Vector Calculus MAT226 Spring 2025
Professor Sormani
Lesson 1 Review of Vectors and Plotting Points in 3D

Please go to your email and make sormanic@gmail.com a contact so that you will receive all my emails. Be sure to read them all semester.

Carefully take notes on pencil and paper while attending class or watching the lesson videos. You will cut and paste the photos of your notes and completed classwork and a selfie taken holding up the first page of your work in a googledoc entitled:

MAT226S25-lesson1-lastname-firstname

Then share editing of that document with me sormanic@gmail.com. You will also put photos of your homework in this googledoc.   If you work with any classmates, be sure to write their names on the problems you completed together.

Today's Playlist 226F21-1 has 22 short videos and photos of the notes for them are below.
Please watch all the videos if you missed class. If you attended class you may choose to watch some videos that might help you understand anything you did not understand in class.

Welcome to Vector Calculus

-Prof Sormani

Lesson I

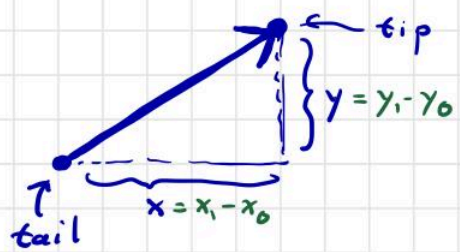
Vectors

* Part I: Vectors in the Plane

Part II: Vectors in Space

A vector in a plane: $\begin{pmatrix} x \\ y \end{pmatrix}$ where $x, y \in \mathbb{R}$ (in; real numbers $(-\infty, \infty)$)

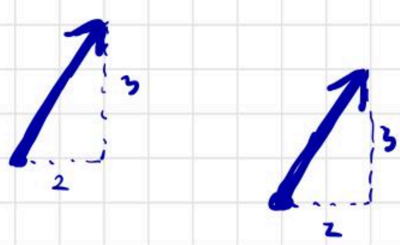
x and y are called components



A vector whose tail is at a point (x_0, y_0) and whose tip is at a point (x_1, y_1) is written $\begin{pmatrix} x_1 - x_0 \\ y_1 - y_0 \end{pmatrix}$ "tip − tail"

a vector can moved to have a new location for its tail, and then the tip moves too.

Example: $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

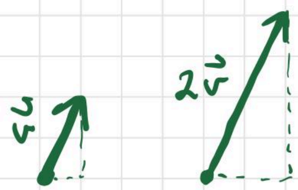


Rescaling a Vector $\vec{v} \in \mathbb{R}^2$

scale by a factor $r \in \mathbb{R}$

Scalar Mult: $r\vec{v} = r\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} r v_1 \\ r v_2 \end{pmatrix}$

Examples: $\vec{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\quad 2\vec{v} = 2\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2\cdot 1 \\ 2\cdot 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$



is twice as long in the same direction!

$-1\vec{v} = -1\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1\cdot 1 \\ -1\cdot 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$

$-1\vec{v}$ is the same length as $\vec{v}$ but opposite direction!

Classwork (1): $\vec{v} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$ (a) plot $\vec{v}$

(b) find and plot $2\vec{v}$

(c) find and plot $3\vec{v}$

(d) find and plot $-\vec{v}$

(e) find and plot $-2\vec{v}$

Classwork (2): repeat for $\vec{v} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

Magnitude of Rescaled Vectors Theorem

$$\|r\vec{v}\| = |r|\,\|\vec{v}\|$$

Consequences: $r=2 \implies \|2\vec{v}\| = |2|\,\|\vec{v}\| = 2\,\|\vec{v}\|$

$r=-1 \implies \|-1\vec{v}\| = |-1|\,\|\vec{v}\| = 1\,\|\vec{v}\| = \|\vec{v}\|$

Proof:

(1) $\|r\vec{v}\| = \left\| r \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\|$ (1) by defn of a vector in $\mathbb{R}^2$

(2) $= \left\| \begin{pmatrix} rv_1 \\ rv_2 \end{pmatrix} \right\|$ (2) by defn of scalar mult.

(3) $= \sqrt{(rv_1)^2 + (rv_2)^2}$ (3) by defn of magnitude

(4) $= \sqrt{r^2 v_1^2 + r^2 \cdot v_2^2}$ (4) by $(ab)^2 = a^2 \cdot b^2$

(5) $= \sqrt{r^2 (v_1^2 + v_2^2)}$ (5) by factoring $ab + ac = a(b+c)$

(6) $= \sqrt{r^2}\,\sqrt{v_1^2 + v_2^2}$ (6) by $\sqrt{ab} = \sqrt{a}\sqrt{b}$

(7) $= |r|\,\sqrt{v_1^2 + v_2^2}$ (7) by $\sqrt{a^2} = |a|$

(8) $= |r|\,\|\vec{v}\|$ (8) by defn of magnitude.

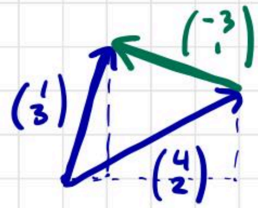
each step has a justifaction explaining that step.

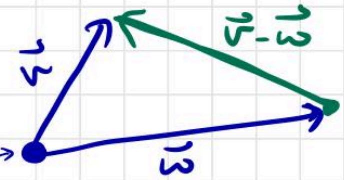
$\underbrace{\text{QED } \square}_{\text{the proof is done}}$

Vector Subtraction

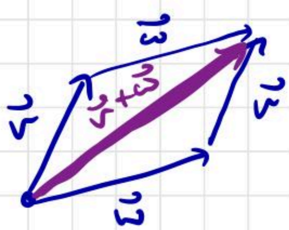
$$\vec{v} - \vec{w} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} - \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} v_1 - w_1 \\ v_2 - w_2 \end{pmatrix}$$

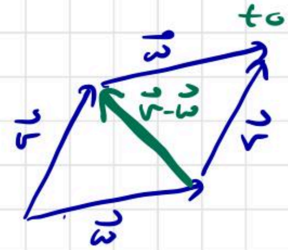
Example: $\begin{pmatrix} 1 \\ 3 \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 1-4 \\ 3-2 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$



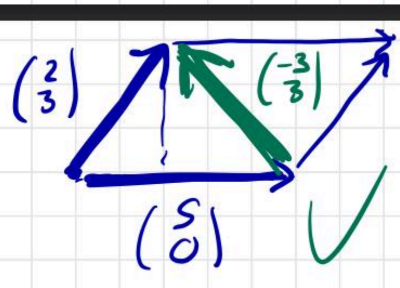


If you put the tails of $\vec{v}$ and $\vec{w}$ together
then $\vec{v}-\vec{w}$ is a vector from the tip of $\vec{w}$
to the tip of $\vec{v}$



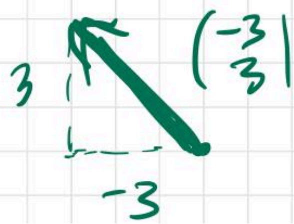


Classwork Plot $\begin{pmatrix} 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 5 \\ 0 \end{pmatrix}$ pause + try



✓

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 5 \\ 0 \end{pmatrix} = \begin{pmatrix} 2-5 \\ 3-0 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \end{pmatrix}$$



✓

$\vec{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

To save space

$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \langle v_1, v_2 \rangle$

Vector addition and scalar multiplication share many properties of ordinary arithmetic, as shown in the following theorem.

THEOREM 11.1 Properties of Vector Operations

Let **u**, **v**, and **w** be vectors in the plane, and let c and d be scalars.

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ — Commutative Property
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ — Associative Property
3. $\mathbf{u} + \mathbf{0} = \mathbf{u}$ — Additive Identity Property
4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ — Additive Inverse Property
5. $c(d\mathbf{u}) = (cd)\mathbf{u}$
6. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ — Distributive Property
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ — Distributive Property
8. $1(\mathbf{u}) = \mathbf{u},\ 0(\mathbf{u}) = \mathbf{0}$

the correct way

Proof The proof of the *Associative Property* of vector addition uses the Associative Property of addition of real numbers.

$$
\begin{aligned}
①\ (\mathbf{u} + \mathbf{v}) + \mathbf{w} &= \boxed{[\langle u_1, u_2\rangle + \langle v_1, v_2\rangle] + \langle w_1, w_2\rangle} = \left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \quad ① \text{ by defn of vector} \\
②\ &= \langle u_1 + v_1, u_2 + v_2\rangle + \langle w_1, w_2\rangle = \begin{pmatrix} u_1+v_1 \\ u_2+v_2 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \quad ② \text{ by vector addition} \\
③\ &= \langle (u_1 + v_1) + w_1, (u_2 + v_2) + w_2\rangle \\
④\ &= \langle u_1 + (v_1 + w_1), u_2 + (v_2 + w_2)\rangle \\
⑤\ &= \langle u_1, u_2\rangle + \langle v_1 + w_1, v_2 + w_2\rangle = \mathbf{u} + (\mathbf{v} + \mathbf{w})
\end{aligned}
$$

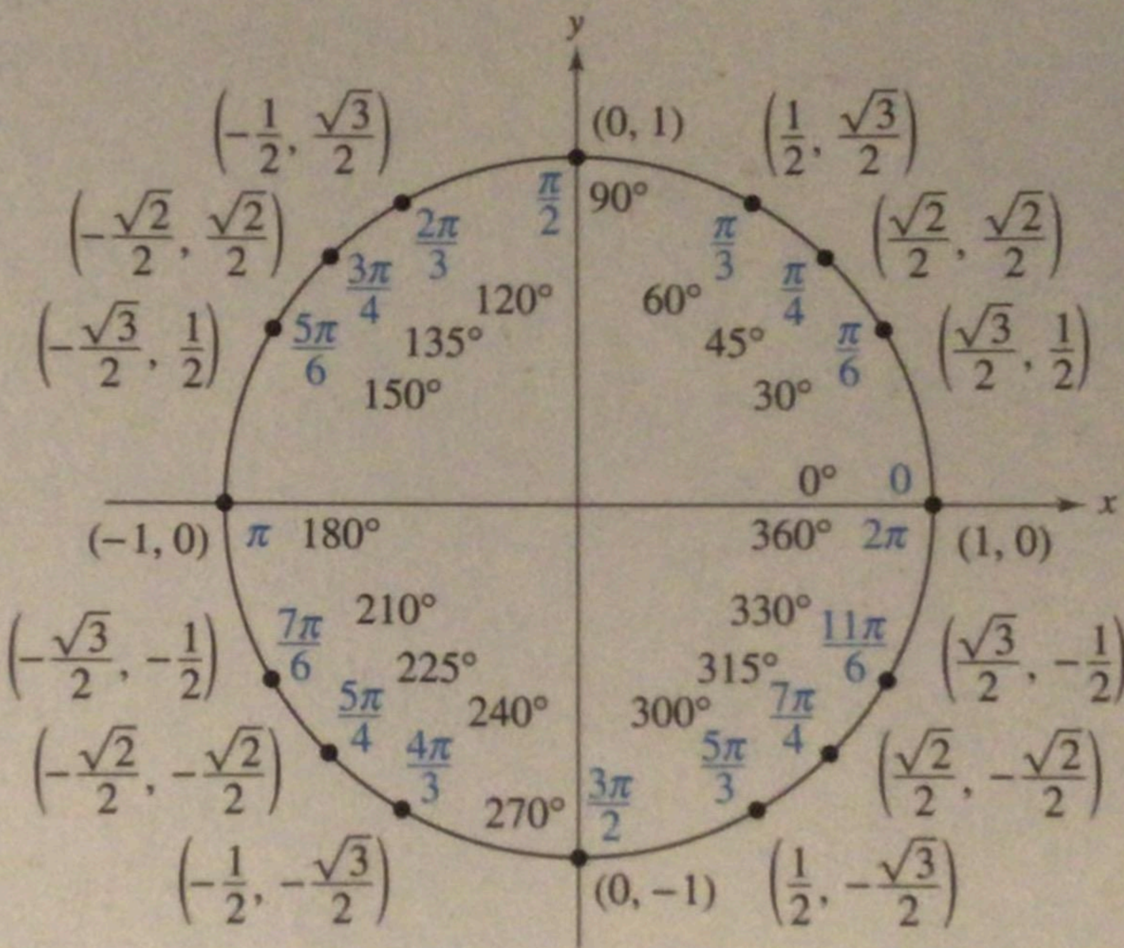
The book skips the justifications!

Similarly, the proof of the *Distributive Property* of vectors depends on the Distributive Property of real numbers.

$$
\begin{aligned}
① \rightarrow (c + d)\mathbf{u} &= (c + d)\langle u_1, u_2\rangle \\
②\ &= \langle (c + d)u_1, (c + d)u_2\rangle \\
③\ &= \langle cu_1 + du_1, cu_2 + du_2\rangle \\
④\ &= \langle cu_1, cu_2\rangle + \langle du_1, du_2\rangle \overset{⑤}{=} c\mathbf{u} + d\mathbf{u}
\end{aligned}
$$

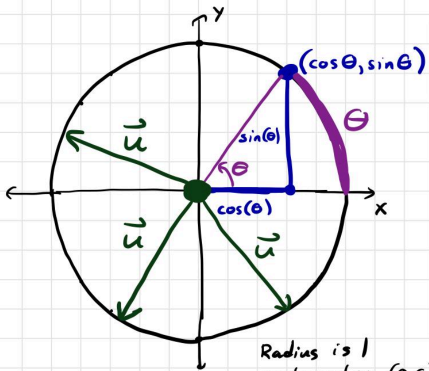
You can write in your own justifications.

The other properties can be proved in a similar manner.



