Vector Calculus MAT226 Spring 2025
Professor Sormani
Lesson 4 Lines and Planes

The deadline for this lesson is in one week,
and lesson 5 is an online only video lesson
which is due before our Lesson 6 meeting.

Carefully take notes while attending class or watching the lesson videos. You will cut and paste the photos of your notes and completed classwork in a googledoc entitled:

MAT226S25-lesson4-lastname-firstname

Then share editing of that document with me sormanic@gmail.com. You will also put photos of your homework in this googledoc. If you work with any classmates, be sure to write their names on the problems you completed together.

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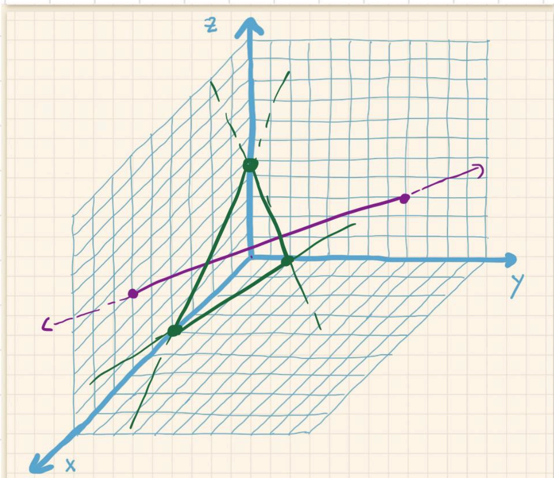
If you missed class, watch the Playlist 226F21-4-1to12 doing ten classwork problems.

Everyone should do the homework as discussed at the bottom of this page.

Vector Calculus Lesson 4
Lines and Planes in $\mathbb{R}^3$



Two points determine a line.

Three points determine a plane.

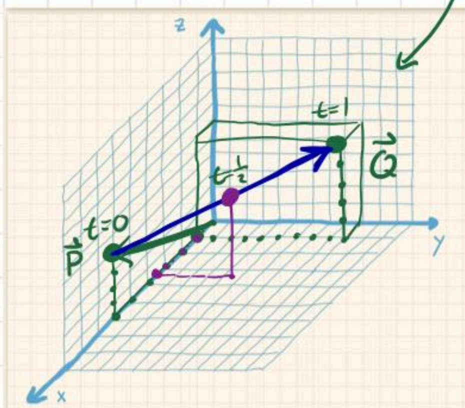
Today we will learn various ways to write down the formulas describing lines and planes in $\mathbb{R}^3$

(Not exactly the same linear algebra in $\mathbb{R}^n$)

Here we use $n=3$

Classwork

① (a) Plot the points $\vec{P} = (6,0,4)$ and $\vec{Q} = (1,9,5)$



(b) find the vector $\vec{v} = \overrightarrow{PQ}$ with tail at P and tip at Q and plot this vector.

$$\vec{v} = \vec{Q} - \vec{P} = \begin{pmatrix} 1 \\ 9 \\ 5 \end{pmatrix} - \begin{pmatrix} 6 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} -5 \\ 9 \\ 1 \end{pmatrix}$$

(c) What is $\vec{x} = \vec{P} + t\vec{v}$ for $t=0$ and $t=1$?

At $t=0$ $\quad \vec{x} = \begin{pmatrix} 6 \\ 0 \\ 4 \end{pmatrix} + 0\begin{pmatrix} -5 \\ 9 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \\ 4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \\ 4 \end{pmatrix} = \vec{P}$

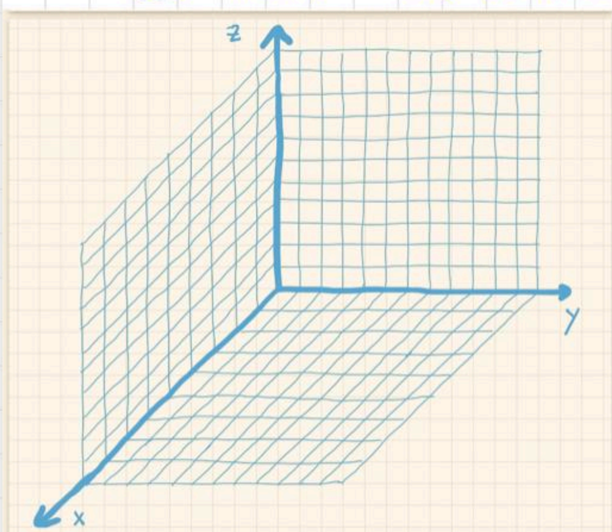
At $t=1$ $\quad \vec{x} = \begin{pmatrix} 6 \\ 0 \\ 4 \end{pmatrix} + 1\begin{pmatrix} -5 \\ 9 \\ 1 \end{pmatrix} = \begin{pmatrix} 6-5 \\ 0+9 \\ 4+1 \end{pmatrix} = \begin{pmatrix} -1 \\ 9 \\ 5 \end{pmatrix} = \vec{Q}$

At $t=\frac{1}{2}$ $\quad \vec{x} = \begin{pmatrix} 6 \\ 0 \\ 4 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -5 \\ 9 \\ 1 \end{pmatrix} = \begin{pmatrix} 6-5/2 \\ 0+9/2 \\ 4+1/2 \end{pmatrix} = \begin{pmatrix} 7/2 \\ 9/2 \\ 9/2 \end{pmatrix}$

As we run through all values of $t \in \mathbb{R}$ we get all the points on the line $\{\vec{x} = \vec{P} + t\vec{v} \mid t \in \mathbb{R}\}$ through $\vec{P}$ and $\vec{Q}$.

② (a) Plot the points $\vec{P} = (4,5,2)$ $\quad \vec{Q} = (2,8,10)$

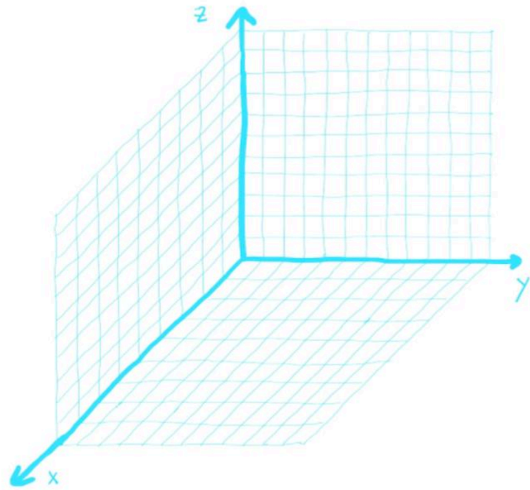
(b) find the direction vector $\vec{v} = \overrightarrow{PQ}$



(c) What is $\vec{x} = \vec{P} + t\vec{v}$ for $t=0$, $t=1$, $t=\frac{1}{2}$, $t=2$, and $t=-1$.

Notice they all lie on the same line.

