.

[latex]C=\frac{5}{9}\left(F - 32\right)[/latex]

Show Solution

[latex]{ C }=\frac{5}{9}\left(F - 32\right)[/latex]

[latex]C\cdot \frac{9}{5}=F - 32[/latex]

[latex]F=\frac{9}{5}C+32[/latex]

By solving in general, we have uncovered the inverse function. If

[latex]C=h\left(F\right)=\frac{5}{9}\left(F - 32\right)[/latex],

then

[latex]F={h}^{-1}\left(C\right)=\frac{9}{5}C+32[/latex].

In this case, we introduced a function [latex]h[/latex] to represent the conversion because the input and output variables are descriptive, and writing [latex]{C}^{-1}[/latex] could get confusing.

Try It

Solve for [latex]x[/latex] in terms of [latex]y[/latex] given [latex]y=\frac{1}{3}\left(x - 5\right)[/latex]

Show Solution

[latex]x=3y+5[/latex]

Example: Solving to Find an Inverse Function

Find the inverse of the function [latex]f\left(x\right)=\dfrac{2}{x - 3}+4[/latex].

Show Solution

[latex]\begin{align}&y=\frac{2}{x - 3}+4 && \text{Change }f(x)\text{ to }y. \\[1.5mm]&x=\frac{2}{y - 3}+4 && \text{Switch }x\text{ and }y. \\[1.5mm] &y - 4=\frac{2}{x - 3} && \text{Subtract 4 from both sides}. \\[1.5mm] &y - 3=\frac{2}{x - 4} && \text{Multiply both sides by }y - 3\text{ and divide by }x - 4. \\[1.5mm] &y=\frac{2}{x - 4}+3 && \text{Add 3 to both sides}.\\[-3mm]&\end{align}[/latex]

So [latex]{f}^{-1}\left(x\right)=\dfrac{2}{x - 4}+3[/latex].

Analysis of the Solution

The domain and range of [latex]f[/latex] exclude the values 3 and 4, respectively. [latex]f[/latex] and [latex]{f}^{-1}[/latex] are equal at two points but are not the same function, as we can see by creating the table below.

[latex]x[/latex]	1	2	5	[latex]{f}^{-1}\left(y\right)[/latex]
[latex]f\left(x\right)[/latex]	3	2	5	[latex]y[/latex]

Example: Solving to Find an Inverse with Radicals

Find the inverse of the function [latex]f\left(x\right)=2+\sqrt{x - 4}[/latex].

Show Solution

[latex]\begin{align}&y=2+\sqrt{x - 4}\\[1.5mm]&x=2+\sqrt{y - 4}\\[1.5mm] &{\left(x - 2\right)}^{2}=y - 4 \\[1.5mm] &y={\left(x- 2\right)}^{2}+4 \end{align}[/latex]

So [latex]{f}^{-1}\left(x\right)={\left(x - 2\right)}^{2}+4[/latex].

The domain of [latex]f[/latex] is [latex]\left[4,\infty \right)[/latex]. Notice that the range of [latex]f[/latex] is [latex]\left[2,\infty \right)[/latex], so this means that the domain of the inverse function [latex]{f}^{-1}[/latex] is also [latex]\left[2,\infty \right)[/latex].

Analysis of the Solution

The formula we found for [latex]{f}^{-1}\left(x\right)[/latex] looks like it would be valid for all real [latex]x[/latex]. However, [latex]{f}^{-1}[/latex] itself must have an inverse (namely, [latex]f[/latex]) so we have to restrict the domain of [latex]{f}^{-1}[/latex] to [latex]\left[2,\infty \right)[/latex] in order to make [latex]{f}^{-1}[/latex] a one-to-one function. This domain of [latex]{f}^{-1}[/latex] is exactly the range of [latex]f[/latex].

Try It

What is the inverse of the function [latex]f\left(x\right)=2-\sqrt{x}[/latex]? State the domains of both the function and the inverse function.

Use an online graphing tool to graph the function, it's inverse, and [latex]f(x) = x[/latex] to check whether you are correct.