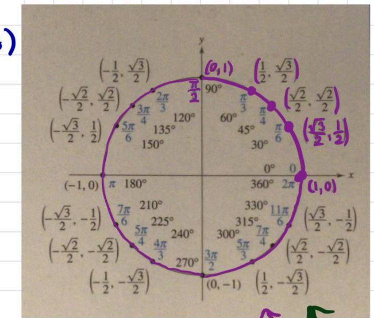
y
x
(0, 1)
$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
(1, 0)
$\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
$\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$
$\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$
(0, −1)
$\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$
$\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$
$\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
(−1, 0)
$\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
$\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
$\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
$\frac{\pi}{2}$ 90°
$\frac{\pi}{3}$ 60°
$\frac{\pi}{4}$ 45°
$\frac{\pi}{6}$ 30°
0° 0
360° 2π
$\frac{11\pi}{6}$ 330°
$\frac{7\pi}{4}$ 315°
$\frac{5\pi}{3}$ 300°
$\frac{3\pi}{2}$ 270°
$\frac{4\pi}{3}$ 240°
$\frac{5\pi}{4}$ 225°
$\frac{7\pi}{6}$ 210°
π 180°
$\frac{5\pi}{6}$ 150°
$\frac{3\pi}{4}$ 135°
$\frac{2\pi}{3}$ 120°

The unit circle and unit vectors



Radius is 1
centered on (0,0)

$x^2 + y^2 = 1$



always use radians in calculus

A unit vector $\vec{u}$ has $\|\vec{u}\| = 1$

$u_1^2 + u_2^2 = 1$

tail on the origin puts tip on unitcircle.

we can always find an angle θ

so that $\vec{u} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$

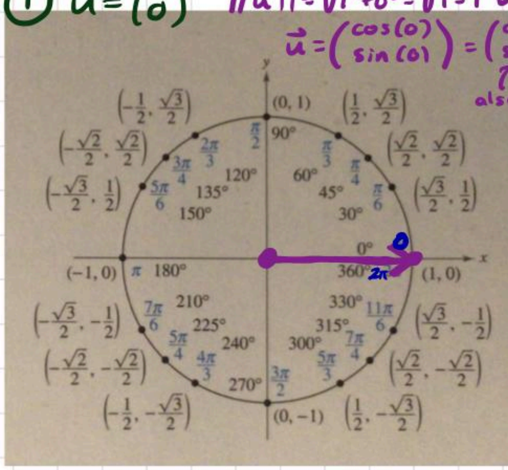
Classwork:

Check the following vectors are unit vectors and find θ for each!

① $\vec{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ② $\vec{u} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ ③ $\vec{u} = \begin{pmatrix} 1/2 \\ -\sqrt{3}/2 \end{pmatrix}$ ④ $\vec{u} = \begin{pmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix}$

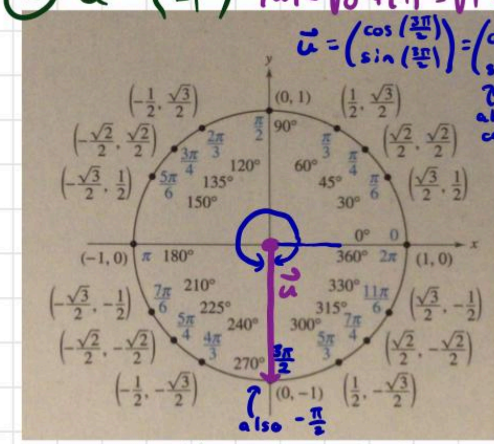
① $\vec{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\|\vec{u}\| = \sqrt{1^2+0^2} = \sqrt{1} = 1$ ✓

$\vec{u} = \begin{pmatrix} \cos(0) \\ \sin(0) \end{pmatrix} = \begin{pmatrix} \cos(2\pi) \\ \sin(2\pi) \end{pmatrix}$ also correct



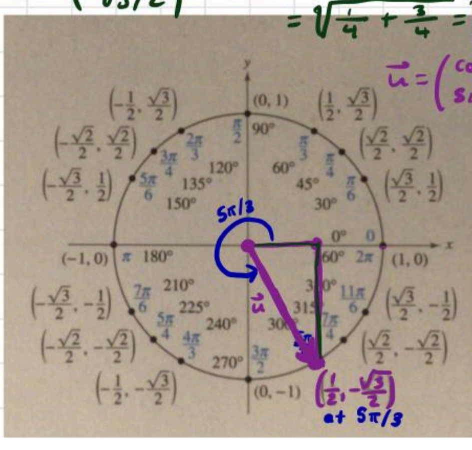
② $u = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ $|\vec{u}| = \sqrt{0^2+(-1)^2} = \sqrt{1} = 1$ ✓

$\vec{u} = \begin{pmatrix} \cos(\frac{3\pi}{2}) \\ \sin(\frac{3\pi}{2}) \end{pmatrix} = \begin{pmatrix} \cos(-\frac{\pi}{2}) \\ \sin(-\frac{\pi}{2}) \end{pmatrix}$ also correct



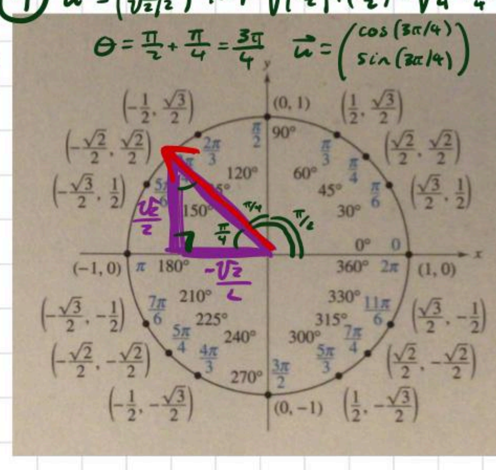
③ $\vec{u} = \begin{pmatrix} 1/2 \\ -\sqrt{3}/2 \end{pmatrix}$ $|\vec{u}| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{-\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = \sqrt{\frac{4}{4}} = \sqrt{1} = 1$ ✓

$\vec{u} = \begin{pmatrix} \cos(5\pi/3) \\ \sin(5\pi/3) \end{pmatrix}$



④ $\vec{u} = \begin{pmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix}$ $|\vec{u}| = \sqrt{\left(\frac{-\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2} = \sqrt{\frac{2}{4} + \frac{2}{4}} = 1$ ✓

$\theta = \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}$ $\vec{u} = \begin{pmatrix} \cos(3\pi/4) \\ \sin(3\pi/4) \end{pmatrix}$



Theorem: $\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v}$ is a unit vector for $\vec{v} \neq \vec{0}$

It is called the unit vector in the direction of $\vec{v}$.

Proof: Check $\|\vec{u}\| = 1$

① $\|\vec{u}\| = \left\| \frac{1}{\|\vec{v}\|}\vec{v} \right\|$ — ① by our defn of $\vec{u}$

② $= \left| \frac{1}{\|\vec{v}\|} \right| \|\vec{v}\|$ — ② by $\|r\vec{v}\| = |r|\,\|\vec{v}\|$ where $r = \frac{1}{\|v\|}$ here

③ $= \frac{1}{\|\vec{v}\|} \cdot \|\vec{v}\|$ — ③ $\left|\frac{1}{a}\right| = \frac{1}{|a|} = \frac{1}{a}$ if $a > 0$ where $a = \|\vec{v}\| > 0$ because $\vec{v} \neq \vec{0}$

final $= 1$ — ④ $\frac{1}{a} \cdot a = 1$ for $a \in \mathbb{R}$ where $a = \|\vec{v}\|$

QED

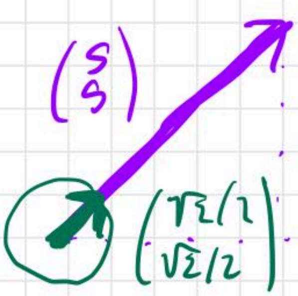
Apply this to find the unit vector in the direction of $\vec{v} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$

$$\vec{u} = \frac{1}{\left\|\begin{pmatrix} 5 \\ 5 \end{pmatrix}\right\|}\begin{pmatrix} 5 \\ 5 \end{pmatrix} = \frac{1}{\sqrt{5^2+5^2}}\begin{pmatrix} 5 \\ 5 \end{pmatrix} = \frac{1}{\sqrt{2 \cdot 25}}\begin{pmatrix} 5 \\ 5 \end{pmatrix} =$$

$$= \frac{1}{5\sqrt{2}}\begin{pmatrix} 5 \\ 5 \end{pmatrix} = \begin{pmatrix} \frac{5}{5\sqrt{2}} \\ \frac{5}{5\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix} = \begin{pmatrix} \cos\left(\frac{\pi}{4}\right) \\ \sin\left(\frac{\pi}{4}\right) \end{pmatrix}$$

answers

textbook: $\langle \sqrt{2}/2, \sqrt{2}/2 \rangle$



points in the same direction

Standard Unit Vectors

The unit vectors $\langle 1, 0\rangle$ and $\langle 0, 1\rangle$ are called the **standard unit vectors** in the plane and are denoted by

$$\mathbf{i} = \langle 1, 0\rangle \quad \text{and} \quad \mathbf{j} = \langle 0, 1\rangle \qquad \text{Standard unit vectors}$$

as shown in Figure 11.10. These vectors can be used to represent any vector uniquely, as follows.

$$\mathbf{v} = \langle v_1, v_2\rangle = \langle v_1, 0\rangle + \langle 0, v_2\rangle = v_1\langle 1, 0\rangle + v_2\langle 0, 1\rangle = v_1\mathbf{i} + v_2\mathbf{j}$$

The vector $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j}$ is called a **linear combination** of **i** and **j**. The scalars v_1 and v_2 are called the **horizontal** and **vertical components of v.**

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \underbrace{\begin{pmatrix} v_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ v_2 \end{pmatrix}}_{\text{by vector addition}} = \underbrace{v_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + v_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\text{by scalar mult}} = \underbrace{v_1\hat{i} + v_2\hat{j}}_{\text{by defn of standard unit vectors } \hat{i} \text{ and } \hat{j}}$$

Vectors in Space

$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \in \mathbb{R}^3$ each component $v_i \in \mathbb{R}$

$= \langle v_1, v_2, v_3 \rangle$ horizontal notation

$= v_1 \hat{i} + v_2 \hat{j} + v_3 \hat{k}$ physics notation

where $\hat{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\hat{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $\hat{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ standard unit vectors in $\mathbb{R}^3$

check

$$v_1 \hat{i} + v_2 \hat{j} + v_3 \hat{k} = v_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + v_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + v_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} v_1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ v_2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_1 + 0 + 0 \\ 0 + v_2 + 0 \\ 0 + 0 + v_3 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$\vec{u}$ is a unit vector if $\|\vec{u}\| = 1$

$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$ direction of $\vec{v}$ $\frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\|\vec{v}\|}\vec{v}$

no unit circle in 3D

scalar multiplication

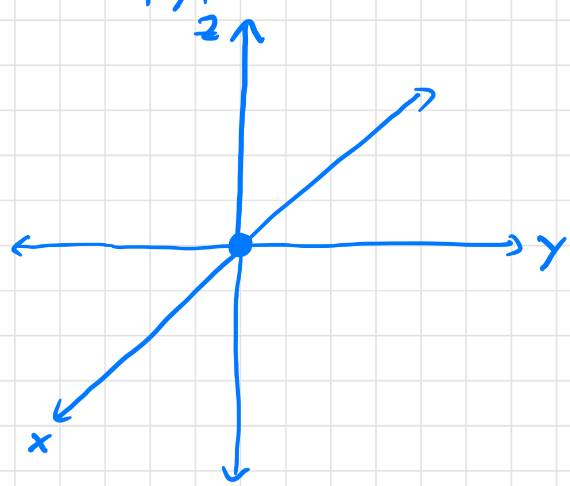
$$r \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} rv_1 \\ rv_2 \\ rv_3 \end{pmatrix}$$

vector addition/subtraction

also adding/subtracting the terms.

Plotting Points in Space

x, y, and z each has an axis



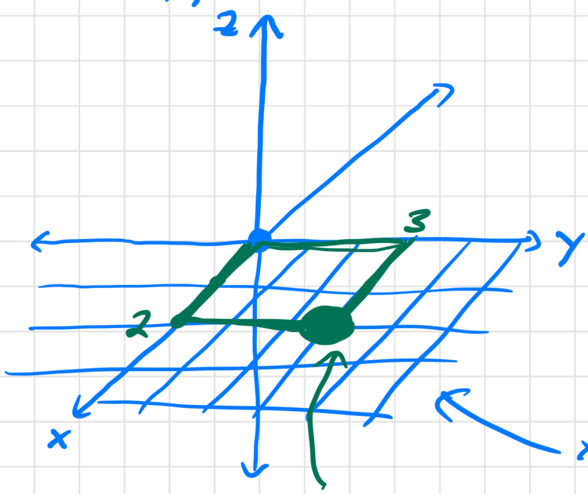
x axis is on the floor coming forwards

y axis is on the floor going to the right

z axis is going upwards in the corner of the room

Plotting Points in Space

x, y, and z each has an axis



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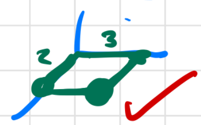
y axis is on the floor going to the right

z axis is going upwards in the corner of the room

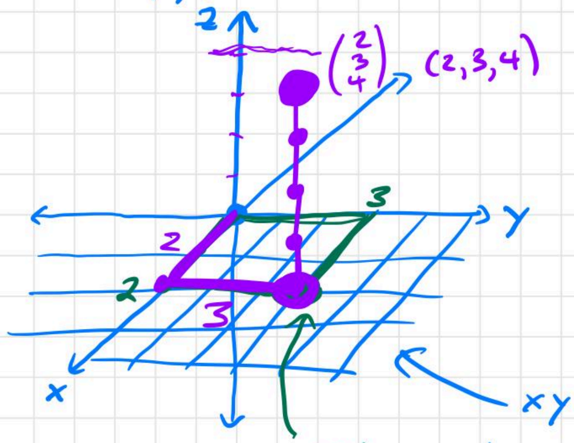
xy plane: $\left\{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \;\middle|\; x, y \in \mathbb{R} \right\}$

plot $\begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$



Plotting Points in Space

x, y, and z each has an axis



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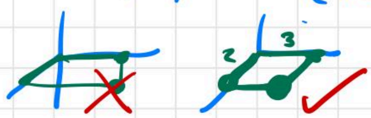
xy plane: $\left\{ \begin{pmatrix}x\\y\\0\end{pmatrix} \;\middle|\; x,y \in \mathbb{R} \right\}$