Classwork 1 is above, try classwork 2 before scrolling down to see the solution: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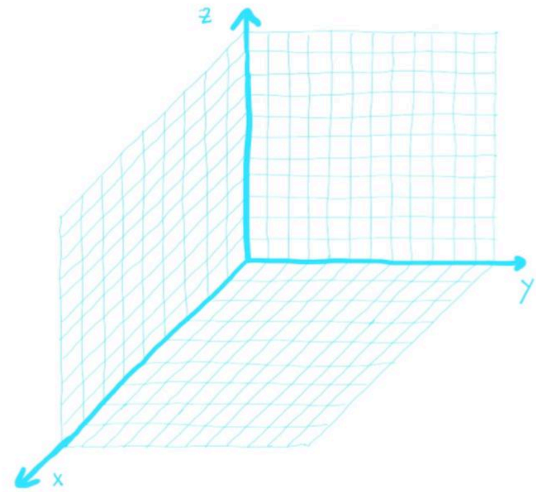
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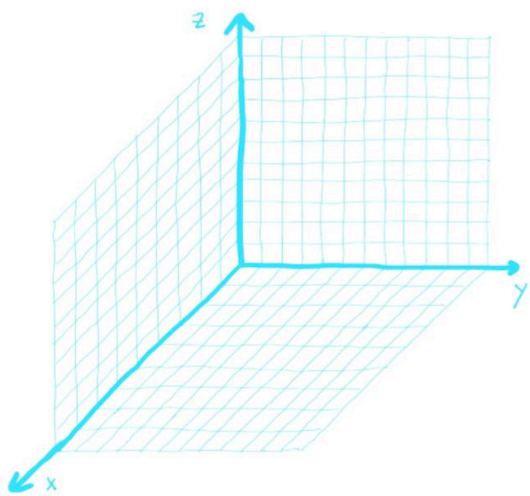
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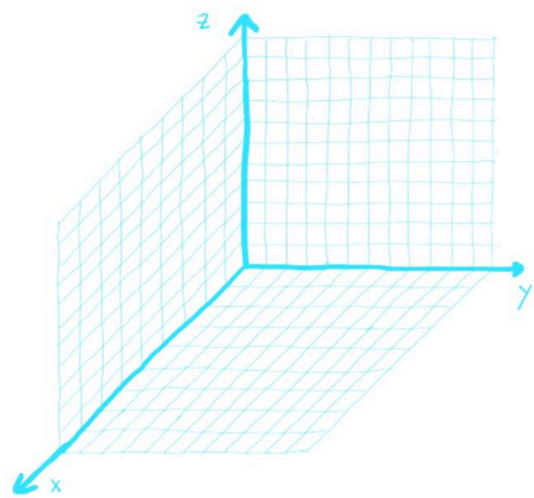
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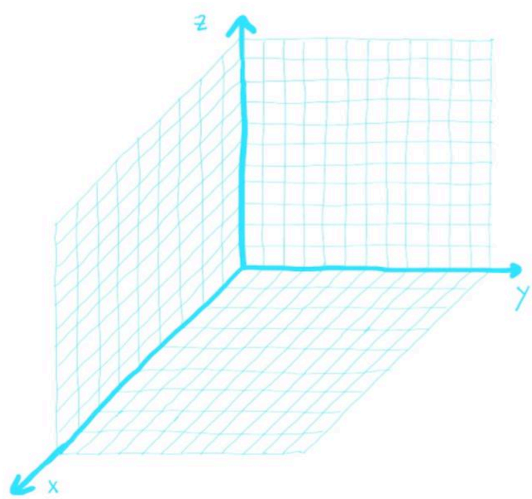
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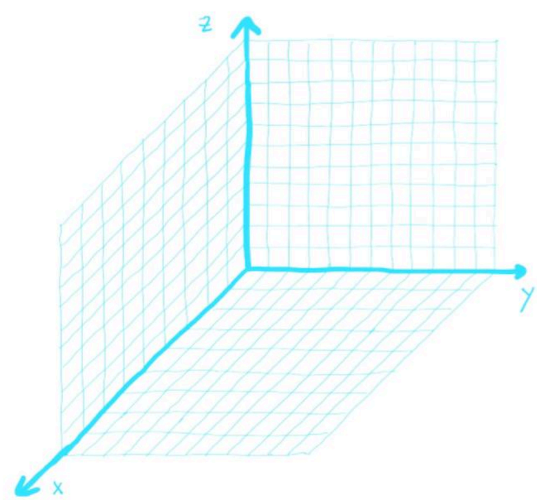

z
y
x

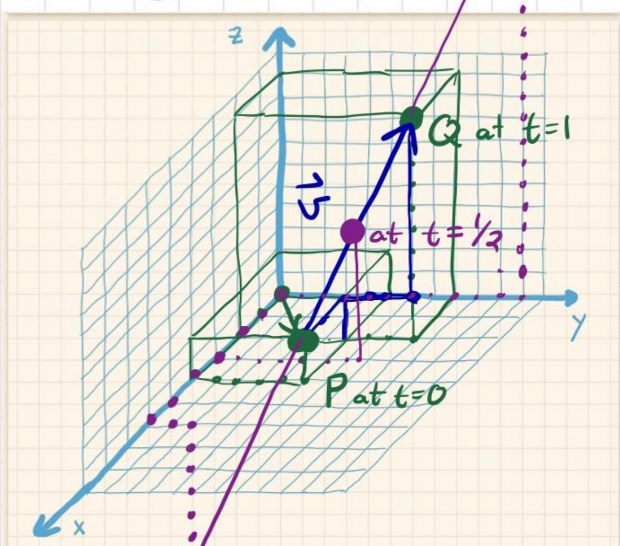
Plotting the line $\overleftrightarrow{PQ}$

$t \to \infty$

at $t=2$

(2) (a) Plot the points $\vec{P} = (4, 5, 2)$ $\vec{Q} = (2, 8, 10)$

(b) find the direction vector $\vec{v} = \overrightarrow{PQ}$



at $t=-1$

$t \to -\infty$

$$\vec{v} = \vec{Q} - \vec{P} = \begin{pmatrix} 2 \\ 8 \\ 10 \end{pmatrix} - \begin{pmatrix} 4 \\ 5 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ 8 \end{pmatrix}$$

$$= \langle -2, 3, 8 \rangle$$

(c) What is $\vec{x} = \vec{P} + t\vec{v}$

at $t=0$ $\quad \vec{x} = \begin{pmatrix} 4 \\ 5 \\ 2 \end{pmatrix} + 0 \begin{pmatrix} -2 \\ 3 \\ 8 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 2 \end{pmatrix} = \vec{P}$

at $t=1$ $\quad \vec{x} = \begin{pmatrix} 4 \\ 5 \\ 2 \end{pmatrix} + 1 \begin{pmatrix} -2 \\ 3 \\ 8 \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \\ 10 \end{pmatrix} = \vec{Q}$

at $t=\frac{1}{2}$ $\quad \vec{x} = \begin{pmatrix} 4 \\ 5 \\ 2 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} -2 \\ 3 \\ 8 \end{pmatrix} = \begin{pmatrix} 3 \\ 13/2 \\ 6 \end{pmatrix}$

at $t=2$ $\quad \vec{x} = \begin{pmatrix} 4 \\ 5 \\ 2 \end{pmatrix} + 2 \begin{pmatrix} -2 \\ 3 \\ 8 \end{pmatrix} = \begin{pmatrix} 0 \\ 11 \\ 18 \end{pmatrix}$

at $t=-1$ $\quad \vec{x} = \begin{pmatrix} 4 \\ 5 \\ 2 \end{pmatrix} + (-1) \begin{pmatrix} -2 \\ 3 \\ 8 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ -6 \end{pmatrix}$

Formulas for lines in Space

Position $\vec{P} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ $\vec{Q} = \begin{pmatrix} x_Q \\ y_Q \\ z_Q \end{pmatrix}$ Direction $\vec{v} = \overrightarrow{PQ} = \vec{Q} - \vec{P} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

Position - Direction Formula (see in linear algebra)

$$\left\{ \vec{x} = \vec{P} + t\vec{v} \mid t \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underbrace{\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}}_{\text{position}} + t \underbrace{\begin{pmatrix} a \\ b \\ c \end{pmatrix}}_{\text{direction}} \;\middle|\; t \in \mathbb{R} \right\}$$

Parametric Equation

$x = x_1 + ta \qquad y = y_1 + tb \qquad z = z_1 + tc$

always works

$x - x_1 = ta \qquad y - y_1 = tb \qquad z - z_1 = tc$

$\frac{x - x_1}{a} = t \qquad \frac{y - y_1}{b} = t \qquad \frac{z - z_1}{c} = t$

remove t to find

Symmetric Equation

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

Warning! only works if $a, b, c \neq 0$.

Classwork 3:

(3) Write the parametric and symmetric equations for the line in Classwork (2)

position $\vec{P} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 2 \end{pmatrix}$ $\vec{v} = \begin{pmatrix} -2 \\ 3 \\ 8 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

Remember this is the set in classwork 2

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 2 \end{pmatrix} + t \begin{pmatrix} -2 \\ 3 \\ 8 \end{pmatrix} \;\middle|\; t \in \mathbb{R} \right\}$$

Solution to classwork 3

③ Write the parametric and symmetric equations for the line in Classwork ②

position $\vec{P} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 2 \end{pmatrix}$ $\quad \vec{v} = \begin{pmatrix} -2 \\ 3 \\ 8 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$x = 4 + t(-2) \qquad y = 5 + t \cdot 3 \qquad z = 2 + t\,8$

parentheses needed

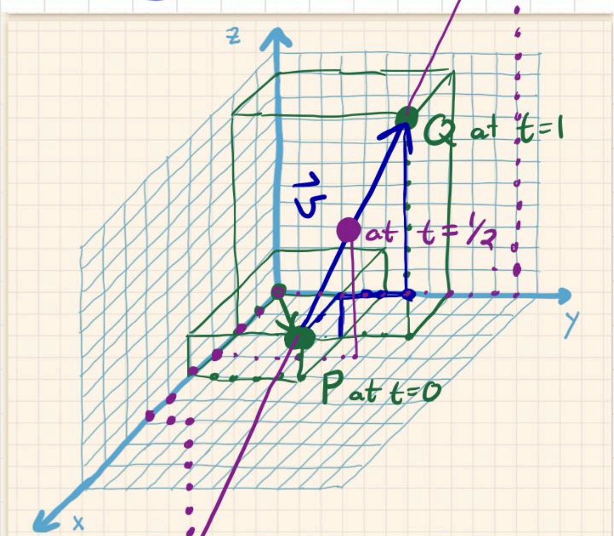
$$\frac{x-4}{-2} = \frac{y-5}{3} = \frac{z-2}{8}$$

$t \to \infty$

at $t=2$

② ⓐ Plot the points $\vec{P} = (4, 5, 2)$ $\quad \vec{Q} = (2, 8, 10)$

ⓑ find the direction vector $\vec{v} = \overrightarrow{PQ}$



$$\vec{v} = \vec{Q} - \vec{P} = \begin{pmatrix} 2 \\ 8 \\ 10 \end{pmatrix} - \begin{pmatrix} 4 \\ 5 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ 8 \end{pmatrix}$$

