1.

$$f(z) = z^4 + 6z^3 - 3z^2 - 96z - 208$$

(a) Use your calculator to solve the equation $f(z) = 0$

(2)

(b) Hence, or otherwise, show that $f(z)$ can be written as

$$(z^2 - a)(z^2 + bz + c)$$

where a, b and c are real constants to be determined.

(3)

Question	Scheme	Marks	AOs
1(a)	$z = \pm 4$	B1	1.1b
	$z = -3 \pm 2\text{i}$	B1	1.1b
		(2)	
(b)	$a = 16$	B1	1.1a
	A complete method to find b and c	M1	3.1a
	$b = 6$ and $c = 13$	A1	1.1b
		(3)	
			(5 marks)

2.

$$zz^* + 3\mathrm{i}z = p + 9\mathrm{i}$$

where z is a complex number and p is a real constant.

Given that this equation has exactly one root, determine the complex number z.

(5)

Question	Scheme	Marks	AOs
2	Using $z = x + y\mathrm{i}$ and $z^* = x - y\mathrm{i}$ $(x + y\mathrm{i})(x - y\mathrm{i}) + 3\mathrm{i}(x + y\mathrm{i}) = p + 9\mathrm{i}$ $\Rightarrow x^2 + y^2 + 3x\mathrm{i} - 3y = p + 9\mathrm{i}$	M1	3.1a
	$x = 3$	B1	1.1b
	Equate real parts $3^2 + y^2 - 3y = p$	M1	1.1b
	Complete method to find the value of y	M1	3.1a
	$z = 3 + \frac{3}{2}\mathrm{i}$	A1	2.2a
		(5)	
			(5 marks)

1.

In this question you may assume the results for

$$\sum_{r=1}^{n} r^3,\ \sum_{r=1}^{n} r^2 \text{ and } \sum_{r=1}^{n} r$$

a. Show that the sum of the cubes of the first n positive odd numbers is

$$n^2(2n^2-1)$$

(5)

The sum of the cubes of 10 consecutive positive odd numbers is 99 800

b. Use the answer to part (a) to determine the smallest of these 10 consecutive positive odd numbers.

(4)

Question	Scheme	Marks	AOs
4(a)	A complete attempt to find the sum of the cubes of the first n odd numbers using three of the standard summation formulae. Attempts to find $\sum(2r+1)^3$ or $\sum(2r-1)^3$ by expanding and using summation formulae	M1	3.1a
	$\sum_{r=1}^{n}(2r-1)^3=\sum_{r=1}^{n}(8r^3-12r^2+6r-1)=8\sum_{r=1}^{n}r^3-12\sum_{r=1}^{n}r^2+6\sum_{r=1}^{n}r-\sum_{r=1}^{n}1$ or $\sum_{r=0}^{n-1}(2r+1)^3=\sum_{r=0}^{n-1}(8r^3+12r^2+6r+1)=8\sum_{r=0}^{n-1}r^3+12\sum_{r=0}^{n-1}r^2+6\sum_{r=0}^{n-1}r+\sum_{r=0}^{n-1}1$	M1	1.1b
	$=8\frac{n^2}{4}(n+1)^2-12\frac{n}{6}(n+1)(2n+1)+6\frac{n}{2}(n+1)-n$ or $=8\frac{(n-1)^2}{4}(n)^2+12\frac{(n-1)}{6}(n)(2n-1)+6\frac{(n-1)}{2}(n)+n$	M1 A1	1.1b 1.1b
	Multiplies out to achieve a correct intermediate line for example $n\ \ n+1\ \ 2n^2-2n+1\ \ -n=2n^4-2n^3+n^2+2n^3-2n^2+n-n$ $2n^4+4n^3+2n^2-4n^3-6n^2-2n+3n^2+3n-n$ leading to $=n^2(2n^2-1)$ cso *	A1 *	2.1
		(5)	