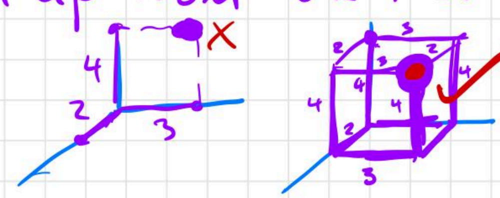
plot $\begin{pmatrix}2\\3\\0\end{pmatrix}$



plot $\begin{pmatrix}2\\3\\4\end{pmatrix}$ 2 forward in x direction

3 to the right in y direction

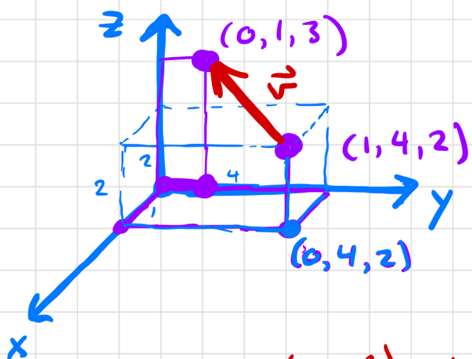
4 up from the floor



front corner is (2,3,4)

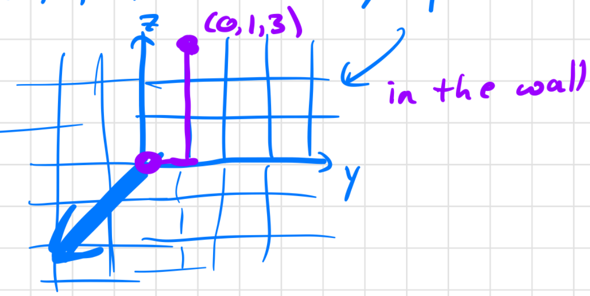
Classwork: ① Plot $(1,4,2)$ and Plot $(0,1,3)$

② Find the vector whose tail is at $(1,4,2)$ and tip is at $(0,1,3)$



$$\vec{v} = \text{tip} - \text{tail} = \begin{pmatrix} 0-1 \\ 1-4 \\ 3-2 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix}$$

$(0,1,3)$ is in the yz plane



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Vector Calculus 226F21

Linear Algebra 0 | Vector Calculus 226F21

Classwork ① Plot (2,5,1) and (4,8,4) and (6,11,7)

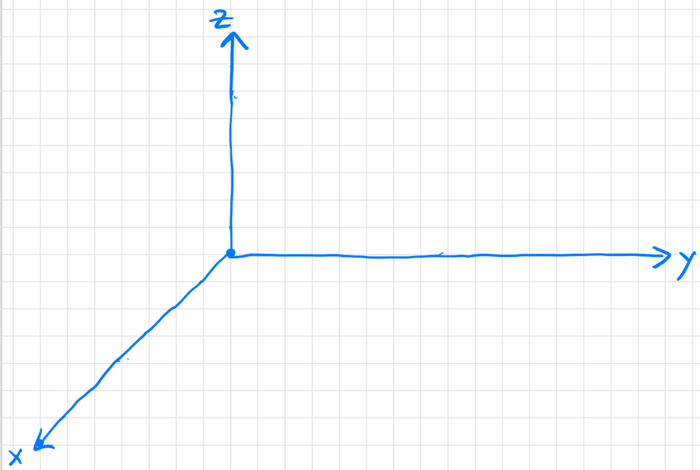
pause & try

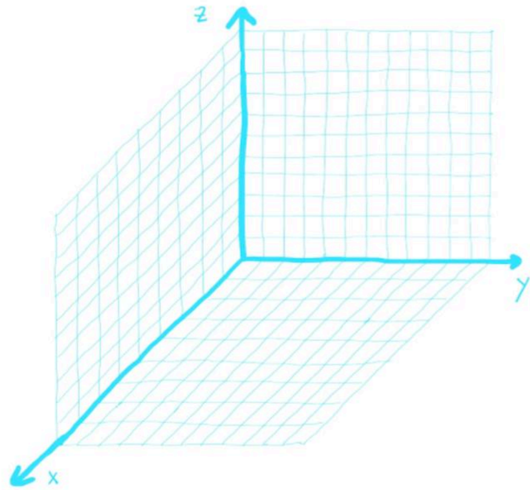
② Find $\vec{v}$ from (2,5,1) to (4,8,4)

③ Find $\vec{w}$ from (2,5,1) to (6,11,7)

④ Are (2,5,1), (4,8,4), and (6,11,17) collinear? That is, are they on a single line

Draw z-axis 8 high so we can fit (6,11,7)
y axis 12 to the right
x axis 7 forward

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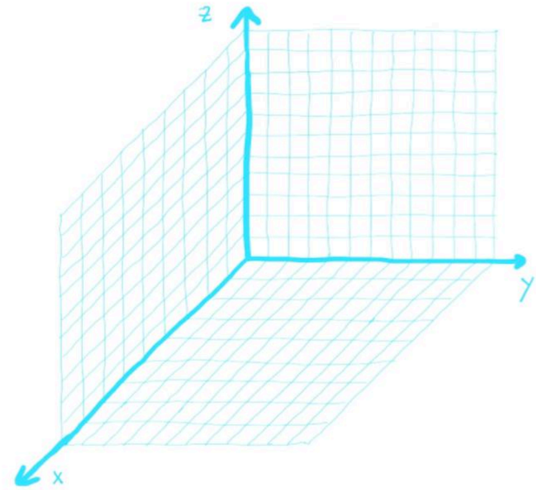
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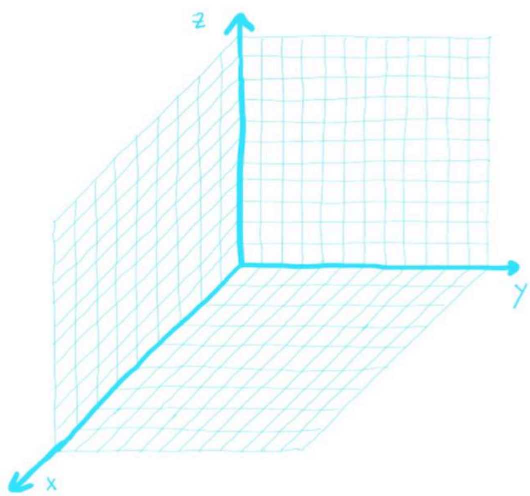
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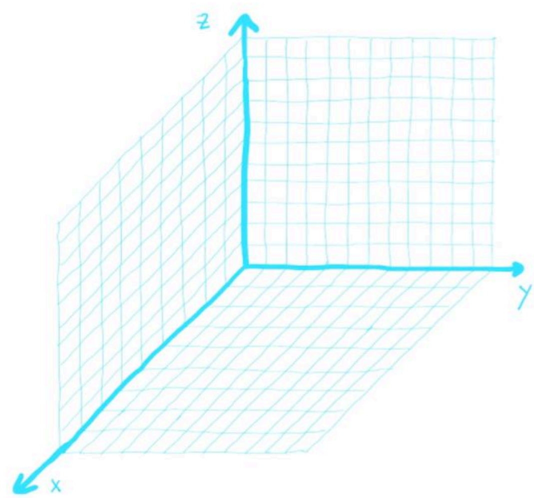
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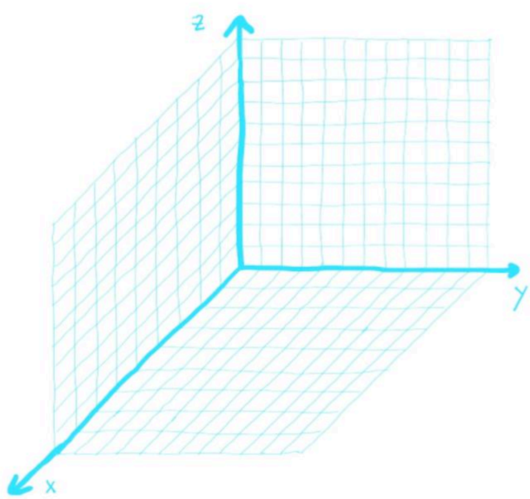
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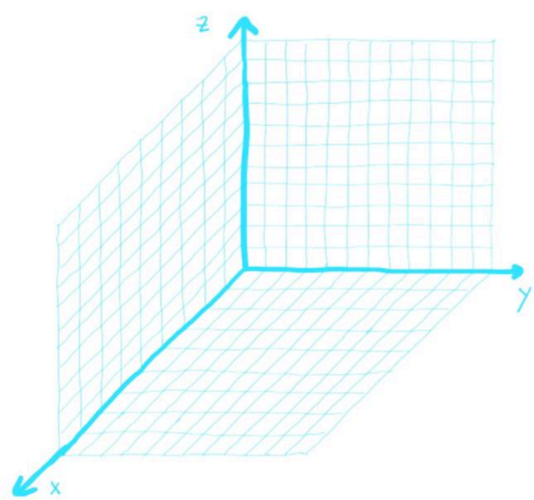
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Classwork ① Plot $(2,5,1)$ and $(4,8,4)$ and $(6,11,7)$

pause & try

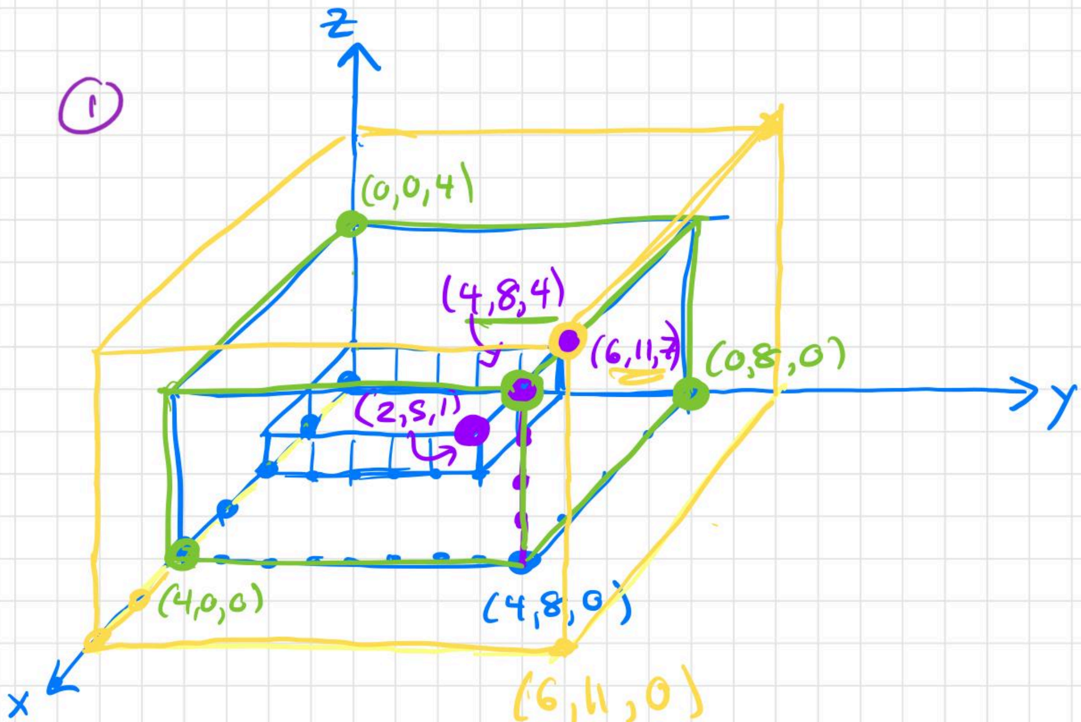
② Find $\vec{v}$ from $(2,5,1)$ to $(4,8,4)$

③ Find $\vec{w}$ from $(2,5,1)$ to $(6,11,7)$

④ Are $(2,5,1)$, $(4,8,4)$, and $(6,11,17)$ collinear? That is, are they on a single line

Draw z-axis 8 high so we can fit $(6,11,7)$
y axis 12 to the right
x axis 7 forward

①



Classwork ① Plot (2,5,1) and (4,8,4) and (6,11,7)

pause & try

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Draw z-axis 8 high so we can fit (6,11,7)
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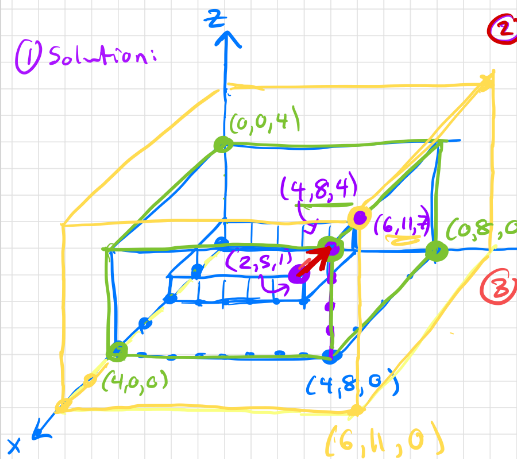
① Solution:



② $\vec{v} = \begin{pmatrix} 4-2 \\ 8-5 \\ 4-1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}$

pause + try ③

③ $\vec{w} = \begin{pmatrix} 6-2 \\ 11-5 \\ 7-1 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ 6 \end{pmatrix}$

Classwork ① Plot (2,5,1) and (4,8,4) and (6,11,7)

pause & try

② Find $\vec{v}$ from (2,5,1) to (4,8,4)

③ Find $\vec{w}$ from (2,5,1) to (6,11,7)

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Draw z-axis 8 high so we can fit (6,11,7)
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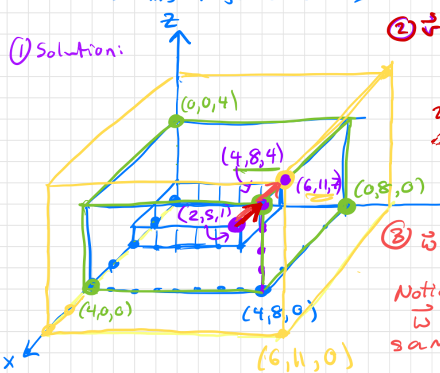
① Solution:



② $\vec{v} = \begin{pmatrix}4-2\\8-5\\4-1\end{pmatrix} = \begin{pmatrix}2\\3\\3\end{pmatrix}$

pause & try ③

③ $\vec{w} = \begin{pmatrix}6-2\\11-5\\7-1\end{pmatrix} = \begin{pmatrix}4\\6\\6\end{pmatrix}$

Notice $\vec{w} = 2\vec{v}$
same direction!