$= \langle -2, 3, 8 \rangle$

ⓒ What is $\vec{x} = \vec{P} + t\vec{v}$

at $t=0$ $\quad \vec{x} = \begin{pmatrix} 4 \\ 5 \\ 2 \end{pmatrix} + 0 \begin{pmatrix} -2 \\ 3 \\ 8 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 2 \end{pmatrix} = \vec{P}$

at $t=1$ $\quad \vec{x} = \begin{pmatrix} 4 \\ 5 \\ 2 \end{pmatrix} + 1 \begin{pmatrix} -2 \\ 3 \\ 8 \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \\ 10 \end{pmatrix} = \vec{Q}$

at $t=\frac{1}{2}$ $\quad \vec{x} = \begin{pmatrix} 4 \\ 5 \\ 2 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} -2 \\ 3 \\ 8 \end{pmatrix} = \begin{pmatrix} 3 \\ 13/2 \\ 6 \end{pmatrix}$

at $t=2$ $\quad \vec{x} = \begin{pmatrix} 4 \\ 5 \\ 2 \end{pmatrix} + 2 \begin{pmatrix} -2 \\ 3 \\ 8 \end{pmatrix} = \begin{pmatrix} 0 \\ 11 \\ 18 \end{pmatrix}$

at $t=-1$ $\quad \vec{x} = \begin{pmatrix} 4 \\ 5 \\ 2 \end{pmatrix} + (-1) \begin{pmatrix} -2 \\ 3 \\ 8 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ -6 \end{pmatrix}$

at $t=-1$

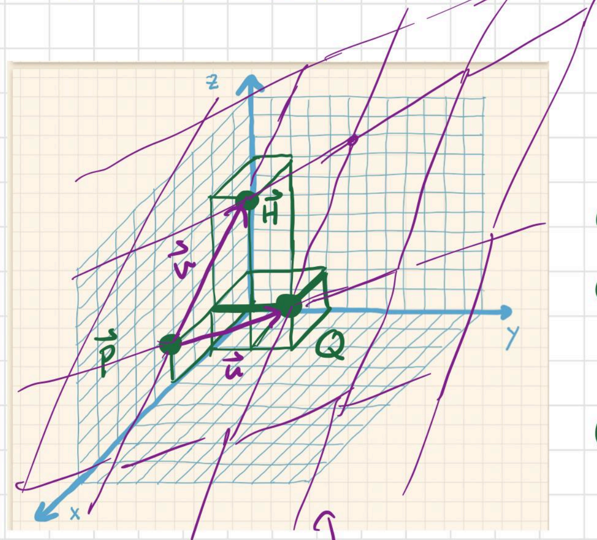
$t \to -\infty$ $\quad \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 2 \end{pmatrix} + t \begin{pmatrix} -2 \\ 3 \\ 8 \end{pmatrix} \;\middle|\; t \in \mathbb{R} \right\}$

i Classwork 4,5,6:

Planes in Space

Three Points determine a plane.

$\vec{P}$ $\vec{Q}$ and $\vec{H}$



Classwork

(4) find $\vec{P}, \vec{Q}$ and $\vec{H}$ on left.

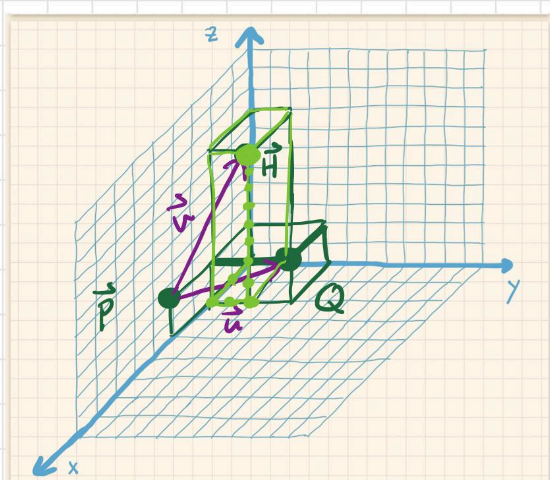
(5) find the line through $\vec{P}$ & $\vec{Q}$ in position vector form

Let $\vec{u} = \overrightarrow{PQ}$ be the direction

(6) find the line through $\vec{P}$ + $\vec{H}$ in position vector form

Let $\vec{v} = \overrightarrow{PH}$ be the direction

position direction form for a plane $\{ \vec{P} + t\vec{v} + s\vec{u} \mid t, s \in \mathbb{R} \}$



Classwork

(4) find $\vec{P}, \vec{Q}$ and $\vec{H}$ on left.

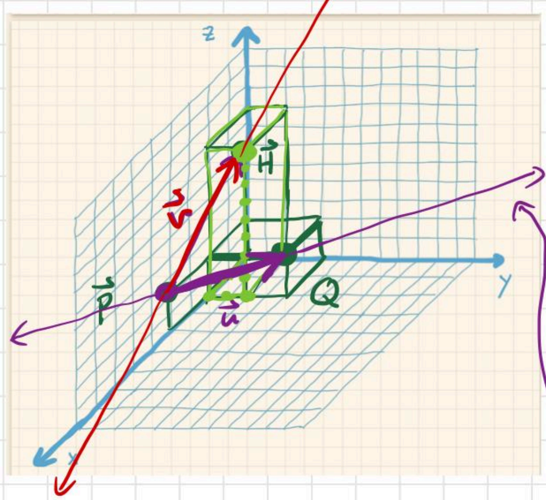
$$\vec{P} = \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix} \quad \vec{Q} = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix} \quad \vec{H} = \begin{pmatrix} 2 \\ 2 \\ 8 \end{pmatrix}$$

(5) find the line through $\vec{P}$ & $\vec{Q}$ in position vector form

Let $\vec{u} = \overrightarrow{PQ}$ be the direction

Planes in Space

Three Points determine a plane.



Classwork

(4) find $\vec{P}$, $\vec{Q}$ and $\vec{H}$ on left.

$$\vec{P} = \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix} \quad \vec{Q} = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix} \quad \vec{H} = \begin{pmatrix} 2 \\ 2 \\ 8 \end{pmatrix}$$

(5) find the line through $\vec{P}$ + $\vec{Q}$ in position vector form

Let $\vec{u} = \overrightarrow{PQ}$ be the direction

$$\vec{u} = \vec{Q} - \vec{P} = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix} - \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \\ 0 \end{pmatrix}$$

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix} + t \begin{pmatrix} -2 \\ 4 \\ 0 \end{pmatrix} \;\middle|\; t \in \mathbb{R} \right\}$$

(6) find the line through $\vec{P}$ + $\vec{H}$ in position vector form

Let $\vec{v} = \overrightarrow{PH}$ be the direction

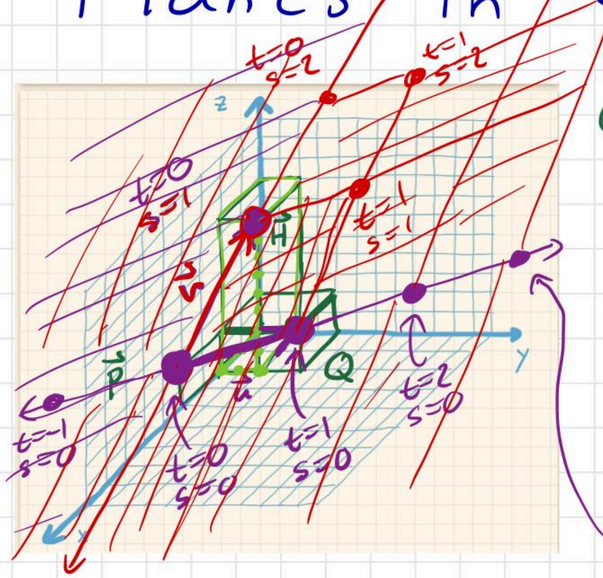
$$\vec{v} = \vec{H} - \vec{P} = \begin{pmatrix} 2 \\ 2 \\ 8 \end{pmatrix} - \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 6 \end{pmatrix}$$

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix} + s \begin{pmatrix} -2 \\ 2 \\ 6 \end{pmatrix} \;\middle|\; s \in \mathbb{R} \right\}$$

change of parameter

Planes in Space

Three Points determine a plane.