Launch Desmos Calculator

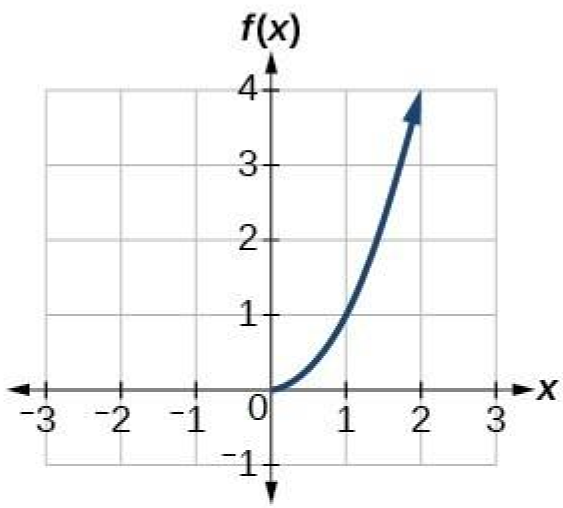
Show Solution

[latex]{f}^{-1}\left(x\right)={\left(2-x\right)}^{2}[/latex]; domain of [latex]f:\left[0,\infty \right)[/latex];

domain of [latex]{ f}^{-1}:\left(-\infty ,2\right][/latex]

Graph a Function’s Inverse

Now that we can find the inverse of a function, we will explore the graphs of functions and their inverses. Let us return to the quadratic function [latex]f\left(x\right)={x}^{2}[/latex] restricted to the domain [latex]\left[0,\infty \right)[/latex], on which this function is one-to-one, and graph it as below.

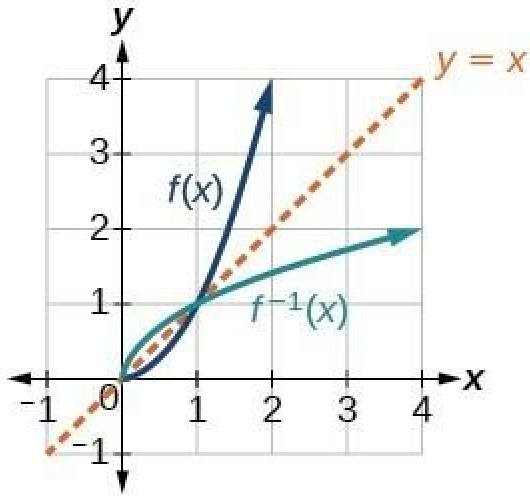


Quadratic function with domain restricted to [0, ∞).

Restricting the domain to [latex]\left[0,\infty \right)[/latex] makes the function one-to-one (it will obviously pass the horizontal line test), so it has an inverse on this restricted domain.

We already know that the inverse of the toolkit quadratic function is the square root function, that is, [latex]{f}^{-1}\left(x\right)=\sqrt{x}[/latex]. What happens if we graph both [latex]f\text{ }[/latex] and [latex]{f}^{-1}[/latex] on the same set of axes, using the [latex]x\text{-}[/latex] axis for the input to both [latex]f\text{ and }{f}^{-1}?[/latex]

We notice a distinct relationship: The graph of [latex]{f}^{-1}\left(x\right)[/latex] is the graph of [latex]f\left(x\right)[/latex] reflected about the diagonal line [latex]y=x[/latex], which we will call the identity line, shown below.