④ Yes they are on the same line because if we start at (2,5,1) and take a straight line to (4,8,4) that has direction $\vec{v}$ continuing in that direction $\vec{w}$ to (6,11,7)

Three points a, b, c are colinear iff

$$\vec{v} = \begin{pmatrix} a_1 - b_1 \\ a_2 - b_2 \\ a_3 - b_3 \end{pmatrix} \text{ and } \vec{w} = \begin{pmatrix} a_1 - c_1 \\ a_2 - c_2 \\ a_3 - c_3 \end{pmatrix}$$

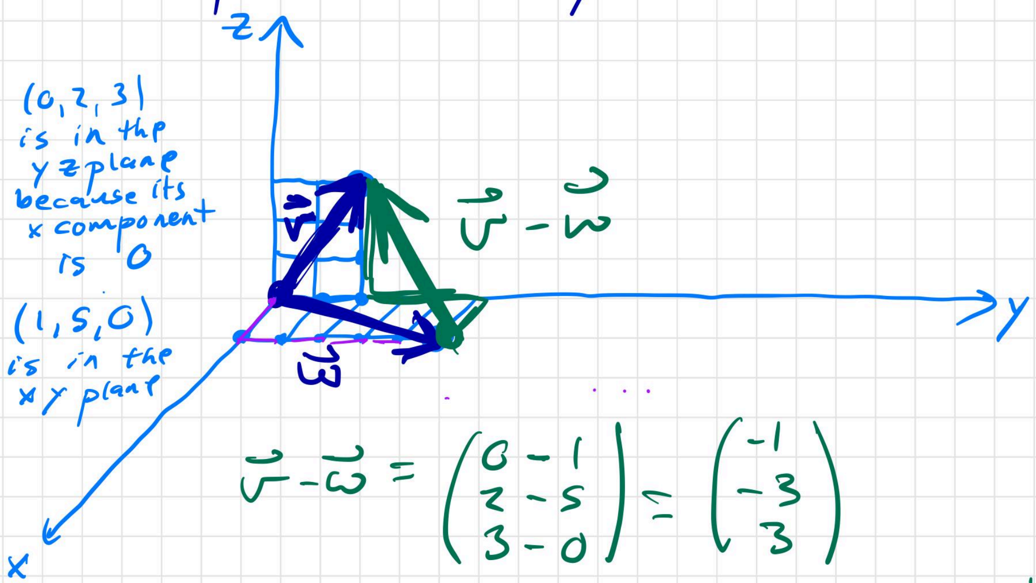
have the same or opposite direction.

Check if there is a real number r such that $\vec{v} = r\vec{w}$

or check that the have the same or opposite unit vectors.

See also the example in the book.

pause + try!



$$\vec{v}-\vec{w} = \begin{pmatrix} 0-1 \\ 2-5 \\ 3-0 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 3 \end{pmatrix}$$

Read about Force + Velocity in Text.

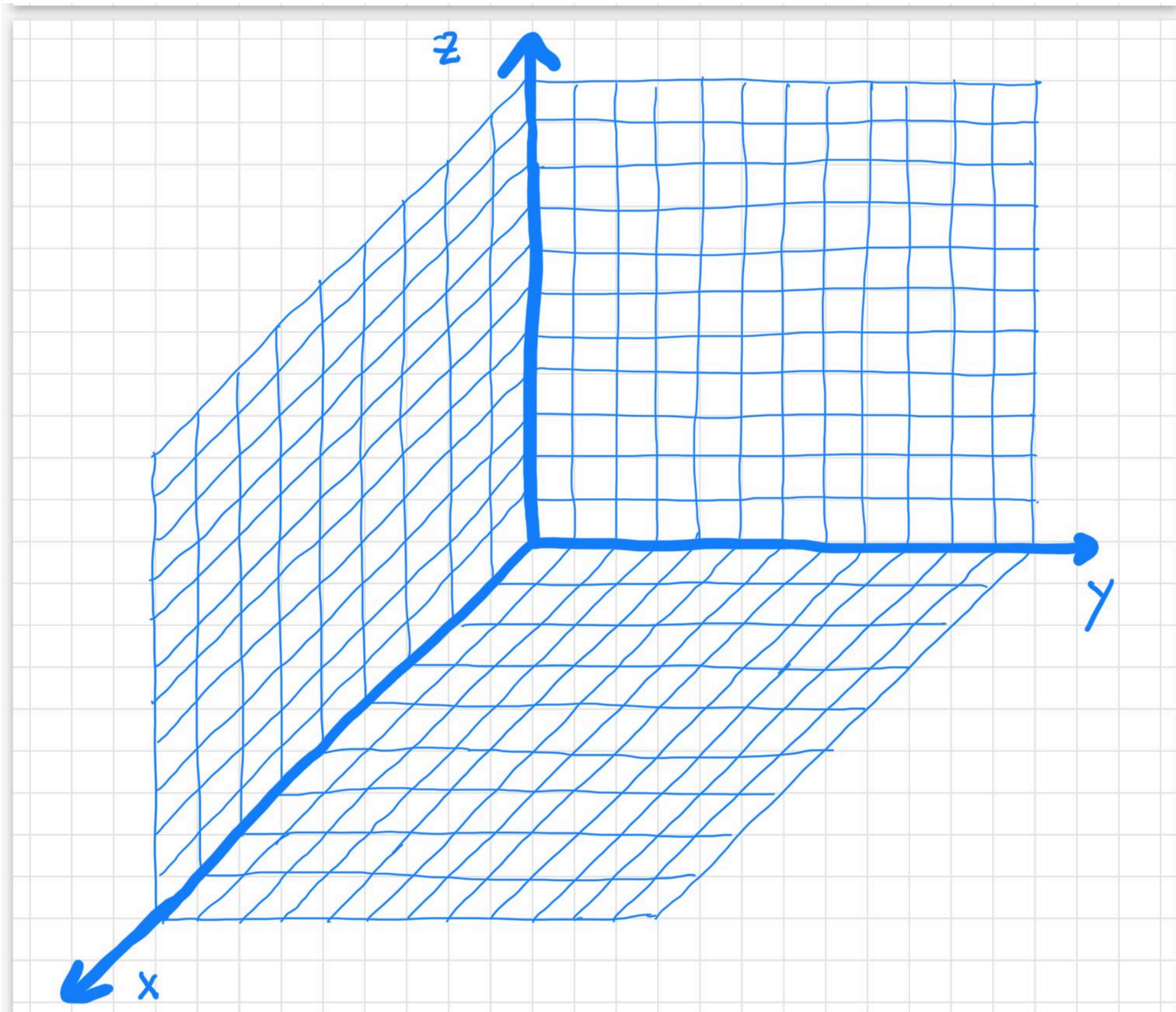
Extra Credit

Prove that when $\vec{v}, \vec{w} \in \mathbb{R}^3$

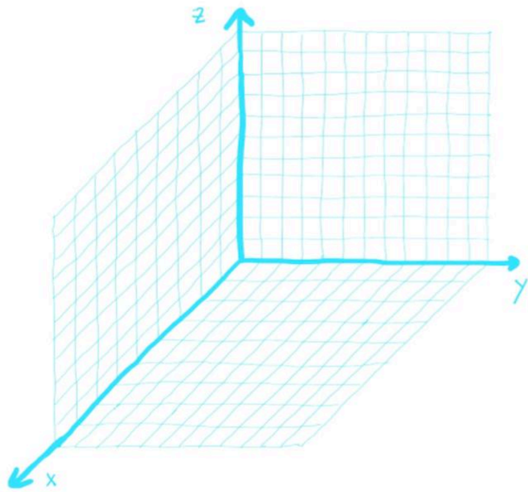
then $r(\vec{v}-\vec{w}) = r\vec{v} - r\vec{w}$

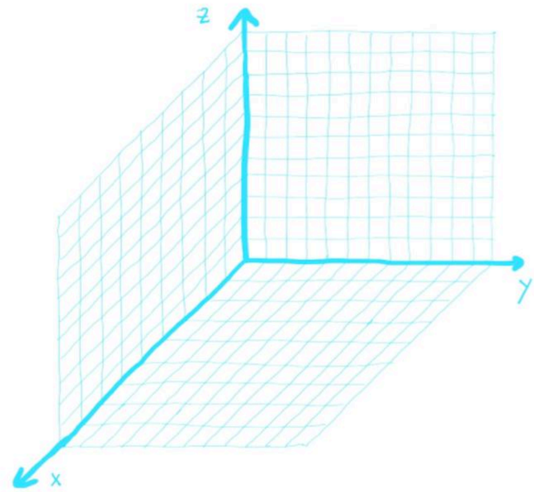
using numbered statements and justifications.

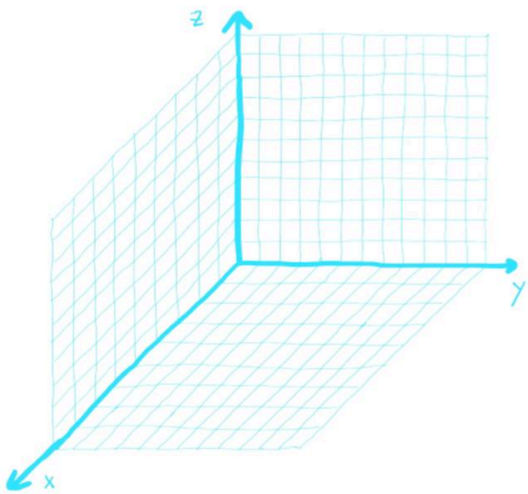
A chart you can print out and use as 3D graph paper:

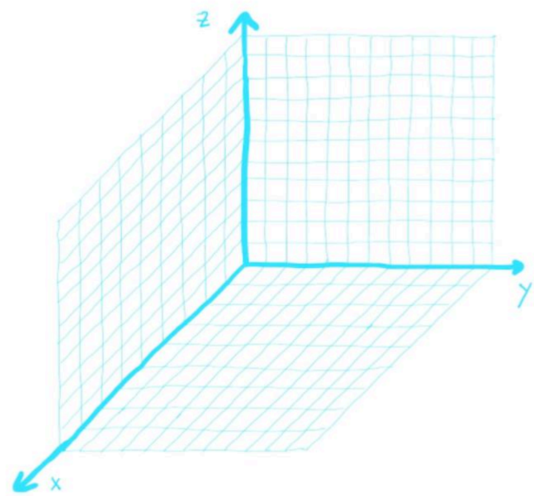


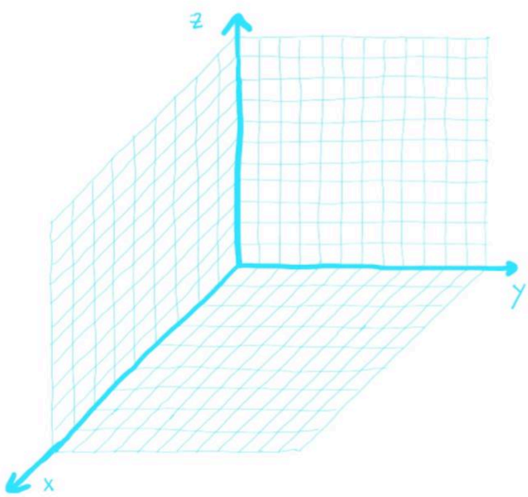
A nice set of videos by RootMath about vectors

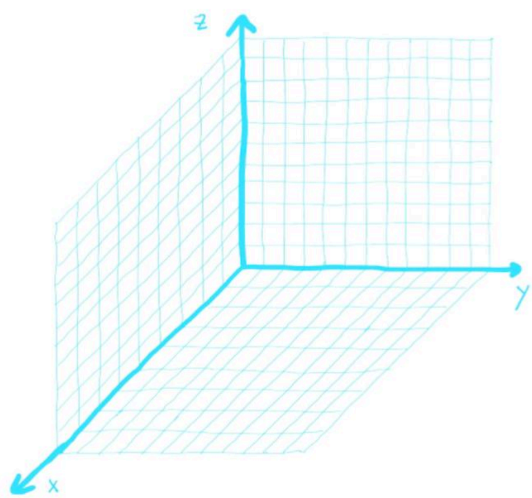












A nice set of videos by RootMath about vectors

How to check if you have watched all the videos

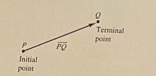
Homework from the department syllabus is
Read 11.1-11.2 and do odd problems in 11.1-11.2
I understand if you do not have time to do all of it. Focus on the ones similar to the classwork that can be done quickly. Physics and engineering majors should read and do the applications with forces and velocities but other students may skip these for now.

Submit the homework in the same googledoc as the classwork. Be sure to write out the questions as well as the answers.