Classwork

(4) find $\vec{P}$, $\vec{Q}$ and $\vec{H}$ on left.

$$\vec{P} = \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix} \quad \vec{Q} = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix} \quad \vec{H} = \begin{pmatrix} 2 \\ 2 \\ 8 \end{pmatrix}$$

(5) find the line through $\vec{P}$ + $\vec{Q}$ in position vector form

Let $\vec{u} = \overrightarrow{PQ}$ be the direction

$$\vec{u} = \vec{Q} - \vec{P} = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix} - \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \\ 0 \end{pmatrix}$$

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix} + t \begin{pmatrix} -2 \\ 4 \\ 0 \end{pmatrix} \;\middle|\; t \in \mathbb{R} \right\}$$

(6) find the line through $\vec{P}$ + $\vec{H}$ in position vector form

Let $\vec{v} = \overrightarrow{PH}$ be the direction

$$\vec{v} = \vec{H} - \vec{P} = \begin{pmatrix} 2 \\ 2 \\ 8 \end{pmatrix} - \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 6 \end{pmatrix}$$

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix} + s \begin{pmatrix} -2 \\ 2 \\ 6 \end{pmatrix} \;\middle|\; s \in \mathbb{R} \right\}$$

change of parameter

Point Direction Formula for a plane

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix} + t \begin{pmatrix} -2 \\ 4 \\ 0 \end{pmatrix} + s \begin{pmatrix} -2 \\ 2 \\ 6 \end{pmatrix} \;\middle|\; t, s \in \mathbb{R} \right\}$$

position $\vec{P}$

direction $\vec{u} = \overrightarrow{PQ}$

direction $\vec{v} = \overrightarrow{PH}$

two parameters (free variables)

Classwork 7 is the work below:

change of parameter

Point Direction Formula for a plane

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix} + t \begin{pmatrix} -2 \\ 4 \\ 0 \end{pmatrix} + s \begin{pmatrix} -2 \\ 2 \\ 6 \end{pmatrix} \;\middle|\; t, s \in \mathbb{R} \right\}$$

position $\vec{p}$ — direction $\vec{u} = \overrightarrow{PQ}$ — direction $\vec{v} = \overrightarrow{PH}$ — two parameters (free variables)

Point Direction Formula for a Plane

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} + t \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + s \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \;\middle|\; t, s \in \mathbb{R} \right\}$$

Given: $\{ \vec{x} = \vec{p} + t\,\vec{u} + s\,\vec{v} \mid t, s \in \mathbb{R} \}$

position — direction $\vec{u} = \overrightarrow{PQ}$ — direction $\vec{v} = \overrightarrow{PH}$

What if we wish to remove the parameters?

① $\vec{x} - \vec{p} = t\vec{u} + s\vec{v}$ ① by subtracting $\vec{p}$

Suppose $\vec{n}$ = perpendicular to the plane "normal"

$\vec{n}$ is $\perp$ to both $\vec{u}$ and $\vec{v}$

$\vec{n} \cdot \vec{u} = 0 \qquad \vec{n} \cdot \vec{v} = 0$

② $(\vec{x} - \vec{p}) \cdot \vec{n} = (t\vec{u} + s\vec{v}) \cdot \vec{n}$ by dot with $\vec{n}$

③ $\vec{x} \cdot \vec{n} - \vec{p} \cdot \vec{n} = t\underbrace{\vec{u} \cdot \vec{n}} + s\underbrace{\vec{v} \cdot \vec{n}}$ by distribution

④ $\vec{x} \cdot \vec{n} - \vec{p} \cdot \vec{n} = 0 + 0$ by $\vec{n} \cdot \vec{u} = 0$, $\vec{n} \cdot \vec{v} = 0$

(5) $\boxed{\vec{x} \cdot \vec{n} = \vec{p} \cdot \vec{n}}$ by adding $\vec{p} \cdot \vec{n}$ to both sides

Taking

$$\vec{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \vec{p} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$$

(6) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ by sub

classwork write out dot product:

(7) $xa + yb + zc = x_1 a + y_1 b + z_1 \cdot c$ by defn dot prod.

Standard Equation of a Plane in $\mathbb{R}^3$

$$\boxed{ax + by + cz = ax_1 + by_1 + cz_1}$$

through position $\vec{P} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ with normal $\vec{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$$\boxed{a(x - x_1) + b(y - y_1) + c(z - z_1) = 0}$$

$$\{ \vec{x} \mid \vec{x} \cdot \vec{n} = \vec{p} \cdot \vec{n} \}$$ ← not in the text

$$\{ \vec{x} \mid (\vec{x} - \vec{p}) \cdot \vec{n} = 0 \}$$

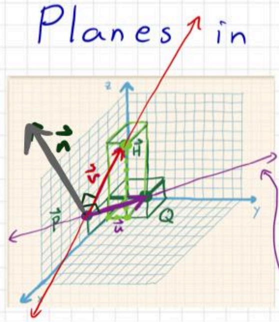
The text has another derivation of the standard equation.

Classwork 8:

Classwork ⑧ Write the standard formula for the plane in classwork ④-⑥.

Planes in Space

Three Points determine a plane.



Classwork

④ find $\vec{P}, \vec{Q}$ and $\vec{H}$ on left.

$\vec{P} = \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix}$ $\vec{Q} = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$ $\vec{H} = \begin{pmatrix} 2 \\ 2 \\ 8 \end{pmatrix}$

⑤ find the line through $\vec{P}$ + $\vec{Q}$ in position vector form

Let $\vec{u} = \overrightarrow{PQ}$ be the direction

$\vec{u} = \vec{Q} - \vec{P} = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix} - \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \\ 0 \end{pmatrix}$

$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix} + t \begin{pmatrix} -2 \\ 4 \\ 0 \end{pmatrix} \;\middle|\; t \in \mathbb{R} \right\}$

⑥ find the line through $\vec{P}$ + $\vec{H}$ in position vector form

$\vec{v} = \vec{H} - \vec{P} = \begin{pmatrix} 2 \\ 2 \\ 8 \end{pmatrix} - \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 6 \end{pmatrix}$

Let $\vec{v} = \overrightarrow{PH}$ be the direction

$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix} + s \begin{pmatrix} -2 \\ 2 \\ 6 \end{pmatrix} \;\middle|\; s \in \mathbb{R} \right\}$

↑ change of parameter

$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

$\vec{P} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix}$ position

normal $\vec{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = ?$

it has to be $\perp$ to $\vec{u}$ + $\vec{v}$

$$\boxed{\vec{n} = \vec{u} \times \vec{v}}$$

found $\vec{u}$ and $\vec{v}$ already in classwork ⑤+⑥

$$\vec{n} = \begin{pmatrix} -2 \\ 4 \\ 0 \end{pmatrix} \times \begin{pmatrix} -2 \\ 2 \\ 6 \end{pmatrix} = \begin{pmatrix} 4 \cdot 6 - 0 \cdot 2 \\ 0 \cdot (-2) - (-2) \cdot 6 \\ -2 \cdot 2 - 4(-2) \end{pmatrix} = \begin{pmatrix} 24 \\ 12 \\ 4 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Check $\vec{n} \cdot \begin{pmatrix} -2 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 24 \\ 12 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 4 \\ 0 \end{pmatrix} = -48 + 48 + 0 = 0$ ✓

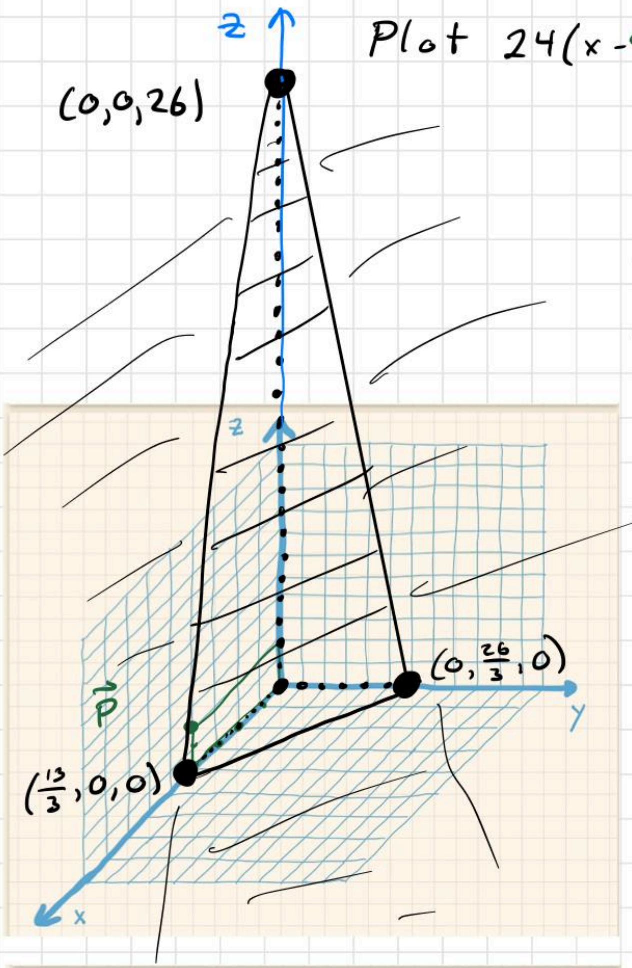
$\vec{n} \cdot \begin{pmatrix} -2 \\ 2 \\ 6 \end{pmatrix} = \begin{pmatrix} 24 \\ 12 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 2 \\ 6 \end{pmatrix} = -48 + 24 + 24 = 0$ ✓

$$\boxed{24(x - 4) + 12(y - 0) + 4(z - 2) = 0}$$

Be sure to do HW like this.