(b)	$\sum_{r=n}^{n+9}(2r-1)^3=\sum_{r=1}^{n+9}(2r-1)^3-\sum_{r=1}^{n-1}(2r-1)^3$ $=(n+9)^2\left(2(n+9)^2-1\right)-(n-1)^2\left(2(n-1)^2-1\right)=99800$ or $\sum_{r=n+1}^{n+10}(2r-1)^3=\sum_{r=1}^{n+10}(2r-1)^3-\sum_{r=1}^{n}(2r-1)^3$ $=(n+10)^2\left(2(n+10)^2-1\right)-(n)^2\left(2n^2-1\right)=99800$ or $\sum_{r=n-9}^{n}(2r-1)^3=\sum_{r=1}^{n}(2r-1)^3-\sum_{r=1}^{n-10}(2r-1)^3$ $=(n)^2\left(2(n)^2-1\right)-(n-10)^2\left(2(n-10)^2-1\right)=99800$	M1	3.1a
	$80n^3+960n^2+5820n-86760=0$ or $80n^3+1200n^2+7980n-79900=0$ or $80n^3-1200n^2+7980n-119700=0$	A1	1.1b
	Solves cubic equation	dM1	1.1b

[redacted]

	Achieves $n=6$ and the smallest number as 11 or Achieves $n=5$ and the smallest number as 11 or Achieves $n=15$ and the smallest number as 11	A1	2.3
		(4)	
			(9 marks)

1.

The cubic equation

$$ax^3+bx^2-19x-b=0$$

where *a* and *b* are constants, has roots α, β and γ

The cubic equation

$$w^3-9w^2-97w+c=0$$

where c is a constant, has roots $(4\alpha-1)$, $(4\beta-1)$ and $(4\gamma-1)$

Without solving either cubic equation, determine the value of a, the value of b and the value of c.

(6)

Question	Scheme	Marks	AOs
3	$w=4x-1 \Rightarrow x=\dfrac{w+1}{4}$	B1	3.1a
	$a\left(\dfrac{w+1}{4}\right)^3+b\left(\dfrac{w+1}{4}\right)^2-19\left(\dfrac{w+1}{4}\right)-b\ (=0)$ or $(4x-1)^3-9(4x-1)^2-97(4x-1)+c\ (=0)$	M1	3.1a
	$aw^3+(3a+4b)w^2+(3a+8b-304)w+(a-60b-304)=0$ or $64x^3-192x^2-304x+87+c=0$	M1	1.1b
	Divides by a and equates the coefficients of w^2 and w $\dfrac{3a+4b}{a}=-9 \quad \dfrac{3a+8b-304}{a}=-97$ and solves simultaneously to find a value for a or a value for b **<u>Note:</u>** $12a+4b=0$ and $100a+8b=304$ or Divides through by '16' leading to values of a and b $4x^3-12x^2-19x+\dfrac{87+c}{19}=0$	M1	3.1a
	$c=\dfrac{a-60b-304}{a}=\ldots$ or $\dfrac{87+c}{19}=12 \Rightarrow c=\ldots$	M1	1.1b
	$a=4 \quad b=-12 \quad c=105$	A1	1.1b
		(6)	
			(6 marks)

3.

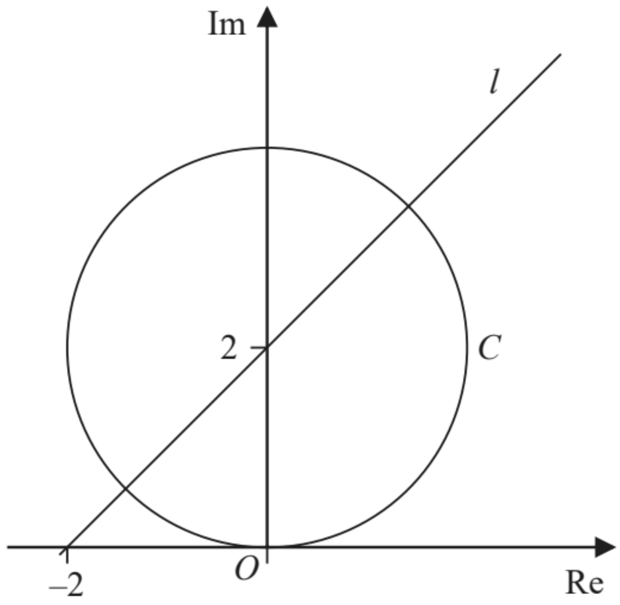


Figure 1

The Argand diagram, shown in Figure 1, shows a circle C and a half-line l.