In case you have not purchased the book, today I include photos of the sections we covered and the solutions:

Section 11.1 Vectors in the Plane

- **Write the component form of a vector.**
- **Perform vector operations and interpret the results geometrically.**
- **Write a vector as a linear combination of standard unit vectors.**
- **Use vectors to solve problems involving force or velocity.**



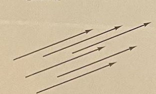
A directed line segment
Figure 11.1

Component Form of a Vector

Many quantities in geometry and physics, such as area, volume, temperature, mass, and time, can be characterized by a single real number scaled to appropriate units of measure. These are called **scalar quantities,** and the real number associated with each is called a **scalar.**

Other quantities, such as force, velocity, and acceleration, involve both magnitude and direction and cannot be characterized completely by a single real number. A **directed line segment** is used to represent such a quantity, as shown in Figure 11.1. The directed line segment $\overrightarrow{PQ}$ has **initial point** P and **terminal point** Q, and its **length** (or **magnitude**) is denoted by $\|\overrightarrow{PQ}\|$. Directed line segments that have the same length and direction are **equivalent,** as shown in Figure 11.2. The set of all directed line segments that are equivalent to a given directed line segment $\overrightarrow{PQ}$ is a **vector in the plane** and is denoted by $\mathbf{v} = \overrightarrow{PQ}$. In typeset material, vectors are usually denoted by lowercase, boldface letters such as **u**, **v**, and **w**. When written by hand, however, vectors are often denoted by letters with arrows above them, such as $\vec{u}$, $\vec{v}$, and $\vec{w}$.

Be sure you see that a vector in the plane can be represented by many different directed line segments—all pointing in the same direction and all of the same length.


Equivalent directed line segments
Figure 11.2

EXAMPLE 1 Vector Representation by Directed Line Segments

Let **v** be represented by the directed line segment from (0, 0) to (3, 2), and let **u** be represented by the directed line segment from (1, 2) to (4, 4). Show that **v** and **u** are equivalent.

Solution Let $P(0, 0)$ and $Q(3, 2)$ be the initial and terminal points of **v**, and let $R(1, 2)$ and $S(4, 4)$ be the initial and terminal points of **u**, as shown in Figure 11.3. You can use the Distance Formula to show that $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ have the *same length.*

$$\|\overrightarrow{PQ}\| = \sqrt{(3-0)^2 + (2-0)^2} = \sqrt{13} \qquad \text{Length of } \overrightarrow{PQ}$$
$$\|\overrightarrow{RS}\| = \sqrt{(4-1)^2 + (4-2)^2} = \sqrt{13} \qquad \text{Length of } \overrightarrow{RS}$$

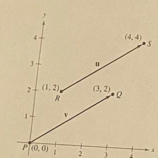
Both line segments have the *same direction,* because they both are directed toward the upper right on lines having the same slope.

$$\text{Slope of } \overrightarrow{PQ} = \frac{2-0}{3-0} = \frac{2}{3}$$

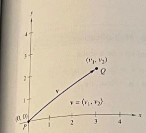
and

$$\text{Slope of } \overrightarrow{RS} = \frac{4-2}{4-1} = \frac{2}{3}$$

Because $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ have the same length and direction, you can conclude that the two vectors are equivalent. That is, **v** and **u** are equivalent.



The vectors **u** and **v** are equivalent.
Figure 11.3



The standard position of a vector
Figure 11.4

The directed line segment whose initial point is the origin is often the most convenient representative of a set of equivalent directed line segments such as those shown in Figure 11.3. This representation of **v** is said to be in **standard position.** A directed line segment whose initial point is the origin can be uniquely represented by the coordinates of its terminal point $Q(v_1, v_2)$, as shown in Figure 11.4.

> **Definition of Component Form of a Vector in the Plane**
>
> If **v** is a vector in the plane whose initial point is the origin and whose terminal point is (v_1, v_2), then the **component form of v** is given by
>
> $\mathbf{v} = \langle v_1, v_2\rangle.$
>
> The coordinates v_1 and v_2 are called the **components of v.** If both the initial point and the terminal point lie at the origin, then **v** is called the **zero vector** and is denoted by $\mathbf{0} = \langle 0, 0\rangle$.

This definition implies that two vectors $\mathbf{u} = \langle u_1, u_2\rangle$ and $\mathbf{v} = \langle v_1, v_2\rangle$ are **equal** if and only if $u_1 = v_1$ and $u_2 = v_2$.

The following procedures can be used to convert directed line segments to component form or vice versa.

NOTE It is important to understand that a vector represents a *set* of directed line segments (each having the same length and direction). In practice, however, it is common not to distinguish between a vector and one of its representatives.

1. If $P(p_1, p_2)$ and $Q(q_1, q_2)$ are the initial and terminal points of a directed line segment, the component form of the vector **v** represented by $\overrightarrow{PQ}$ is $\langle v_1, v_2\rangle = \langle q_1 - p_1, q_2 - p_2\rangle$. Moreover, the **length** (or **magnitude**) **of v** is

$$\|\mathbf{v}\| = \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2} = \sqrt{v_1^2 + v_2^2}. \qquad \text{Length of a vector}$$

2. If $\mathbf{v} = \langle v_1, v_2\rangle$, **v** can be represented by the directed line segment, in standard position, from $P(0, 0)$ to $Q(v_1, v_2)$.

The length of **v** is also called the **norm of v**. If $\|\mathbf{v}\| = 1$, **v** is a **unit vector.** Moreover, $\|\mathbf{v}\| = 0$ if and only if **v** is the zero vector **0**.

EXAMPLE 2 Finding the Component Form and Length of a Vector

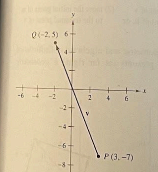
Find the component form and length of the vector **v** that has initial point $(3, -7)$ and terminal point $(-2, 5)$.

Solution Let $P(3, -7) = (p_1, p_2)$ and $Q(-2, 5) = (q_1, q_2)$. Then the components of $\mathbf{v} = \langle v_1, v_2\rangle$ are

$$v_1 = q_1 - p_1 = -2 - 3 = -5$$
$$v_2 = q_2 - p_2 = 5 - (-7) = 12.$$

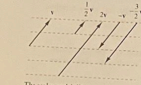
So, as shown in Figure 11.5, $\mathbf{v} = \langle -5, 12\rangle$, and the length of **v** is

$$\|\mathbf{v}\| = \sqrt{(-5)^2 + 12^2} = \sqrt{169} = 13.$$



Component form of **v**: $\mathbf{v} = \langle -5, 12\rangle$
Figure 11.5

Vector Operations



The scalar multiplication of **v**
Figure 11.6

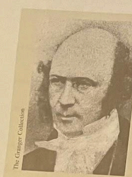
Definitions of Vector Addition and Scalar Multiplication

Let $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ be vectors and let c be a scalar.

1. The **vector sum** of $\mathbf{u}$ and $\mathbf{v}$ is the vector $\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2 \rangle$.
2. The **scalar multiple** of c and $\mathbf{u}$ is the vector $c\mathbf{u} = \langle cu_1, cu_2 \rangle$.
3. The **negative** of $\mathbf{v}$ is the vector
$$-\mathbf{v} = (-1)\mathbf{v} = \langle -v_1, -v_2 \rangle.$$
4. The **difference** of $\mathbf{u}$ and $\mathbf{v}$ is
$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}) = \langle u_1 - v_1, u_2 - v_2 \rangle.$$

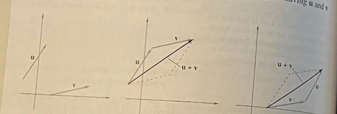
Geometrically, the scalar multiple of a vector $\mathbf{v}$ and a scalar c is the vector that is $|c|$ times as long as $\mathbf{v}$, as shown in Figure 11.6. If c is positive, $c\mathbf{v}$ has the same direction as $\mathbf{v}$. If c is negative, $c\mathbf{v}$ has the opposite direction.

The sum of two vectors can be represented geometrically by positioning the vectors (without changing their magnitudes or directions) so that the initial point of one coincides with the terminal point of the other, as shown in Figure 11.7. The vector $\mathbf{u} + \mathbf{v}$, called the **resultant vector,** is the diagonal of a parallelogram having $\mathbf{u}$ and $\mathbf{v}$ as its adjacent sides.



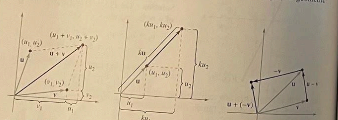
ISAAC WILLIAM ROWAN HAMILTON (1805–1865)

Some of the earliest work with vectors was done by the Irish mathematician William Rowan Hamilton. Hamilton spent many years developing a system of vector-like quantities called *quaternions*. Although Hamilton was convinced of the benefits of quaternions, the operations he defined did not produce good models for physical phenomena. It wasn't until the latter half of the nineteenth century that the Scottish physicist James Maxwell (1831–1879) restructured Hamilton's quaternions in a form useful for representing physical quantities such as force, velocity, and acceleration.



To find **u** + **v**, (1) move the initial point of **v** to the terminal point of **u**, or (2) move the initial point of **u** to the terminal point of **v**.
Figure 11.7

Figure 11.8 shows the equivalence of the geometric and algebraic definitions of vector addition and scalar multiplication, and presents (at far right) a geometric interpretation of $\mathbf{u} - \mathbf{v}$.



Vector addition
Figure 11.8
Scalar multiplication
Vector subtraction

EXAMPLE 3 Vector Operations

Given $\mathbf{v} = \langle -2, 5 \rangle$ and $\mathbf{w} = \langle 3, 4 \rangle$, find each of the vectors.

a. $\frac{1}{2}\mathbf{v}$ **b.** $\mathbf{w} - \mathbf{v}$ **c.** $\mathbf{v} + 2\mathbf{w}$

Solution

a. $\frac{1}{2}\mathbf{v} = \left\langle \frac{1}{2}(-2), \frac{1}{2}(5) \right\rangle = \left\langle -1, \frac{5}{2} \right\rangle$

b. $\mathbf{w} - \mathbf{v} = \langle w_1 - v_1, w_2 - v_2 \rangle = \langle 3 - (-2), 4 - 5 \rangle = \langle 5, -1 \rangle$

c. Using $2\mathbf{w} = \langle 6, 8 \rangle$, you have
$$\begin{aligned} \mathbf{v} + 2\mathbf{w} &= \langle -2, 5 \rangle + \langle 6, 8 \rangle \\ &= \langle -2 + 6, 5 + 8 \rangle \\ &= \langle 4, 13 \rangle. \end{aligned}$$

Vector addition and scalar multiplication share many properties of ordinary arithmetic, as shown in the following theorem.

THEOREM 11.1 Properties of Vector Operations

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in the plane, and let c and d be scalars.

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ — Commutative Property
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ — Associative Property
3. $\mathbf{u} + \mathbf{0} = \mathbf{u}$ — Additive Identity Property
4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ — Additive Inverse Property
5. $c(d\mathbf{u}) = (cd)\mathbf{u}$
6. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ — Distributive Property
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ — Distributive Property
8. $1(\mathbf{u}) = \mathbf{u}$, $0(\mathbf{u}) = \mathbf{0}$

Proof The proof of the *Associative Property* of vector addition uses the Associative Property of addition of real numbers.
$$\begin{aligned} (\mathbf{u} + \mathbf{v}) + \mathbf{w} &= [\langle u_1, u_2 \rangle + \langle v_1, v_2 \rangle] + \langle w_1, w_2 \rangle \\ &= \langle u_1 + v_1, u_2 + v_2 \rangle + \langle w_1, w_2 \rangle \\ &= \langle (u_1 + v_1) + w_1, (u_2 + v_2) + w_2 \rangle \\ &= \langle u_1 + (v_1 + w_1), u_2 + (v_2 + w_2) \rangle \\ &= \langle u_1, u_2 \rangle + \langle v_1 + w_1, v_2 + w_2 \rangle = \mathbf{u} + (\mathbf{v} + \mathbf{w}) \end{aligned}$$
Similarly, the proof of the *Distributive Property* of vectors depends on the Distributive Property of real numbers.
$$\begin{aligned} (c + d)\mathbf{u} &= (c + d)\langle u_1, u_2 \rangle \\ &= \langle (c + d)u_1, (c + d)u_2 \rangle \\ &= \langle cu_1 + du_1, cu_2 + du_2 \rangle \\ &= \langle cu_1, cu_2 \rangle + \langle du_1, du_2 \rangle = c\mathbf{u} + d\mathbf{u} \end{aligned}$$
The other properties can be proved in a similar manner.

EMMY NOETHER (1882–1935)

One person who contributed to our knowledge of axiomatic systems was the German mathematician Emmy Noether. Noether is generally recognized as the leading woman mathematician in recent history.

FOR FURTHER INFORMATION For more information on Emmy Noether, see the article "Emmy Noether, Greatest Woman Mathematician" by Clark Kimberling in *The Mathematics Teacher*. To view this article, go to the website *www.matharticles.com*.

Any set of vectors (with an accompanying set of scalars) that satisfies the eight properties given in Theorem 11.1 is a **vector space.*** The eight properties are the *vector space axioms*. So, this theorem states that the set of vectors in the plane (with the set of real numbers) forms a vector space.

THEOREM 11.2 Length of a Scalar Multiple

Let $\mathbf{v}$ be a vector and let c be a scalar. Then
$$\|c\mathbf{v}\| = |c|\,\|\mathbf{v}\|. \qquad |c| \text{ is the absolute value of } c.$$

Proof Because $c\mathbf{v} = \langle cv_1, cv_2 \rangle$, it follows that
$$\begin{aligned} \|c\mathbf{v}\| = \|\langle cv_1, cv_2 \rangle\| &= \sqrt{(cv_1)^2 + (cv_2)^2} \\ &= \sqrt{c^2v_1^2 + c^2v_2^2} \\ &= \sqrt{c^2(v_1^2 + v_2^2)} \\ &= |c|\sqrt{v_1^2 + v_2^2} \\ &= |c|\,\|\mathbf{v}\|. \end{aligned}$$

In many applications of vectors, it is useful to find a unit vector that has the same direction as a given vector. The following theorem gives a procedure for doing this.

THEOREM 11.3 Unit Vector in the Direction of v

If $\mathbf{v}$ is a nonzero vector in the plane, then the vector
$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\|\mathbf{v}\|}\mathbf{v}$$
has length 1 and the same direction as $\mathbf{v}$.