Classwork 9:

Classwork ⑨



Plot $24(x-4) + 12(y-0) + 4(z-2) = 0$

this includes

$\vec{P} = \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix}$

$24x - 96 + 12y - 0 + 4z - 8 = 0$

$\boxed{24x + 12y + 4z = 96 + 0 + 8 = 104}$

what if $y = 0$ and $z = 0$?

$24x + 0 + 0 = 104$

So $x = \frac{104}{24} = \frac{52}{12} = \frac{26}{6} = \frac{13}{3}$

$(\frac{13}{3}, 0, 0)$ is in the plane

What if $x = 0$ and $z = 0$?

$24 \cdot 0 + 12y + 4 \cdot 0 = 104$

$12y = 104$

$y = \frac{104}{12} = \frac{26}{3}$

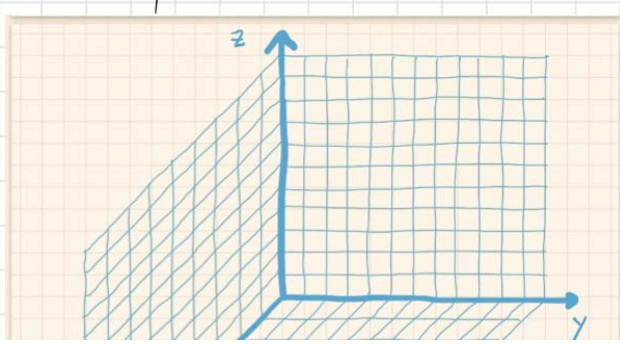
$(0, \frac{26}{3}, 0)$ is on the plane

What if $x = 0$ and $y = 0$?

$24 \cdot 0 + 12 \cdot 0 + 4z = 104$

$4z = 104 \quad z = 26$

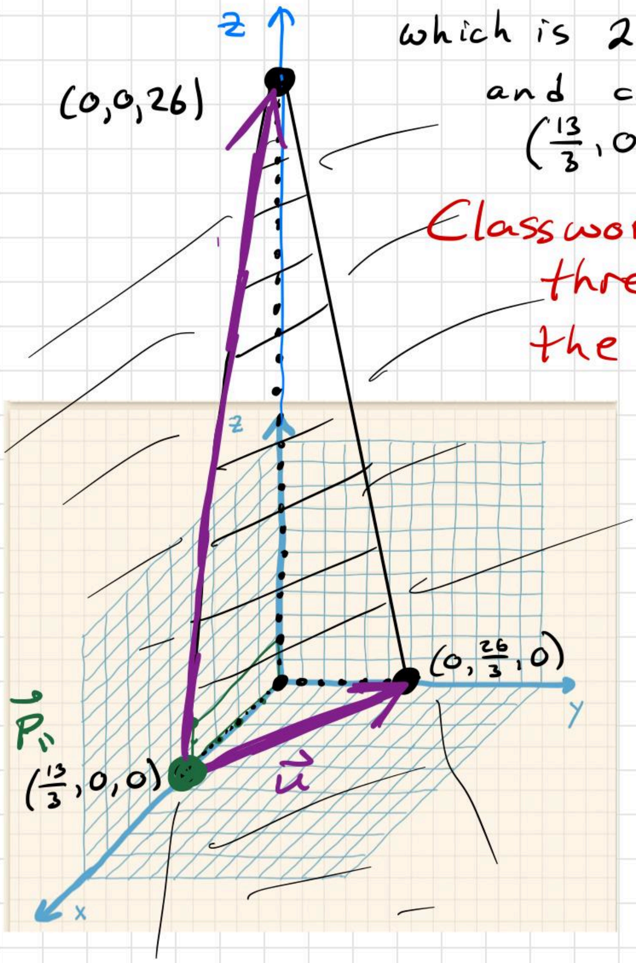
So $(0, 0, 26)$ is on the plane



Classwork ⑩

Check these three points determine the same plane.

Classwork (9) Plot $24(x-4)+12(y-0)+4(z-2)=0$



which is $24x+12y+4z=104$

and contains

$\left(\frac{13}{3},0,0\right),\ \left(0,\frac{26}{3},0\right),\ \left(0,0,26\right)$

Classwork (10) Check these three points determine the same plane.

$\vec{p}=\begin{pmatrix}13/3\\0\\0\end{pmatrix}\quad \vec{u}=\begin{pmatrix}0\\26/3\\0\end{pmatrix}-\begin{pmatrix}13/3\\0\\0\end{pmatrix}=\begin{pmatrix}-13/3\\26/3\\0\end{pmatrix}$

$\vec{v}=\begin{pmatrix}0\\0\\26\end{pmatrix}-\begin{pmatrix}13/3\\0\\0\end{pmatrix}=\begin{pmatrix}-13/3\\0\\26\end{pmatrix}$

$\vec{n}=\vec{u}\times\vec{v}=$

$\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}-13/3\\26/3\\0\end{pmatrix}\times\begin{pmatrix}-13/3\\0\\26\end{pmatrix}=\begin{pmatrix}26\cdot26/3\\13\cdot26/3\\26\cdot13/9\end{pmatrix}$

$a(x-x_1)+b(y-y_1)+c(z-z_1)=0$

Looks very different so simplify it $\longrightarrow$ $\frac{26^2}{3}\left(x-\frac{13}{3}\right)+\frac{13\cdot26}{3}(y-0)+\frac{26\cdot13}{9}(z-0)=0$

mult by 9 $\rightarrow$ $26^2\cdot3\left(x-\frac{13}{3}\right)+13\cdot26\cdot3(y)+26\cdot13z=0$

distributed $\rightarrow$ $26^2\cdot3x-26^2\cdot13+26\cdot13\cdot3y+26\cdot13z=0$

divide by 26 $\rightarrow$ $26\cdot3x-26\cdot13+13\cdot3y+13z=0$

divide by 13 $\rightarrow$ $2\cdot3x-26+3y+1z=0$

add 26

$\boxed{6x+3y+1z=26}$

simpler formula for the same plane.

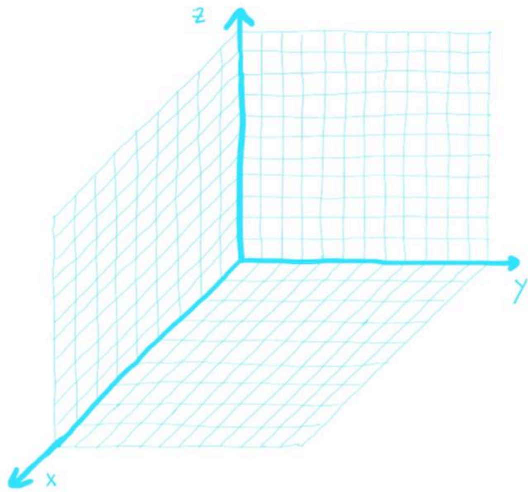
Classwork (9) Old formula

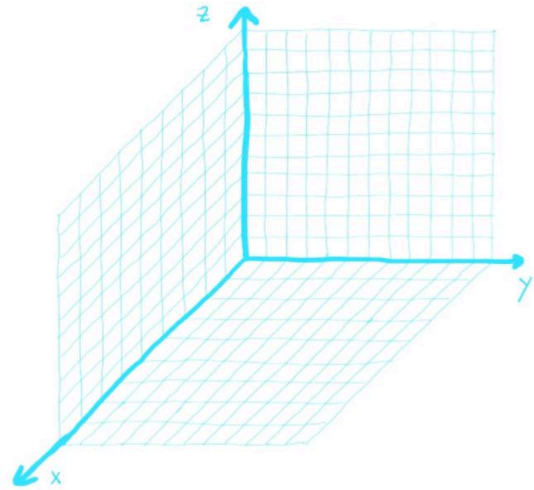
$24x+12y+4z=104$

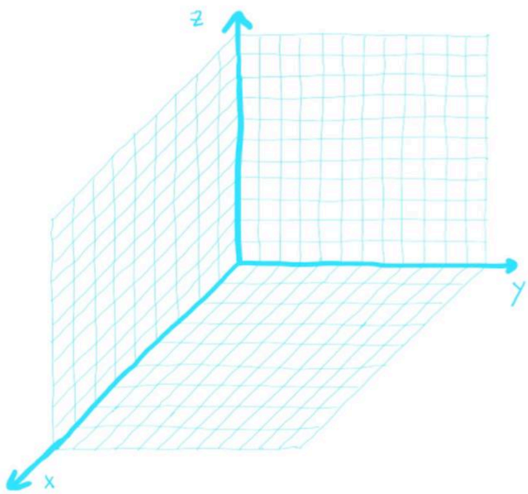
div by 4

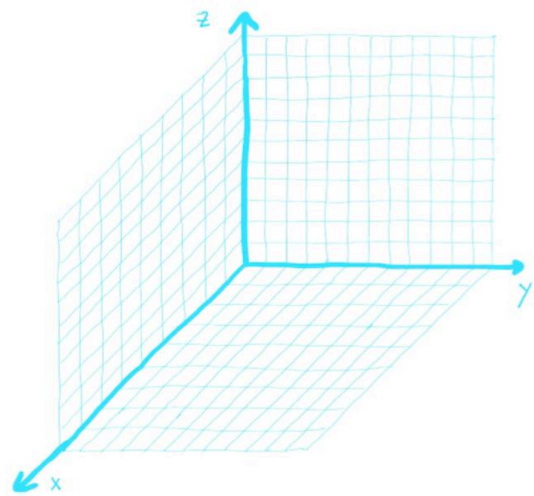
$\boxed{6x+3y+z=26}$ $\leftarrow$ It is the same plane!