(a) Write down the equation of the locus of points represented in the complex plane by

(i) the circle C,

(ii) the half-line l.

(2)

(b) Use set notation to describe the set of points that lie on both C and l.

(1)

(c) Find the complex numbers that lie on both C and l, giving your answers in the form $a + \text{i}b$, where $a, b \in \mathbb{R}$.

(3)

Question	Scheme	Marks	AOs
3(a) (i)	$\lvert z-2\mathrm{i}\rvert = 2$	B1	1.1b
(ii)	$\arg(z+2) = \frac{\pi}{4}$	B1	1.1b
		(2)	
(b)	$\{z \in \mathbb{C}: \lvert z-2\mathrm{i}\rvert = 2\} \cap \left\{z \in \mathbb{C}: \arg(z+2) = \frac{\pi}{4}\right\}$	B1ft	2.5
		(1)	
(c)	Solves $x^2+(y-2)^2=4$ and $y=x+2$ to reach $x=\ldots$ or $y=\ldots$ Alternatively uses Pythagoras to find the length of triangle $\sqrt{2}$ and uses to reach $x=\ldots$ or $y=\ldots$	M1	3.1a
	Finds a complete coordinate or complex number	dM1	1.1b
	$z = \sqrt{2}+\left(2+\sqrt{2}\right)\mathrm{i}$ and $z = -\sqrt{2}+\left(2-\sqrt{2}\right)\mathrm{i}$	A1	1.1b
		(3)	
			(6 marks)

The cubic equation

$$2x^3 + 6x^2 - 3x + 12 = 0$$

has roots α, β and γ.

Without solving the equation, find the cubic equation whose roots are $(\alpha + 3)$, $(\beta + 3)$ and $(\gamma + 3)$, giving your answer in the form $pw^3 + qw^2 + rw + s = 0$, where p, q, r and s are integers to be found.

(5)

Question	Scheme	Marks	AOs
2.	$\{w = x + 3 \Rightarrow\}\ x = w - 3$	B1	3.1a
	$2(w-3)^3 + 6(w-3)^2 - 3(w-3) + 12\ (= 0)$	M1	1.1b
	$2w^3 - 18w^2 + 54w - 54 + 6(w^2 - 6w + 9) - 3w + 9 + 12 (= 0)$		
	$2w^3 - 12w^2 + 15w + 21 = 0$	M1	3.1a
	(So $p = 2$, $q = -12$, $r = 15$ and $s = 21$)	A1	1.1b
		A1	1.1b
		(5)	

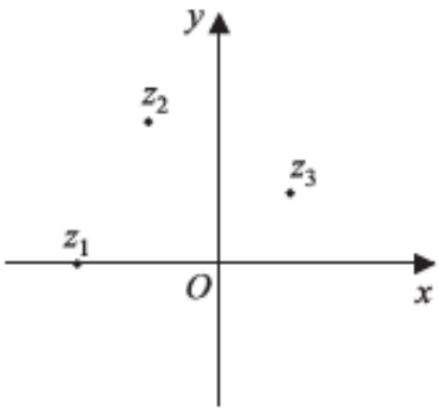


Figure 1

The complex numbers $z_1 = -2$, $z_2 = -1 + 2\text{i}$ and $z_3 = 1 + \text{i}$ are plotted in Figure 1, on an Argand diagram for the complex plane with $z = x + \text{i}y$

(a) Explain why z_1, z_2 and z_3 cannot all be roots of a quartic polynomial equation with real coefficients.

(2)

(b) Show that $\arg\left(\dfrac{z_2 - z_1}{z_3 - z_1}\right) = \dfrac{\pi}{4}$

(3)

(c) Hence show that $\arctan(2) - \arctan\left(\dfrac{1}{3}\right) = \dfrac{\pi}{4}$

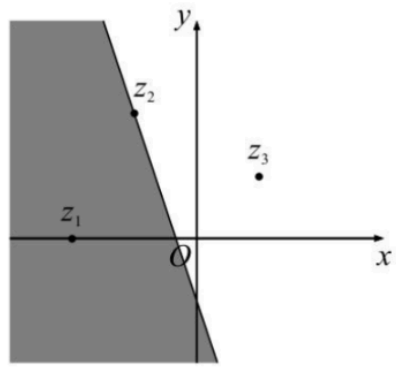
(2)

A copy of Figure 1, labelled Diagram 1, is given on page 12.