Proof Because $1/\|\mathbf{v}\|$ is positive and $\mathbf{u} = (1/\|\mathbf{v}\|)\mathbf{v}$, you can conclude that $\mathbf{u}$ has the same direction as $\mathbf{v}$. To see that $\|\mathbf{u}\| = 1$, note that
$$\begin{aligned} \|\mathbf{u}\| &= \left\|\left(\frac{1}{\|\mathbf{v}\|}\right)\mathbf{v}\right\| \\ &= \left|\frac{1}{\|\mathbf{v}\|}\right|\,\|\mathbf{v}\| \\ &= \frac{1}{\|\mathbf{v}\|}\,\|\mathbf{v}\| \\ &= 1. \end{aligned}$$
So, $\mathbf{u}$ has length 1 and the same direction as $\mathbf{v}$.

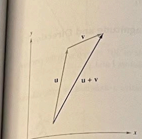
In Theorem 11.3, $\mathbf{u}$ is called a **unit vector in the direction of v.** The process of multiplying $\mathbf{v}$ by $1/\|\mathbf{v}\|$ to get a unit vector is called **normalization of v.**

* *For more information about vector spaces, see* Elementary Linear Algebra, *Fifth Edition, by Larson, Edwards, and Falvo (Boston: Houghton Mifflin Company, 2004).*

EXAMPLE 4 Finding a Unit Vector

Find a unit vector in the direction of $\mathbf{v} = \langle -2, 5 \rangle$ and verify that it has length 1.

Solution From Theorem 11.3, the unit vector in the direction of $\mathbf{v}$ is
$$\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\langle -2, 5 \rangle}{\sqrt{(-2)^2 + (5)^2}} = \frac{1}{\sqrt{29}}\langle -2, 5 \rangle = \left\langle \frac{-2}{\sqrt{29}}, \frac{5}{\sqrt{29}} \right\rangle.$$
This vector has length 1, because
$$\sqrt{\left(\frac{-2}{\sqrt{29}}\right)^2 + \left(\frac{5}{\sqrt{29}}\right)^2} = \sqrt{\frac{4}{29} + \frac{25}{29}} = \sqrt{\frac{29}{29}} = 1.$$

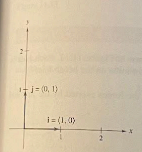


Triangle inequality
Figure 11.9

Generally, the length of the sum of two vectors is not equal to the sum of their lengths. To see this, consider the vectors $\mathbf{u}$ and $\mathbf{v}$ as shown in Figure 11.9. By considering $\mathbf{u}$ and $\mathbf{v}$ as two sides of a triangle, you can see that the length of the third side is $\|\mathbf{u} + \mathbf{v}\|$, and you have
$$\|\mathbf{u} + \mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|.$$
Equality occurs only if the vectors $\mathbf{u}$ and $\mathbf{v}$ have the *same direction*. This result is called the **triangle inequality** for vectors. (You are asked to prove this in Exercise 89, Section 11.3.)

Standard Unit Vectors

The unit vectors $\langle 1, 0 \rangle$ and $\langle 0, 1 \rangle$ are called the **standard unit vectors** in the plane and are denoted by
$$\mathbf{i} = \langle 1, 0 \rangle \quad \text{and} \quad \mathbf{j} = \langle 0, 1 \rangle \qquad \text{Standard unit vectors}$$
as shown in Figure 11.10. These vectors can be used to represent any vector uniquely, as follows.
$$\mathbf{v} = \langle v_1, v_2 \rangle = \langle v_1, 0 \rangle + \langle 0, v_2 \rangle = v_1\langle 1, 0 \rangle + v_2\langle 0, 1 \rangle = v_1\mathbf{i} + v_2\mathbf{j}$$
The vector $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j}$ is called a **linear combination** of $\mathbf{i}$ and $\mathbf{j}$. The scalars v_1 and v_2 are called the **horizontal** and **vertical components of v.**



Standard unit vectors **i** and **j**
Figure 11.10

EXAMPLE 5 Writing a Linear Combination of Unit Vectors

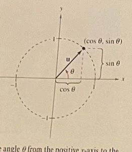
Let $\mathbf{u}$ be the vector with initial point $(2, -5)$ and terminal point $(-1, 3)$, and let $\mathbf{v} = 2\mathbf{i} - \mathbf{j}$. Write each vector as a linear combination of $\mathbf{i}$ and $\mathbf{j}$.

a. $\mathbf{u}$ **b.** $\mathbf{w} = 2\mathbf{u} - 3\mathbf{v}$

Solution

a. $\begin{aligned}[t] \mathbf{u} &= \langle q_1 - p_1, q_2 - p_2 \rangle \\ &= \langle -1 - 2, 3 - (-5) \rangle \\ &= \langle -3, 8 \rangle = -3\mathbf{i} + 8\mathbf{j} \end{aligned}$

b. $\begin{aligned}[t] \mathbf{w} = 2\mathbf{u} - 3\mathbf{v} &= 2(-3\mathbf{i} + 8\mathbf{j}) - 3(2\mathbf{i} - \mathbf{j}) \\ &= -6\mathbf{i} + 16\mathbf{j} - 6\mathbf{i} + 3\mathbf{j} \\ &= -12\mathbf{i} + 19\mathbf{j} \end{aligned}$



The angle θ from the positive x-axis to the vector **u**
Figure 11.11

If **u** is a unit vector and θ is the angle (measured counterclockwise) from the positive x-axis to **u**, then the terminal point of **u** lies on the unit circle, and you have

$$\mathbf{u} = \langle \cos\theta, \sin\theta \rangle = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j} \qquad \text{Unit vector}$$

as shown in Figure 11.11. Moreover, it follows that any other nonzero vector **v** making an angle θ with the positive x-axis has the same direction as **u**, and you can write

$$\mathbf{v} = \|\mathbf{v}\|\langle \cos\theta, \sin\theta \rangle = \|\mathbf{v}\|\cos\theta\,\mathbf{i} + \|\mathbf{v}\|\sin\theta\,\mathbf{j}.$$

EXAMPLE 6 Writing a Vector of Given Magnitude and Direction

The vector **v** has a magnitude of 3 and makes an angle of $30° = \pi/6$ with the positive x-axis. Write **v** as a linear combination of the unit vectors **i** and **j**.

Solution Because the angle between **v** and the positive x-axis is $\theta = \pi/6$, you can write the following.

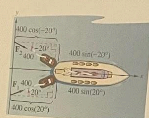
$$\begin{aligned}\mathbf{v} &= \|\mathbf{v}\|\cos\theta\,\mathbf{i} + \|\mathbf{v}\|\sin\theta\,\mathbf{j}\\ &= 3\cos\frac{\pi}{6}\mathbf{i} + 3\sin\frac{\pi}{6}\mathbf{j}\\ &= \frac{3\sqrt{3}}{2}\mathbf{i} + \frac{3}{2}\mathbf{j}\end{aligned}$$

Applications of Vectors

Vectors have many applications in physics and engineering. One example is force. A vector can be used to represent force because force has both magnitude and direction. If two or more forces are acting on an object, then the **resultant force** on the object is the vector sum of the vector forces.

EXAMPLE 7 Finding the Resultant Force

Two tugboats are pushing an ocean liner, as shown in Figure 11.12. Each boat is exerting a force of 400 pounds. What is the resultant force on the ocean liner?



The resultant force on the ocean liner that is exerted by the two tugboats.
Figure 11.12

Solution Using Figure 11.12, you can represent the forces exerted by the first and second tugboats as

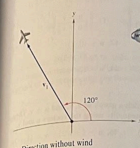
$$\begin{aligned}\mathbf{F}_1 &= 400\langle \cos 20°, \sin 20° \rangle\\ &= 400\cos(20°)\mathbf{i} + 400\sin(20°)\mathbf{j}\\ \mathbf{F}_2 &= 400\langle \cos(-20°), \sin(-20°) \rangle\\ &= 400\cos(20°)\mathbf{i} - 400\sin(20°)\mathbf{j}.\end{aligned}$$

The resultant force on the ocean liner is

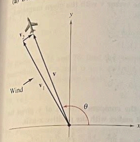
$$\begin{aligned}\mathbf{F} &= \mathbf{F}_1 + \mathbf{F}_2\\ &= [400\cos(20°)\mathbf{i} + 400\sin(20°)\mathbf{j}] + [400\cos(20°)\mathbf{i} - 400\sin(20°)\mathbf{j}]\\ &= 800\cos(20°)\mathbf{i}\\ &\approx 752\mathbf{i}.\end{aligned}$$

So, the resultant force on the ocean liner is approximately 752 pounds in the direction of the positive x-axis.

In surveying and navigation, a bearing is a direction that measures the acute angle that a path or line of sight makes with a fixed north-south line. In air navigation, bearings are measured in degrees clockwise from north.



(a) Direction without wind



(b) Direction with wind
Figure 11.13

EXAMPLE 8 Finding a Velocity

An airplane is traveling at a fixed altitude with a negligible wind factor. The airplane is traveling at a speed of 500 miles per hour with a bearing of 330°, as shown in Figure 11.13(a). As the airplane reaches a certain point, it encounters wind with a velocity of 70 miles per hour in the direction N 45° E (45° east of north), as shown in Figure 11.13(b). What are the resultant speed and direction of the airplane?

Solution Using Figure 11.13(a), represent the velocity of the airplane (alone) as

$$\mathbf{v}_1 = 500\cos(120°)\mathbf{i} + 500\sin(120°)\mathbf{j}.$$

The velocity of the wind is represented by the vector

$$\mathbf{v}_2 = 70\cos(45°)\mathbf{i} + 70\sin(45°)\mathbf{j}.$$

The resultant velocity of the airplane (in the wind) is

$$\begin{aligned}\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 &= 500\cos(120°)\mathbf{i} + 500\sin(120°)\mathbf{j} + 70\cos(45°)\mathbf{i} + 70\sin(45°)\mathbf{j}\\ &\approx -200.5\mathbf{i} + 482.5\mathbf{j}.\end{aligned}$$

To find the resultant speed and direction, write $\mathbf{v} = \|\mathbf{v}\|(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j})$. Because $\|\mathbf{v}\| \approx \sqrt{(-200.5)^2 + (482.5)^2} \approx 522.5$, you can write

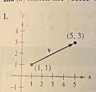
$$\mathbf{v} \approx 522.5\left(\frac{-200.5}{522.5}\mathbf{i} + \frac{482.5}{522.5}\mathbf{j}\right) \approx 522.5[\cos(112.6°)\mathbf{i} + \sin(112.6°)\mathbf{j}].$$

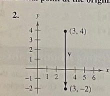
The new speed of the airplane, as altered by the wind, is approximately 522.5 miles per hour in a path that makes an angle of 112.6° with the positive x-axis.

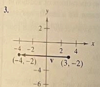
Exercises for Section 11.1

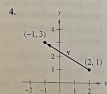
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, (a) find the component form of the vector v and (b) sketch the vector with its initial point at the origin.

1. 

2. 

3. 

4. 

In Exercises 5–8, find the vectors u and v whose initial and terminal points are given. Show that u and v are equivalent.

5. **u**: (3, 2), (5, 6)
v: (−1, 4), (1, 8)
6. **u**: (−4, 0), (1, 8)
v: (2, −1), (7, 7)
7. **u**: (0, 3), (6, −2)
v: (3, 10), (9, 5)
8. **u**: (−4, −1), (11, −4)
v: (10, 13), (25, 10)

In Exercises 9–16, the initial and terminal points of a vector v are given. (a) Sketch the given directed line segment, (b) write the vector in component form, and (c) sketch the vector with its initial point at the origin.

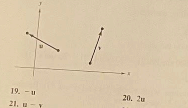
	Initial Point	*Terminal Point*
9.	(1, 2)	(5, 5)
10.	(2, −6)	(3, 6)
11.	(10, 2)	(6, −1)
12.	(0, −4)	(−5, −1)
13.	(6, 2)	(6, 6)
14.	(7, −1)	(−3, −1)
15.	$(\frac{3}{2}, \frac{4}{3})$	$(\frac{1}{2}, 3)$
16.	(0.12, 0.60)	(0.84, 1.25)

indicates that in the HM mathSpace® CD-ROM and the online Eduspace® system for this text, you will find an Open Exploration, which further explores this example using the computer algebra systems Maple, Mathcad, Mathematica, and Derive.

In Exercises 17 and 18, sketch each scalar multiple of v.

17. $\mathbf{v} = \langle 2, 3 \rangle$
(a) $2\mathbf{v}$ (b) $-3\mathbf{v}$ (c) $\frac{7}{2}\mathbf{v}$ (d) $\frac{2}{3}\mathbf{v}$
18. $\mathbf{v} = \langle -1, 5 \rangle$
(a) $4\mathbf{v}$ (b) $-\frac{1}{2}\mathbf{v}$ (c) $0\mathbf{v}$ (d) $-6\mathbf{v}$

In Exercises 19–22, use the figure to sketch a graph of the vector. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.



19. $-\mathbf{u}$
20. $2\mathbf{u}$
21. $\mathbf{u} - \mathbf{v}$
22. $\mathbf{u} + 2\mathbf{v}$

In Exercises 23 and 24, find (a) $\frac{2}{3}\mathbf{u}$, (b) $\mathbf{v} - \mathbf{u}$, and (c) $2\mathbf{u} + 5\mathbf{v}$.

23. $\mathbf{u} = \langle 4, 9 \rangle$
$\mathbf{v} = \langle 2, -5 \rangle$
24. $\mathbf{u} = \langle -3, -8 \rangle$
$\mathbf{v} = \langle 8, 25 \rangle$

In Exercises 25–28, find the vector v where $\mathbf{u} = \langle 2, -1 \rangle$ and $\mathbf{w} = \langle 1, 2 \rangle$. Illustrate the vector operations geometrically.

25. $\mathbf{v} = \frac{3}{2}\mathbf{u}$
26. $\mathbf{v} = \mathbf{u} + \mathbf{w}$
27. $\mathbf{v} = \mathbf{u} + 2\mathbf{w}$
28. $\mathbf{v} = 5\mathbf{u} - 3\mathbf{w}$

In Exercises 29 and 30, the vector v and its initial point are given. Find the terminal point.