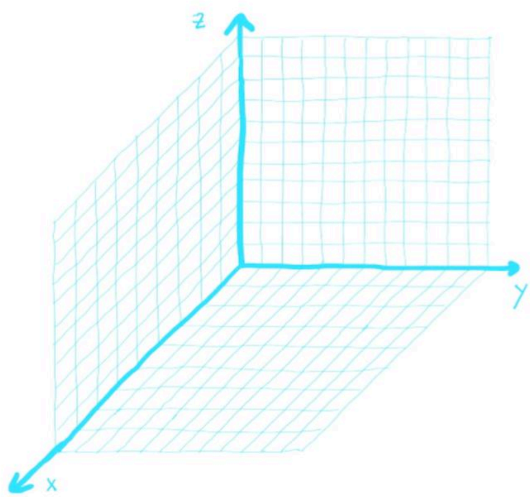
Some graph papers:

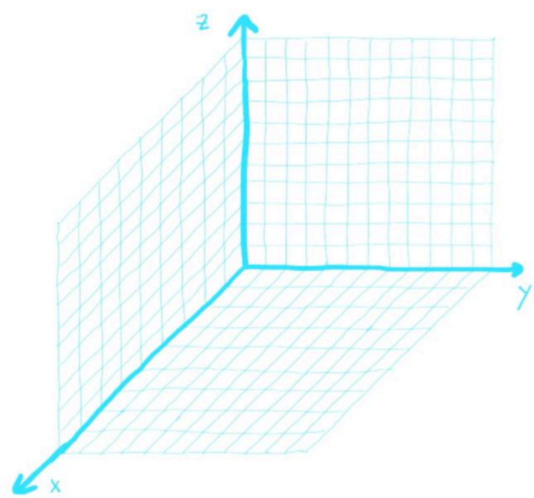






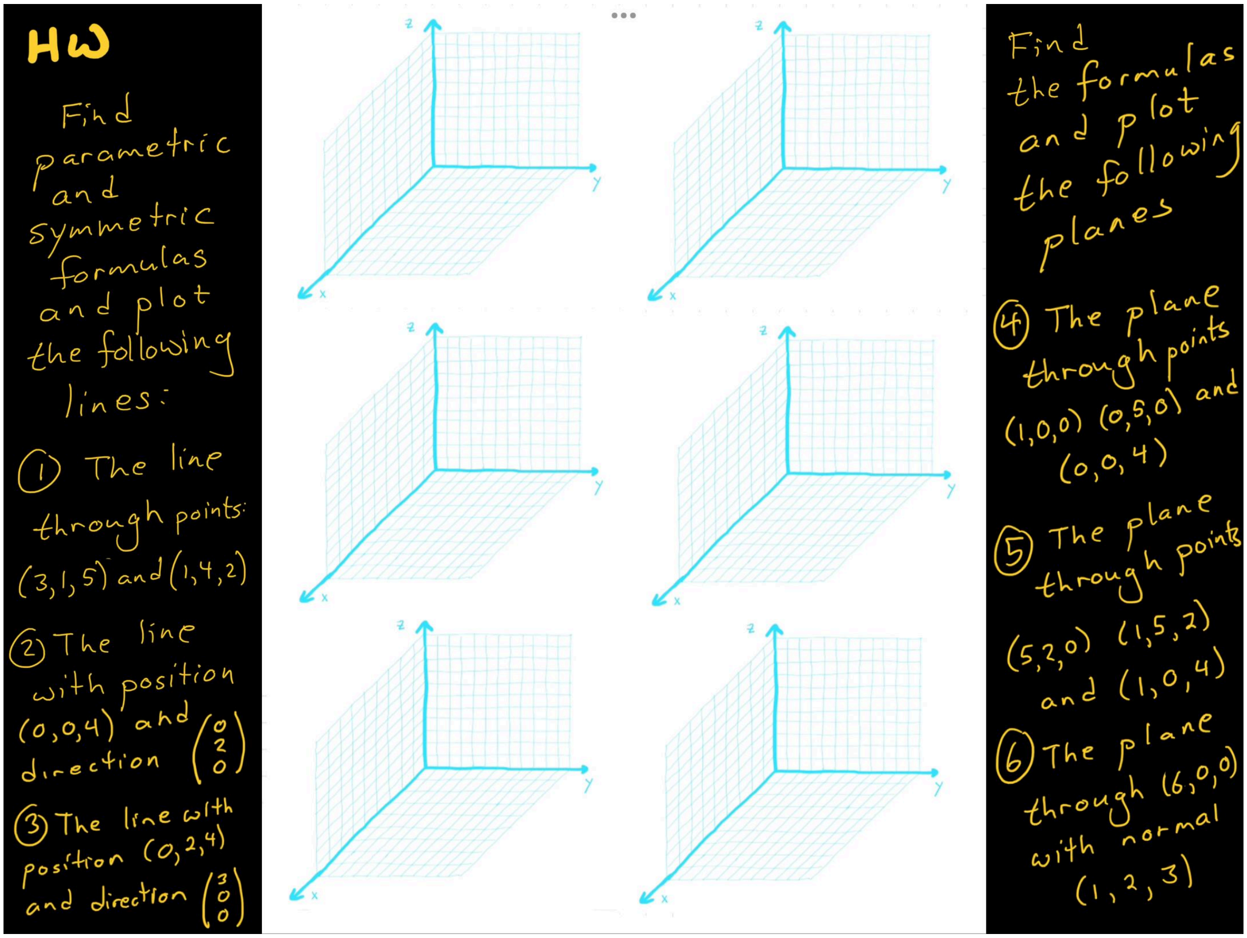


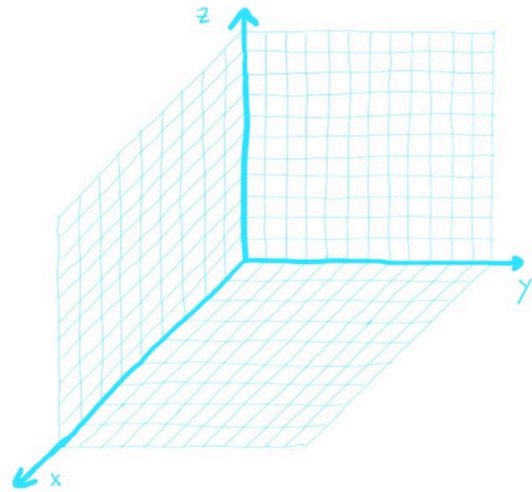


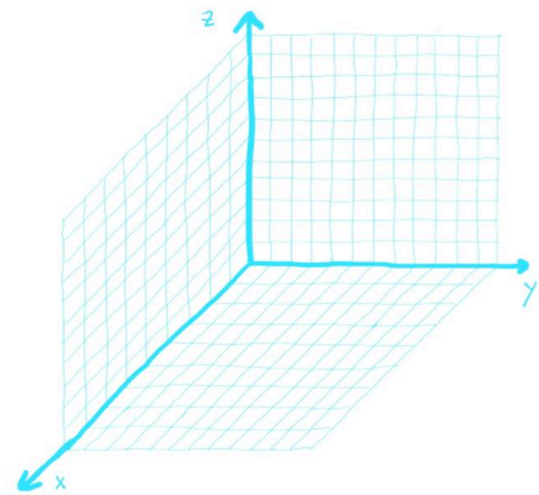


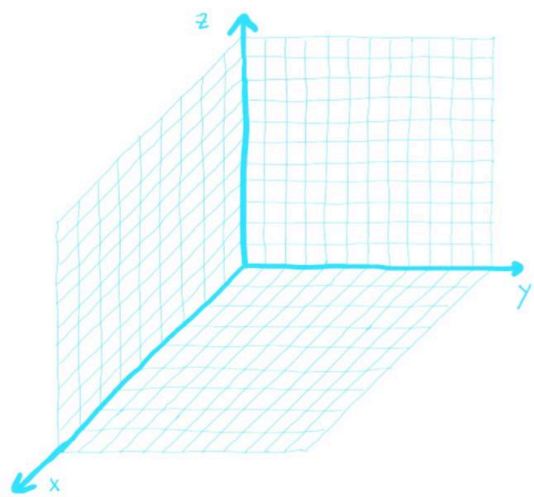
Homework: Lehman had all odd problems in 11.5 but must read the chapter to learn all these problems.