(d) Shade, on Diagram 1, the set of points of the complex plane that satisfy the inequality

$$|z + 2| \leqslant |z - 1 - \text{i}|$$

(2)

Question	Scheme	Marks	AOs
5 (a)	Complex roots of a real polynomial occur in conjugate pairs	M1	1.2
	so a polynomial with z_1, z_2 and z_3 as roots also needs z_2* and z_3* as roots, so 5 roots in total, but a quartic has at most 4 roots, so no quartic can have z_1, z_2 and z_3 as roots.	A1	2.4
		(2)	
(b)	$\frac{z_2 - z_1}{z_3 - z_1} = \frac{-1+2i-(-2)}{1+i-(-2)} = \frac{1+2i}{3+i} \times \frac{3-i}{3-i} = \ldots$	M1	1.1b
	$= \frac{3-i+6i+2}{9+1} = \frac{5+5i}{10} = \frac{1}{2} + \frac{1}{2}i$ oe	A1	1.1b
	As $\frac{1}{2} + \frac{1}{2}i$ is in the first quadrant (may be shown by diagram), hence $\arg\left(\frac{z_2 - z_1}{z_3 - z_1}\right) = \arctan\left(\frac{1/2}{1/2}\right)\left(= \arctan(1)\right) = \frac{\pi}{4}$ *	A1*	2.1
		(3)	
(c)	$\arg\left(\frac{z_2 - z_1}{z_3 - z_1}\right) = \arg(z_2 - z_1) - \arg(z_3 - z_1) = \arg(1+2i) - \arg(3+i)$	M1	1.1b
	Hence $\arctan(2) - \arctan\left(\frac{1}{3}\right) = \frac{\pi}{4}$ *	A1*	2.1
		(2)	
(d)	 Line passing through z_2 and the negative imaginary axis drawn.	B1	1.1b
	Area below and left of their line shaded, where the line must have negative gradient passing through negative imaginary axis but need not pass through z_2	B1	1.1b
	Unless otherwise indicated by the student mark Diagram 1(if used) if there are multiple attempts.		
		(2)	
	(9 marks)		

7.

$$f(z) = z^3 - 8z^2 + pz - 24$$

where p is a real constant.

Given that the equation $f(z) = 0$ has distinct roots

$$\alpha, \beta \text{ and } \left(\alpha + \frac{12}{\alpha} - \beta\right)$$

(a) solve completely the equation $f(z) = 0$

(6)

(b) Hence find the value of p.

(2)

Question	Scheme	Marks	AOs
7. (a)	$\alpha + \beta + \left(\alpha + \frac{12}{\alpha} - \beta\right) = 8$ so $2\alpha + \frac{12}{\alpha} = 8$	M1	1.1b
		A1	1.1b
	$\Rightarrow 2\alpha^2 - 8\alpha + 12 = 0$ or $\alpha^2 - 4\alpha + 6 = 0$ $\Rightarrow \alpha = \frac{4 \pm \sqrt{(-4)^2 - 4(1)(6)}}{2(1)}$ or $(\alpha - 2)^2 - 4 + 6 = 0 \Rightarrow \alpha = \ldots$	M1	1.1b
	$\Rightarrow \alpha = 2 \pm i\sqrt{2}$ are the two complex roots	A1	1.1b
	A correct full method to find the third root. Common methods are: Sum of roots = 8 $\Rightarrow$ third root $= 8 - \left(2 + i\sqrt{2}\right) - \left(2 - i\sqrt{2}\right) = \ldots$ third root $= 2 + i\sqrt{2} + \frac{12}{2 + i\sqrt{2}} - \left(2 - i\sqrt{2}\right) = \ldots$ Product of roots = 24 $\Rightarrow$ third root $= \frac{24}{\left(2 + i\sqrt{2}\right)\left(2 - i\sqrt{2}\right)} = \ldots$ $(z - \alpha)(z - \beta) = z^2 - 4z + 6 \Rightarrow f(z) = \left(z^2 - 4z + 6\right)(z - \gamma) \Rightarrow \gamma = \ldots$ (or long division to find third factor).	M1	3.1a
	Hence the roots of $f(z) = 0$ are $2 \pm i\sqrt{2}$ and 4	A1	1.1b
		(6)	
(b)	E.g. f(4) = 0 $\Rightarrow 4^3 - 8 \times 4^2 + 4p - 24 = 0 \Rightarrow p = \ldots$ Or $p = \left(2 + i\sqrt{2}\right)\left(2 - i\sqrt{2}\right) + 4\left(2 + i\sqrt{2}\right) + 4\left(2 - i\sqrt{2}\right) \Rightarrow p = \ldots$ Or $f(z) = (z - 4)(z^2 - 4z + 6) \Rightarrow p = \ldots$	M1	3.1a
	$\Rightarrow p = 22$ **cso**	A1	1.1b
		(2)	
	(8 marks)		