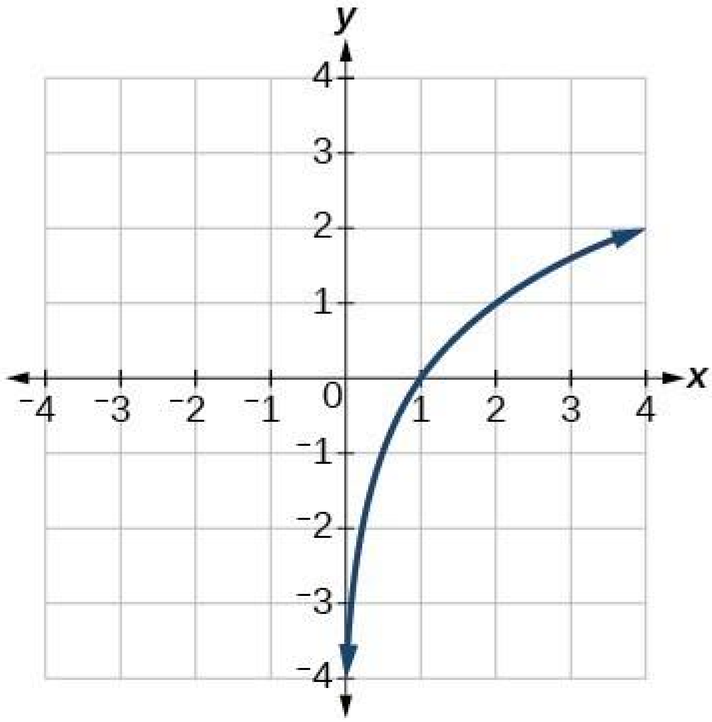


Square and square-root functions on the non-negative domain

This relationship will be observed for all one-to-one functions, because it is a result of the function and its inverse swapping inputs and outputs. This is equivalent to interchanging the roles of the vertical and horizontal axes.

Example: Finding the Inverse of a Function Using Reflection about the Identity Line

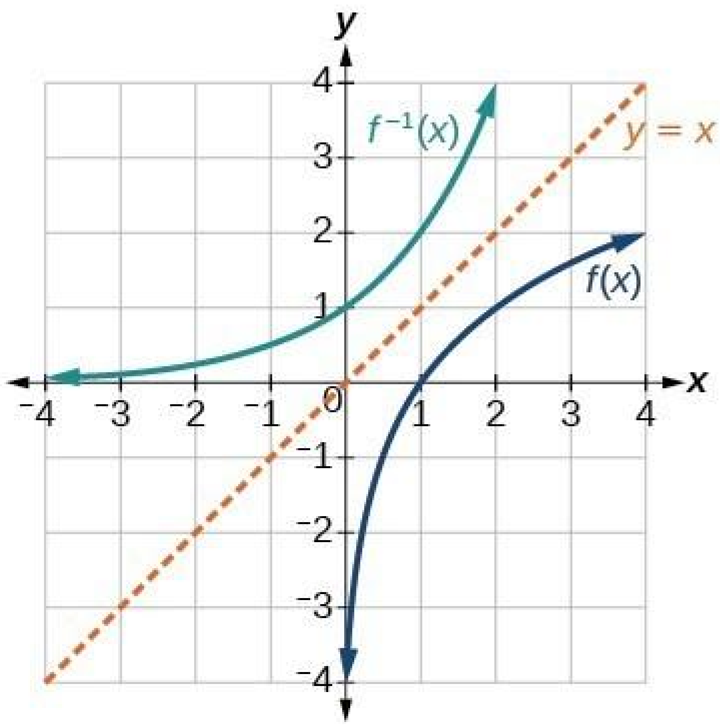
Given the graph of [latex]f\left(x\right)[/latex], sketch a graph of [latex]{f}^{-1}\left(x\right)[/latex].



Show Solution

This is a one-to-one function, so we will be able to sketch an inverse. Note that the graph shown has an apparent domain of [latex]\left(0,\infty \right)[/latex] and range of [latex]\left(-\infty ,\infty \right)[/latex], so the inverse will have a domain of [latex]\left(-\infty ,\infty \right)[/latex] and range of [latex]\left(0,\infty \right)[/latex].

If we reflect this graph over the line [latex]y=x[/latex], the point [latex]\left(1,0\right)[/latex] reflects to [latex]\left(0,1\right)[/latex] and the point [latex]\left(4,2\right)[/latex] reflects to [latex]\left(2,4\right)[/latex]. Sketching the inverse on the same axes as the original graph gives us the result in the graph below.



The function and its inverse, showing reflection about the identity line

Q & A

Is there any function that is equal to its own inverse?

Yes. If [latex]f={f}^{-1}[/latex], then [latex]f\left(f\left(x\right)\right)=x[/latex], and we can think of several functions that have this property. The identity function does, and so does the reciprocal function, because

[latex]\frac{1}{\frac{1}{x}}=x[/latex]

Any function [latex]f\left(x\right)=c-x[/latex], where [latex]c[/latex] is a constant, is also equal to its own inverse.

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