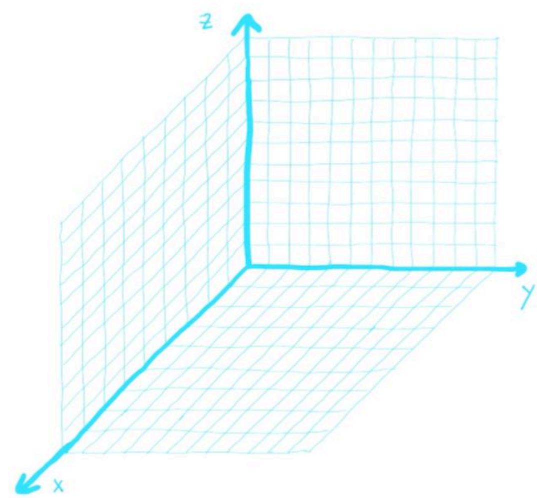
Our class can focus on these six problems

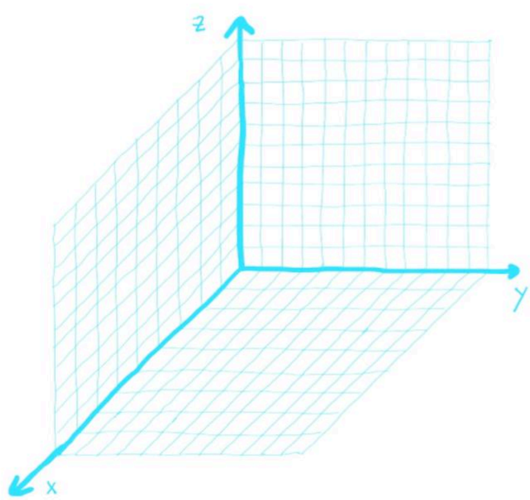


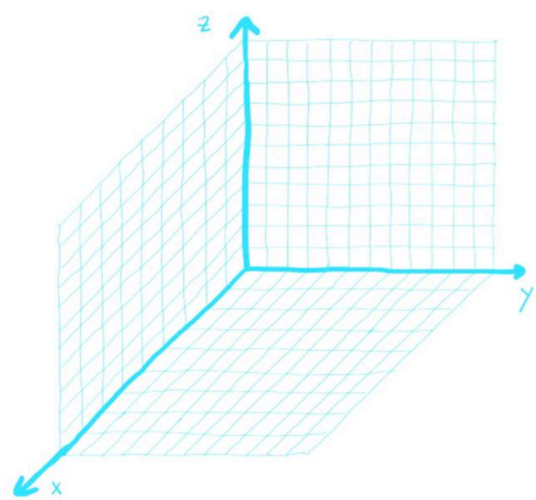








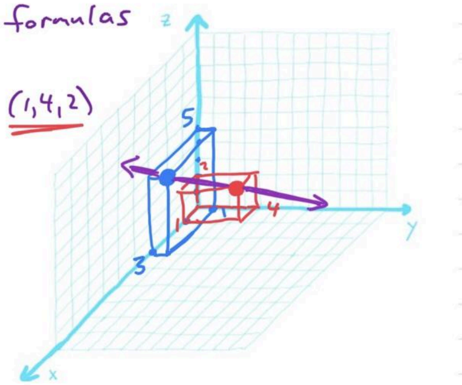




Solutions:

For lines:

Find parametric and symenetric formulas and plot the following lines.



① The line through $(3,1,5)$ and $(1,4,2)$

position $\begin{pmatrix}3\\1\\5\end{pmatrix}$

direction: $\begin{pmatrix}1\\4\\2\end{pmatrix} - \begin{pmatrix}3\\1\\5\end{pmatrix} = \begin{pmatrix}1-3\\4-1\\2-5\end{pmatrix} = \begin{pmatrix}-2\\3\\-3\end{pmatrix}$

$$\left\{ \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}3\\1\\5\end{pmatrix} + t\begin{pmatrix}-2\\3\\-3\end{pmatrix} \;\middle|\; t \in \mathbb{R} \right\}$$

↑ position direction form

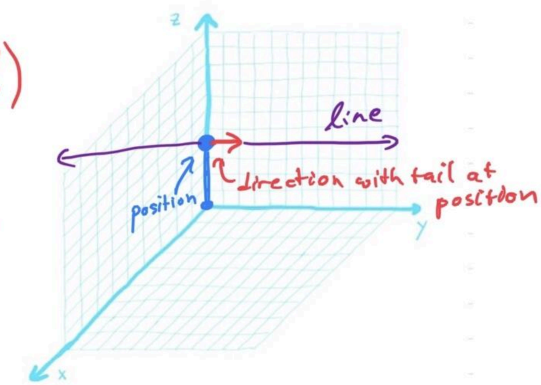
$$\boxed{\begin{aligned} x &= 3 + (-2)t \\ y &= 1 + (3)t \\ z &= 5 + (-3)t \end{aligned}}$$

← parametric formula.

Solve for t: $\boxed{\begin{aligned} x-3 &= -2t \\ y-1 &= 3t \\ z-5 &= -3t \end{aligned}}$ Keep solving for t $\boxed{\begin{aligned} \frac{(x-3)}{-2} &= t \\ \frac{(y-1)}{3} &= t \\ \frac{z-5}{-3} &= t \end{aligned}}$ Eliminate the parameter: $\boxed{\frac{x-3}{-2} = \frac{y-1}{3} = \frac{z-5}{-3}}$

↑ symmetric formula

② The line with position $\begin{pmatrix}0\\0\\4\end{pmatrix}$ and direction $\begin{pmatrix}0\\2\\0\end{pmatrix}$



$$\left\{ \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\0\\4\end{pmatrix} + t\begin{pmatrix}0\\2\\0\end{pmatrix} \;\middle|\; t \in \mathbb{R} \right\}$$

← position/direction form

$$\boxed{\begin{aligned} x &= 0 + 0t \\ y &= 0 + 2t \\ z &= 4 + 0t \end{aligned}}$$

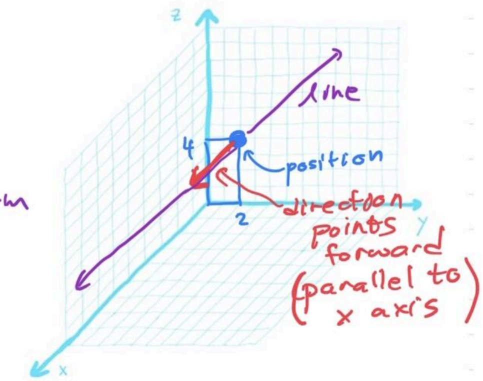
← parametric form

Solve for t: $\boxed{\begin{aligned} x-0 &= 0t \\ y-0 &= 2t \\ z-4 &= 0t \end{aligned}}$ Keep solving for t $\begin{aligned} \frac{x-0}{0} &= t \\ \frac{y-0}{2} &= t \\ \frac{z-4}{0} &= t \end{aligned}$

division by zero error!

So no symmetric form!

③ The line with position $\begin{pmatrix}0\\2\\4\end{pmatrix}$ and direction $\begin{pmatrix}3\\0\\0\end{pmatrix}$



$\left\{ \begin{pmatrix}x\\y\\z\end{pmatrix} = \quad + t \;\middle|\; t \in \mathbb{R} \right\}$

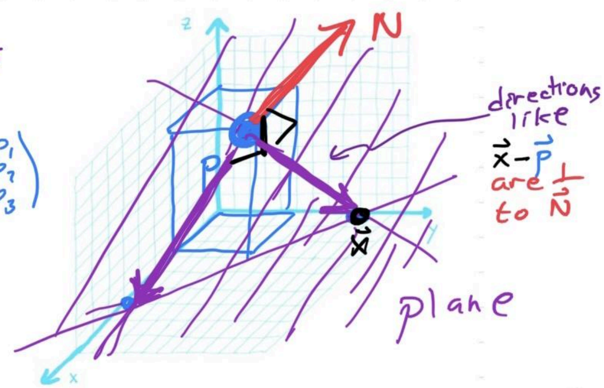
position direction form

$\boxed{\begin{aligned} x &= 0 + 3t \\ y &= 2 + 0t \\ z &= 4 + 0t \end{aligned}}$ parametric form

No symmetric form again due to division by zero error.