Given that

$$z_1 = 2 + 3\text{i}$$
$$|z_1 z_2| = 39\sqrt{2}$$
$$\arg(z_1 z_2) = \frac{\pi}{4}$$

where z_1 and z_2 are complex numbers,

(a) write z_1 in the form $r(\cos\theta + \text{i}\sin\theta)$

Give the exact value of r and give the value of θ in radians to 4 significant figures.

(2)

(b) Find z_2 giving your answer in the form $a + \text{i}b$ where a and b are integers.

(6)

Question	Scheme	Marks	AOs
2(a)	$\lvert z_1\rvert=\sqrt{13}$ and $\arg z_1=\tan^{-1}\left(\frac{3}{2}\right)$	B1	1.1b
	$z_1=\sqrt{13}\left(\cos 0.9828+\mathrm{i}\sin 0.9828\right)$	B1ft	1.1b
		(2)	
(b)	A complete method to find the modulus of z_2 e.g. $\lvert z_1\rvert=\sqrt{13}$ and uses $\lvert z_1z_2\rvert=\lvert z_1\rvert\times\lvert z_2\rvert=39\sqrt{2}\Rightarrow\lvert z_2\rvert=3\sqrt{26}$ or $\sqrt{234}$	M1 A1	3.1a 1.1b
	A complete method to find the argument of z_2 e.g. $\arg(z_1z_2)=\arg(z_1)+\arg(z_2)=\frac{\pi}{4}\Rightarrow\arg(z_2)=...$ $\arg(z_2)=\frac{\pi}{4}-\tan^{-1}\left(\frac{3}{2}\right)$ or $\frac{\pi}{4}-0.9828$ or $-0.1974...$	M1 A1	3.1a 1.1b
	$z_2=3\sqrt{26}\left(\cos('-0.1974...')+\mathrm{i}\sin('-0.1974...')\right)$ or $z_2=a+b\mathrm{i}\ \Rightarrow a^2+b^2=234$ and $\tan^{-1}(-0.1974)=\frac{b}{a}\Rightarrow\frac{b}{a}=-0.2$ $\Rightarrow a=...$ and $b=...$	ddM1	1.1b
	Deduces that $z_2=15-3\mathrm{i}$ only	A1	2.2a
	Alternative $z_1z_2=(a+b\mathrm{i})(2+3\mathrm{i})=(2a-3b)+(3a+2b)\mathrm{i}$ $(2a-3b)^2+(3a+2b)^2=\left(39\sqrt{2}\right)^2$ or 3042 $\Rightarrow a^2+b^2=234$ or $\lvert z_1z_2\rvert=\lvert z_1\rvert\times\lvert z_2\rvert=39\sqrt{2}\Rightarrow\lvert z_2\rvert=3\sqrt{26}$ or $\sqrt{234}$ $\Rightarrow a^2+b^2=234$	M1 A1	3.1a 1.1b
	$\arg\left[(2a-3b)+(3a+2b)\mathrm{i}\right]=\frac{\pi}{4}\Rightarrow\tan^{-1}\left(\frac{3a+2b}{2a-3b}\right)=\frac{\pi}{4}\Rightarrow\frac{3a+2b}{2a-3b}=1$ $\Rightarrow a=-5b$	M1 A1	3.1a 1.1b
	Solves $a=-5b$ and $a^2+b^2=234$ to find values for a and b	ddM1	1.1b
	Deduces that $z_2=15-3\mathrm{i}$ only	A1	2.2a
		(6)	
			(8 marks)