29. $\mathbf{v} = \langle -1, 3 \rangle$; Initial point: (4, 2)
30. $\mathbf{v} = \langle 4, -9 \rangle$; Initial point: (3, 2)

In Exercises 31–36, find the magnitude of v.

31. $\mathbf{v} = \langle 4, 3 \rangle$
32. $\mathbf{v} = \langle 12, -5 \rangle$
33. $\mathbf{v} = 6\mathbf{i} - 5\mathbf{j}$
34. $\mathbf{v} = -10\mathbf{i} + 3\mathbf{j}$
35. $\mathbf{v} = 4\mathbf{j}$
36. $\mathbf{v} = \mathbf{i} - \mathbf{j}$

In Exercises 37–40, find the unit vector in the direction of u and verify that it has length 1.

37. $\mathbf{u} = \langle 3, 12 \rangle$
38. $\mathbf{u} = \langle 5, 15 \rangle$
39. $\mathbf{u} = \left\langle \frac{3}{2}, \frac{5}{2} \right\rangle$
40. $\mathbf{u} = \langle -6.2, 3.4 \rangle$

In Exercises 41–44, find the following.

(a) $\|\mathbf{u}\|$ (b) $\|\mathbf{v}\|$ (c) $\|\mathbf{u} + \mathbf{v}\|$
(d) $\left\|\frac{\mathbf{u}}{\|\mathbf{u}\|}\right\|$ (e) $\left\|\frac{\mathbf{v}}{\|\mathbf{v}\|}\right\|$ (f) $\left\|\frac{\mathbf{u} + \mathbf{v}}{\|\mathbf{u} + \mathbf{v}\|}\right\|$

41. $\mathbf{u} = \langle 1, -1 \rangle$
$\mathbf{v} = \langle -1, 2 \rangle$
42. $\mathbf{u} = \langle 0, 1 \rangle$
$\mathbf{v} = \langle 3, -3 \rangle$
43. $\mathbf{u} = \left\langle 1, \frac{1}{2} \right\rangle$
$\mathbf{v} = \langle 2, 3 \rangle$
44. $\mathbf{u} = \langle 2, -4 \rangle$
$\mathbf{v} = \langle 5, 5 \rangle$

In Exercises 45 and 46, sketch a graph of u, v, and u + v. Then demonstrate the triangle inequality using the vectors u and v.

45. $\mathbf{u} = \langle 2, 1 \rangle$, $\mathbf{v} = \langle 5, 4 \rangle$
46. $\mathbf{u} = \langle -3, 2 \rangle$, $\mathbf{v} = \langle 1, -2 \rangle$

In Exercises 47–50, find the vector v with the given magnitude and the same direction as u.

	Magnitude	*Direction*
47.	$\|\mathbf{v}\| = 4$	$\mathbf{u} = \langle 1, 1 \rangle$
48.	$\|\mathbf{v}\| = 4$	$\mathbf{u} = \langle -1, 1 \rangle$
49.	$\|\mathbf{v}\| = 2$	$\mathbf{u} = \langle \sqrt{3}, 3 \rangle$
50.	$\|\mathbf{v}\| = 3$	$\mathbf{u} = \langle 0, 3 \rangle$

In Exercises 51–54, find the component form of v given its magnitude and the angle it makes with the positive x-axis.

51. $\|\mathbf{v}\| = 3$, $\theta = 0°$
52. $\|\mathbf{v}\| = 5$, $\theta = 120°$
53. $\|\mathbf{v}\| = 2$, $\theta = 150°$
54. $\|\mathbf{v}\| = 1$, $\theta = 3.5°$

In Exercises 55–58, find the component form of u + v given the lengths of u and v and the angles that u and v make with the positive x-axis.

55. $\|\mathbf{u}\| = 1$, $\theta_u = 0°$
$\|\mathbf{v}\| = 3$, $\theta_v = 45°$
56. $\|\mathbf{u}\| = 4$, $\theta_u = 0°$
$\|\mathbf{v}\| = 2$, $\theta_v = 60°$
57. $\|\mathbf{u}\| = 2$, $\theta_u = 4$
$\|\mathbf{v}\| = 1$, $\theta_v = 2$
58. $\|\mathbf{u}\| = 5$, $\theta_u = -0.5$
$\|\mathbf{v}\| = 5$, $\theta_v = 0.5$

Writing About Concepts

59. In your own words, state the difference between a scalar and a vector. Give examples of each.
60. Give geometric descriptions of the operations of addition of vectors and multiplication of a vector by a scalar.
61. Identify the quantity as a scalar or as a vector. Explain your reasoning.
(a) The muzzle velocity of a gun
(b) The price of a company's stock
62. Identify the quantity as a scalar or as a vector. Explain your reasoning.
(a) The air temperature in a room
(b) The weight of a car

In Exercises 63–68, find a and b such that $\mathbf{v} = a\mathbf{u} + b\mathbf{w}$, where $\mathbf{u} = \langle 1, 2 \rangle$ and $\mathbf{w} = \langle 1, -1 \rangle$.

63. $\mathbf{v} = \langle 2, 1 \rangle$
64. $\mathbf{v} = \langle 0, 3 \rangle$
65. $\mathbf{v} = \langle 3, 0 \rangle$
66. $\mathbf{v} = \langle 3, 3 \rangle$
67. $\mathbf{v} = \langle 1, 1 \rangle$
68. $\mathbf{v} = \langle -1, 7 \rangle$

In Exercises 69–74, find a unit vector (a) parallel to and (b) normal to the graph of $f(x)$ at the given point. Then sketch a graph of the vectors and the function.

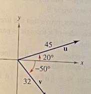
	Function	*Point*
69.	$f(x) = x^2$	(3, 9)
70.	$f(x) = -x^2 + 5$	(1, 4)
71.	$f(x) = x^3$	(1, 1)
72.	$f(x) = x^3$	(−2, −8)
73.	$f(x) = \sqrt{25 - x^2}$	(3, 4)
74.	$f(x) = \tan x$	$\left(\frac{\pi}{4}, 1\right)$

In Exercises 75 and 76, find the component form of v given the magnitudes of u and u + v and the angles that u and u + v make with the positive x-axis.

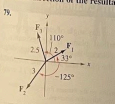
75. $\|\mathbf{u}\| = 1$, $\theta = 45°$
$\|\mathbf{u} + \mathbf{v}\| = \sqrt{2}$, $\theta = 90°$
76. $\|\mathbf{u}\| = 4$, $\theta = 30°$
$\|\mathbf{u} + \mathbf{v}\| = 6$, $\theta = 120°$

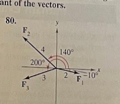
77. ***Programming*** You are given the magnitudes of **u** and **v** and the angles **u** and **v** make with the positive x-axis. Write a program for a graphing utility in which the output is the following.
(a) $\mathbf{u} + \mathbf{v}$ (b) $\|\mathbf{u} + \mathbf{v}\|$
(c) The angle $\mathbf{u} + \mathbf{v}$ makes with the positive x-axis

78. ***Programming*** Use the program you wrote in Exercise 77 to find the magnitude and direction of the resultant of the vectors shown.



In Exercises 79 and 80, use a graphing utility to find the magnitude and direction of the resultant of the vectors.

79. 

80. 

81. ***Numerical and Graphical Analysis*** Forces with magnitudes of 180 newtons and 275 newtons act on a hook (see figure). The angle between the two forces is θ degrees.
(a) If $\theta = 30°$, find the direction and magnitude of the resultant force.
(b) Write the magnitude M and direction α of the resultant force as functions of θ, where $0° \le \theta \le 180°$.
(c) Use a graphing utility to complete the table.

θ	0°	30°	60°	90°	120°	150°	180°
M							
α							

(d) Use a graphing utility to graph the two functions M and α.
(e) Explain why one of the functions decreases for increasing values of θ whereas the other does not.

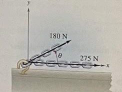


Figure for 81

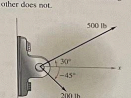


Figure for 82

82. ***Resultant Force*** Forces with magnitudes of 500 pounds and 200 pounds act on a machine part at angles of 30° and −45°, respectively, with the x-axis (see figure). Find the direction and magnitude of the resultant force.

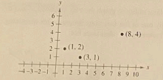
83. ***Resultant Force*** Three forces with magnitudes of 75 pounds, 100 pounds, and 125 pounds act on an object at angles of 30°, 45°, and 120°, respectively, with the positive x-axis. Find the direction and magnitude of the resultant force.

84. ***Resultant Force*** Three forces with magnitudes of 400 newtons, 280 newtons, and 350 newtons act on an object at angles of −30°, 45°, and 135°, respectively, with the positive x-axis. Find the direction and magnitude of the resultant force.

85. ***Think About It*** Consider two forces of equal magnitude acting on a point.
(a) If the magnitude of the resultant is the sum of the magnitudes of the two forces, make a conjecture about the angle between the forces.
(b) If the resultant of the forces is **0**, make a conjecture about the angle between the forces.
(c) Can the magnitude of the resultant be greater than the sum of the magnitudes of the two forces? Explain.

86. ***Graphical Reasoning*** Consider two forces $\mathbf{F}_1 = \langle 20, 0\rangle$ and $\mathbf{F}_2 = 10\langle \cos\theta, \sin\theta\rangle$.
(a) Find $\|\mathbf{F}_1 + \mathbf{F}_2\|$.
(b) Determine the magnitude of the resultant as a function of θ. Use a graphing utility to graph the function for $0 \le \theta < 2\pi$.
(c) Use the graph in part (b) to determine the range of the function. What is its maximum and for what value of θ does it occur? What is its minimum and for what value of θ does it occur?
(d) Explain why the magnitude of the resultant is never 0.

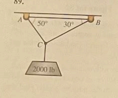
87. Three vertices of a parallelogram are $(1, 2)$, $(3, 1)$, and $(8, 4)$. Find the three possible fourth vertices (see figure).



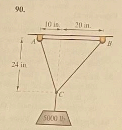
88. Use vectors to find the points of trisection of the line segment with endpoints $(1, 2)$ and $(7, 5)$.

Cable Tension **In Exercises 89 and 90, use the figure to determine the tension in each cable supporting the given load.**

89.



90.



91. ***Projectile Motion*** A gun with a muzzle velocity of 1200 feet per second is fired at an angle of 6° above the horizontal. Find the vertical and horizontal components of the velocity.

92. ***Shared Load*** To carry a 100-pound cylindrical weight, two workers lift on the ends of short ropes tied to an eyelet on the top center of the cylinder. One rope makes a 20° angle away from the vertical and the other makes a 30° angle (see figure).
(a) Find each rope's tension if the resultant force is vertical.
(b) Find the vertical component of each worker's force.

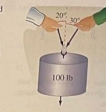


Figure for 92

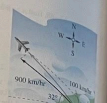


Figure for 93

93. ***Navigation*** A plane is flying in the direction 302°. Its speed with respect to the air is 900 kilometers per hour. The wind at the plane's altitude is from the southwest at 100 kilometers per hour (see figure). What is the true direction of the plane, and what is its speed with respect to the ground?

94. ***Navigation*** A plane flies at a constant groundspeed of 400 miles per hour due east and encounters a 50-mile-per-hour wind from the northwest. Find the airspeed and compass direction that will allow the plane to maintain its groundspeed and eastward direction.

True or False? **In Exercises 95–100, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

95. If $\mathbf{u}$ and $\mathbf{v}$ have the same magnitude and direction, then $\mathbf{u}$ and $\mathbf{v}$ are equivalent.

96. If $\mathbf{u}$ is a unit vector in the direction of $\mathbf{v}$, then $\mathbf{v} = \|\mathbf{v}\|\,\mathbf{u}$.

97. If $\mathbf{u} = a\mathbf{i} + b\mathbf{j}$ is a unit vector, then $a^2 + b^2 = 1$.

98. If $\mathbf{v} = a\mathbf{i} + b\mathbf{j} = \mathbf{0}$, then $a = -b$.

99. If $a = b$, then $\|a\mathbf{i} + b\mathbf{j}\| = \sqrt{2}a$.

100. If $\mathbf{u}$ and $\mathbf{v}$ have the same magnitude but opposite directions, then $\mathbf{u} + \mathbf{v} = \mathbf{0}$.

101. Prove that $\mathbf{u} = (\cos\theta)\mathbf{i} - (\sin\theta)\mathbf{j}$ and $\mathbf{v} = (\sin\theta)\mathbf{i} + (\cos\theta)\mathbf{j}$ are unit vectors for any angle θ.

102. ***Geometry*** Using vectors, prove that the line segment joining the midpoints of two sides of a triangle is parallel to, and one-half the length of, the third side.

103. ***Geometry*** Using vectors, prove that the diagonals of a parallelogram bisect each other.

104. Prove that the vector $\mathbf{w} = \|\mathbf{u}\|\mathbf{v} + \|\mathbf{v}\|\mathbf{u}$ bisects the angle between $\mathbf{u}$ and $\mathbf{v}$.

105. Consider the vector $\mathbf{u} = \langle x, y\rangle$. Describe the set of all points (x, y) such that $\|\mathbf{u}\| = 5$.

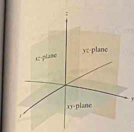
Putnam Exam Challenge

106. A coast artillery gun can fire at any angle of elevation between 0° and 90° in a fixed vertical plane. If air resistance is neglected and the muzzle velocity is constant ($= v_0$), determine the set H of points in the plane and above the horizontal which can be hit.

This problem was composed by the Committee on the Putnam Prize Competition.

Section 11.2 Space Coordinates and Vectors in Space

- **Understand the three-dimensional rectangular coordinate system.**
- **Analyze vectors in space.**
- **Use three-dimensional vectors to solve real-life problems.**



The three-dimensional coordinate system
Figure 11.14

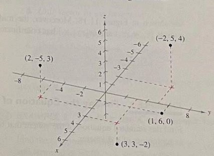
Coordinates in Space

Up to this point in the text, you have been primarily concerned with the two-dimensional coordinate system. Much of the remaining part of your study of calculus will involve the three-dimensional coordinate system.