For planes:

Find the formulas and plot the following planes



Note: The position can be any of the points. $P = \begin{pmatrix}P_1\\P_2\\P_3\end{pmatrix}$

The normal can be given $\vec{N} = \begin{pmatrix}N_1\\N_2\\N_3\end{pmatrix}$

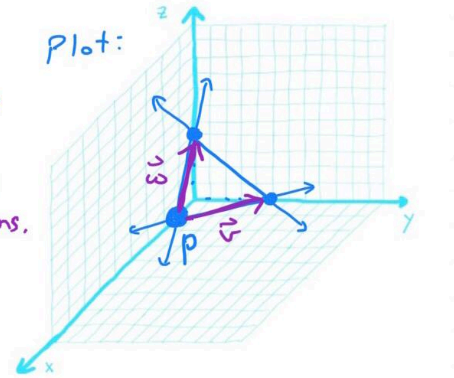
Then formula is $0 = \vec{N} \cdot (\vec{x} - \vec{P})$ which is $N_1(x_1 - P_1) + N_2(x_2 - P_2) + N_3(x_3 - P_3) = 0$

Two directions can be found by subtraction of points on the plane

The Normal can be found taking cross product of the two directions.

④ Find the plane through points $\begin{pmatrix}1\\0\\0\end{pmatrix}$ $\begin{pmatrix}0\\5\\0\end{pmatrix}$ and $\begin{pmatrix}0\\0\\4\end{pmatrix}$

Plot:



Any of these three can be the position. I choose $\vec{P} = \begin{pmatrix}1\\0\\0\end{pmatrix}$

No normal is given so I need to take cross product of directions.

Directions: $\vec{v} = \begin{pmatrix}0\\5\\0\end{pmatrix} - \begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0-1\\5-0\\0-0\end{pmatrix} = \begin{pmatrix}-1\\5\\0\end{pmatrix}$

$\vec{w} = \begin{pmatrix}0\\0\\4\end{pmatrix} - \begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0-1\\0-0\\4-0\end{pmatrix} = \begin{pmatrix}-1\\0\\4\end{pmatrix}$

$$\left\{ \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix} + t\begin{pmatrix}-1\\5\\0\end{pmatrix} + s\begin{pmatrix}-1\\0\\4\end{pmatrix} \,\middle|\, t,s \in \mathbb{R} \right\}$$ ← position direction form

Normal

$$\vec{N} = \vec{v} \times \vec{w} = \begin{pmatrix}-1\\5\\0\end{pmatrix} \times \begin{pmatrix}-1\\0\\4\end{pmatrix} = \begin{pmatrix}5\cdot4 - 0\cdot0\\0(-1)-(-1)4\\-1(0)-5(-1)\end{pmatrix} = \begin{pmatrix}20\\4\\5\end{pmatrix}$$

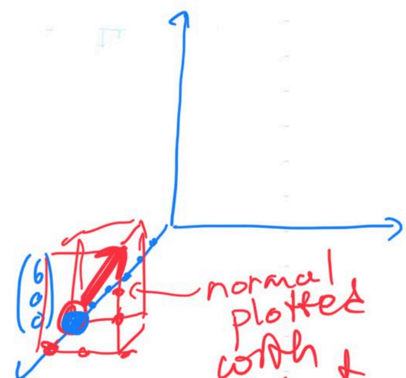
Check $\vec{N}\cdot\vec{v} = \begin{pmatrix}20\\4\\5\end{pmatrix} \cdot \begin{pmatrix}-1\\5\\0\end{pmatrix} = 20(-1) + 4(5) + 5\cdot0 = 0$ so perpendicular ✓

$\vec{N}\cdot\vec{w} = \begin{pmatrix}20\\4\\5\end{pmatrix} \cdot \begin{pmatrix}-1\\0\\4\end{pmatrix} = 20(-1) + 4(0) + 5(4) = 0$ ✓ so perpendicular

The plane is $\vec{N}\cdot(\vec{x}-\vec{P}) = 0$ so $20(x_1-1)+4(x_2-0)+5(x_3-0) = 0$

It is also correct to write $20(x-1) + 4(y-0) + 5(z-0) = 0$

⑥ The plane through $\begin{pmatrix}6\\0\\0\end{pmatrix}$ with normal $\begin{pmatrix}1\\2\\3\end{pmatrix}$



normal plotted with tail at the position.

$$1(x_1-6) + 2(x_2-0) + 3(x_3-0) = 0$$

or

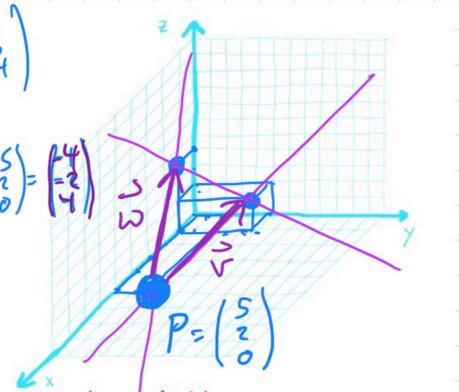
$$1(x-6) + 2(y-0) + 3(z-0) = 0$$

How to plot the plane?

We need more positions:

Find where the plane crosses each axis

⑤ The plane through $\begin{pmatrix}5\\2\\0\end{pmatrix}$ $\begin{pmatrix}1\\5\\2\end{pmatrix}$ and $\begin{pmatrix}1\\0\\4\end{pmatrix}$



Choose position $\vec{p}=\begin{pmatrix}5\\2\\0\end{pmatrix}$

Directions $\vec{v}=\begin{pmatrix}1\\5\\2\end{pmatrix}-\begin{pmatrix}5\\2\\0\end{pmatrix}=\begin{pmatrix}-4\\3\\2\end{pmatrix}$ $\vec{w}=\begin{pmatrix}1\\0\\4\end{pmatrix}-\begin{pmatrix}5\\2\\0\end{pmatrix}=\begin{pmatrix}-4\\-2\\4\end{pmatrix}$

Position Direction form:

$$\left\{\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}5\\2\\0\end{pmatrix}+t\begin{pmatrix}-4\\3\\2\end{pmatrix}+s\begin{pmatrix}-4\\-2\\4\end{pmatrix}\,\middle|\, t,s\in\mathbb{R}\right\}$$

Normal

$$\vec{N}=\vec{v}\times\vec{w}=\begin{pmatrix}-4\\3\\2\end{pmatrix}\times\begin{pmatrix}-4\\-2\\4\end{pmatrix}=\begin{pmatrix}3\cdot4-2(-2)\\2(-4)-(-4)4\\-4(-2)-3(-4)\end{pmatrix}=\begin{pmatrix}12+4\\-8+16\\8+12\end{pmatrix}=\begin{pmatrix}16\\8\\20\end{pmatrix}$$

Check the normal is perpendicular to the direction:

$$\vec{N}\bullet\vec{v}=\begin{pmatrix}16\\8\\20\end{pmatrix}\bullet\begin{pmatrix}-4\\3\\2\end{pmatrix}=-64+24+40=0 \checkmark \qquad \vec{N}\bullet\vec{w}=\begin{pmatrix}16\\8\\20\end{pmatrix}\bullet\begin{pmatrix}-4\\-2\\4\end{pmatrix}=-64-16+80=0 \checkmark$$

So

$$\vec{N}\bullet(\vec{x}-\vec{p})=0 \qquad 16(x_1-5)+8(x_2-2)+20(x_3-0)=0$$

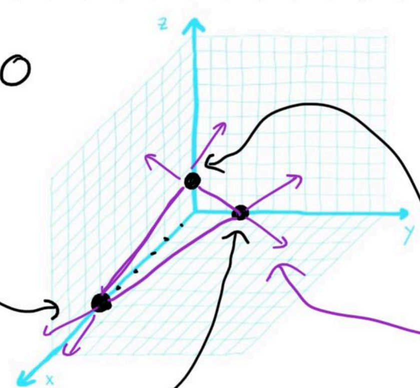
Also $16(x-5)+8(y-2)+20(z-0)=0$

x axis has $y=0$ and $z=0$

$$1(x-6)+2(0-0)+3(0-0)=0$$

solve for x:

$$1(x-6)+0+0=0$$
$$\Rightarrow x-6=0$$
$$\Rightarrow x=6 \quad \text{So } \begin{pmatrix}6\\0\\0\end{pmatrix}$$



y axis has $x=0$ and $z=0$

$$1(0-6)+2(y-0)+3(0-0)=0$$

solve for y: $-6+2y+0=0$ $\Rightarrow 2y=6$ $\Rightarrow y=3$ So $\begin{pmatrix}0\\3\\0\end{pmatrix}$

z axis has $x=0$ and $y=0$

$$1(0-6)+2(0-0)+3(z-0)=0$$

solve for z: $-6+0+3z=0\Rightarrow 3z=6\Rightarrow z=2$ So $\begin{pmatrix}0\\0\\2\end{pmatrix}$

Then draw plane

Feedback to students:

Vector Calculus MAT226 Spring 2025
Professor Sormani
Lesson 4 Lines and Planes

Score: +1 A or +.75B or +½ C or +¼ D or +0 F

The classwork and notes are completed +½
Only some classwork is done +¼
No notes/classwork is done +0 (this is half the credit for each lesson)

The homework is tried +½
Only some homework is tried +¼
No homework is tried +0 (this is half the credit for each lesson)

The solutions to the homework are now at the end of Lesson 4: here. There are other correct answers so feel free to ask if you want to know if your answer is also correct. I can help with graphing at office hours.