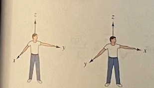
Before extending the concept of a vector to three dimensions, you must be able to identify points in the **three-dimensional coordinate system.** You can construct this system by passing a z-axis perpendicular to both the x- and y-axes at the origin. Figure 11.14 shows the positive portion of each coordinate axis. Taken as pairs, the axes determine three **coordinate planes:** the ***xy*-plane,** the ***xz*-plane,** and the ***yz*-plane.** These three coordinate planes separate three-space into eight **octants.** The first octant is the one for which all three coordinates are positive. In this three-dimensional system, a point P in space is determined by an ordered triple (x, y, z) where x, y, and z are as follows.

x = directed distance from yz-plane to P
y = directed distance from xz-plane to P
z = directed distance from xy-plane to P

Several points are shown in Figure 11.15.



Points in the three-dimensional coordinate system are represented by ordered triples.
Figure 11.15

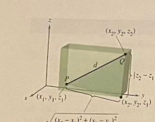


Right-handed system
Figure 11.16

Left-handed system

A three-dimensional coordinate system can have either a **left-handed** or a **right-handed** orientation. To determine the orientation of a system, imagine that you are standing at the origin, with your arms pointing in the direction of the positive x- and y-axes, and with the z-axis pointing up, as shown in Figure 11.16. The system is right-handed or left-handed depending on which hand points along the x-axis. In this text, you will work exclusively with the right-handed system.

NOTE The three-dimensional rotatable graphs that are available in the *HM mathSpace®* CD-ROM and the online *Eduspace®* system for this text will help you visualize points or objects in a three-dimensional coordinate system.



The distance between two points in space
Figure 11.17

Many of the formulas established for the two-dimensional coordinate system can be extended to three dimensions. For example, to find the distance between two points in space, you can use the Pythagorean Theorem twice, as shown in Figure 11.17. By doing this, you will obtain the formula for the distance between the points (x_1, y_1, z_1) and (x_2, y_2, z_2).

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Distance Formula

EXAMPLE 1 Finding the Distance Between Two Points in Space

The distance between the points $(2, -1, 3)$ and $(1, 0, -2)$ is

$$\begin{aligned} d &= \sqrt{(1-2)^2 + (0+1)^2 + (-2-3)^2} \\ &= \sqrt{1+1+25} \\ &= \sqrt{27} \\ &= 3\sqrt{3}. \end{aligned}$$

Distance Formula

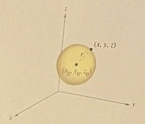


Figure 11.18

A **sphere** with center at (x_0, y_0, z_0) and radius r is defined to be the set of all points (x, y, z) such that the distance between (x, y, z) and (x_0, y_0, z_0) is r. You can use the Distance Formula to find the **standard equation of a sphere** of radius r, centered at (x_0, y_0, z_0). If (x, y, z) is an arbitrary point on the sphere, the equation of the sphere is

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$$

Equation of sphere

as shown in Figure 11.18. Moreover, the midpoint of the line segment joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2) has coordinates

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$

Midpoint Rule

EXAMPLE 2 Finding the Equation of a Sphere

Find the standard equation of the sphere that has the points $(5, -2, 3)$ and $(0, 4, -3)$ as endpoints of a diameter.

Solution By the Midpoint Rule, the center of the sphere is

$$\left(\frac{5+0}{2}, \frac{-2+4}{2}, \frac{3-3}{2}\right) = \left(\frac{5}{2}, 1, 0\right).$$

Midpoint Rule

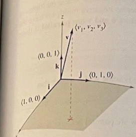
By the Distance Formula, the radius is

$$r = \sqrt{\left(0 - \frac{5}{2}\right)^2 + (4-1)^2 + (-3-0)^2} = \sqrt{\frac{97}{4}} = \frac{\sqrt{97}}{2}.$$

Therefore, the standard equation of the sphere is

$$\left(x - \frac{5}{2}\right)^2 + (y - 1)^2 + z^2 = \frac{97}{4}.$$

Equation of sphere



The standard unit vectors in space
Figure 11.19

Vectors in Space

In space, vectors are denoted by ordered triples $\mathbf{v} = \langle v_1, v_2, v_3\rangle$. The **zero vector** is denoted by $\mathbf{0} = \langle 0, 0, 0\rangle$. Using the unit vectors $\mathbf{i} = \langle 1, 0, 0\rangle$, $\mathbf{j} = \langle 0, 1, 0\rangle$, and $\mathbf{k} = \langle 0, 0, 1\rangle$ in the direction of the positive z-axis, the **standard unit vector notation** for $\mathbf{v}$ is

$$\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$$

as shown in Figure 11.19. If $\mathbf{v}$ is represented by the directed line segment from $P(p_1, p_2, p_3)$ to $Q(q_1, q_2, q_3)$, as shown in Figure 11.20, the component form of $\mathbf{v}$ is given by subtracting the coordinates of the initial point from the coordinates of the terminal point, as follows.

$$\mathbf{v} = \langle v_1, v_2, v_3\rangle = \langle q_1 - p_1, q_2 - p_2, q_3 - p_3\rangle$$

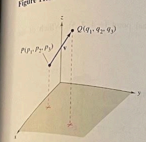


Figure 11.20

Vectors in Space

Let $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ be vectors in space and let c be a scalar.

1. *Equality of Vectors:* $\mathbf{u} = \mathbf{v}$ if and only if $u_1 = v_1$, $u_2 = v_2$, and $u_3 = v_3$.
2. *Component Form:* If $\mathbf{v}$ is represented by the directed line segment from $P(p_1, p_2, p_3)$ to $Q(q_1, q_2, q_3)$, then
$$\mathbf{v} = \langle v_1, v_2, v_3\rangle = \langle q_1 - p_1, q_2 - p_2, q_3 - p_3\rangle.$$
3. *Length:* $\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$
4. *Unit Vector in the Direction of* $\mathbf{v}$: $\dfrac{\mathbf{v}}{\|\mathbf{v}\|} = \left(\dfrac{1}{\|\mathbf{v}\|}\right)\langle v_1, v_2, v_3\rangle, \quad \mathbf{v} \neq \mathbf{0}$
5. *Vector Addition:* $\mathbf{v} + \mathbf{u} = \langle v_1 + u_1, v_2 + u_2, v_3 + u_3\rangle$
6. *Scalar Multiplication:* $c\mathbf{v} = \langle cv_1, cv_2, cv_3\rangle$

NOTE The properties of vector addition and scalar multiplication given in Theorem 11.1 are also valid for vectors in space.

EXAMPLE 3 Finding the Component Form of a Vector in Space

Find the component form and magnitude of the vector $\mathbf{v}$ having initial point $(-2, 3, 1)$ and terminal point $(0, -4, 4)$. Then find a unit vector in the direction of $\mathbf{v}$.

Solution The component form of $\mathbf{v}$ is

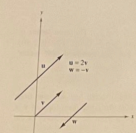
$$\begin{aligned} \mathbf{v} = \langle q_1 - p_1, q_2 - p_2, q_3 - p_3\rangle &= \langle 0 - (-2), -4 - 3, 4 - 1\rangle \\ &= \langle 2, -7, 3\rangle \end{aligned}$$

which implies that its magnitude is

$$\|\mathbf{v}\| = \sqrt{2^2 + (-7)^2 + 3^2} = \sqrt{62}.$$

The unit vector in the direction of $\mathbf{v}$ is

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{62}}\langle 2, -7, 3\rangle.$$



Parallel vectors
Figure 11.21

Recall from the definition of scalar multiplication that positive scalar multiples of a nonzero vector **v** have the same direction as **v**, whereas negative multiples have the direction opposite of **v**. In general, two nonzero vectors **u** and **v** are **parallel** if there is some scalar c such that $\mathbf{u} = c\mathbf{v}$.

Definition of Parallel Vectors

Two nonzero vectors **u** and **v** are **parallel** if there is some scalar c such that $\mathbf{u} = c\mathbf{v}$.

For example, in Figure 11.21, the vectors **u**, **v**, and **w** are parallel because $\mathbf{u} = 2\mathbf{v}$ and $\mathbf{w} = -\mathbf{v}$.

EXAMPLE 4 Parallel Vectors

Vector **w** has initial point $(2, -1, 3)$ and terminal point $(-4, 7, 5)$. Which of the following vectors is parallel to **w**?

a. $\mathbf{u} = \langle 3, -4, -1\rangle$

b. $\mathbf{v} = \langle 12, -16, 4\rangle$

Solution Begin by writing **w** in component form.

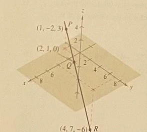
$$\mathbf{w} = \langle -4-2, 7-(-1), 5-3\rangle = \langle -6, 8, 2\rangle$$

a. Because $\mathbf{u} = \langle 3, -4, -1\rangle = -\frac{1}{2}\langle -6, 8, 2\rangle = -\frac{1}{2}\mathbf{w}$, you can conclude that **u** is parallel to **w**.

b. In this case, you want to find a scalar c such that

$$\begin{aligned}\langle 12, -16, 4\rangle &= c\langle -6, 8, 2\rangle.\\ 12 &= -6c \rightarrow c = -2\\ -16 &= 8c \rightarrow c = -2\\ 4 &= 2c \rightarrow c = 2\end{aligned}$$

Because there is no c for which the equation has a solution, the vectors are not parallel.



The points P, Q, and R lie on the same line.
Figure 11.22

EXAMPLE 5 Using Vectors to Determine Collinear Points

Determine whether the points $P(1, -2, 3)$, $Q(2, 1, 0)$, and $R(4, 7, -6)$ are collinear.

Solution The component forms of $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are

$$\overrightarrow{PQ} = \langle 2-1, 1-(-2), 0-3\rangle = \langle 1, 3, -3\rangle$$

and

$$\overrightarrow{PR} = \langle 4-1, 7-(-2), -6-3\rangle = \langle 3, 9, -9\rangle.$$

These two vectors have a common initial point. So, P, Q, and R lie on the same line if and only if $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are parallel—which they are because $\overrightarrow{PR} = 3\overrightarrow{PQ}$, as shown in Figure 11.22.

EXAMPLE 6 Standard Unit Vector Notation

a. Write the vector $\mathbf{v} = 4\mathbf{i} - 5\mathbf{k}$ in component form.

b. Find the terminal point of the vector $\mathbf{v} = 7\mathbf{i} - \mathbf{j} + 3\mathbf{k}$, given that the initial point is $P(-2, 3, 5)$.

Solution

a. Because **j** is missing, its component is 0 and

$$\mathbf{v} = 4\mathbf{i} - 5\mathbf{k} = \langle 4, 0, -5\rangle.$$

b. You need to find $Q(q_1, q_2, q_3)$ such that $\mathbf{v} = \overrightarrow{PQ} = 7\mathbf{i} - \mathbf{j} + 3\mathbf{k}$. This implies that $q_1 - (-2) = 7$, $q_2 - 3 = -1$, and $q_3 - 5 = 3$. The solution of these three equations is $q_1 = 5$, $q_2 = 2$, and $q_3 = 8$. Therefore, Q is $(5, 2, 8)$.

Application

EXAMPLE 7 Measuring Force

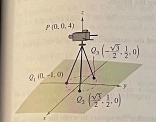


Figure 11.23

A television camera weighing 120 pounds is supported by a tripod, as shown in Figure 11.23. Represent the force exerted on each leg of the tripod as a vector.

Solution Let the vectors $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$ represent the forces exerted on the three legs. From Figure 11.23, you can determine the directions of $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$ to be as follows.

$$\begin{aligned}\overrightarrow{PQ}_1 &= \langle 0-0, -1-0, 0-4\rangle = \langle 0, -1, -4\rangle\\ \overrightarrow{PQ}_2 &= \left\langle \frac{\sqrt{3}}{2} - 0, \frac{1}{2} - 0, 0-4\right\rangle = \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2}, -4\right\rangle\\ \overrightarrow{PQ}_3 &= \left\langle -\frac{\sqrt{3}}{2} - 0, \frac{1}{2} - 0, 0-4\right\rangle = \left\langle -\frac{\sqrt{3}}{2}, \frac{1}{2}, -4\right\rangle\end{aligned}$$

Because each leg has the same length, and the total force is distributed equally among the three legs, you know that $\|\mathbf{F}_1\| = \|\mathbf{F}_2\| = \|\mathbf{F}_3\|$. So, there exists a constant c such that

$$\mathbf{F}_1 = c\langle 0, -1, -4\rangle, \quad \mathbf{F}_2 = c\left\langle \frac{\sqrt{3}}{2}, \frac{1}{2}, -4\right\rangle, \quad \text{and} \quad \mathbf{F}_3 = c\left\langle -\frac{\sqrt{3}}{2}, \frac{1}{2}, -4\right\rangle.$$

Let the total force exerted by the object be given by $\mathbf{F} = -120\mathbf{k}$. Then, using the fact that

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3$$

you can conclude that $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$ all have a vertical component of -40. This implies that $c(-4) = -40$ and $c = 10$. Therefore, the forces exerted on the legs can be represented by

$$\begin{aligned}\mathbf{F}_1 &= \langle 0, -10, -40\rangle\\ \mathbf{F}_2 &= \langle 5\sqrt{3}, 5, -40\rangle\\ \mathbf{F}_3 &= \langle -5\sqrt{3}, 5, -40\rangle.\end{aligned}$$

Exercises for Section 11.2

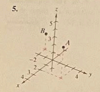
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, plot the points on the same three-dimensional coordinate system.

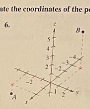
1. (a) $(2, 1, 3)$ (b) $(-1, 2, 1)$
2. (a) $(3, -2, 5)$ (b) $\left(\frac{3}{2}, 4, -2\right)$
3. (a) $(5, -2, 2)$ (b) $(5, -2, -2)$
4. (a) $(0, 4, -5)$ (b) $(4, 0, 5)$

In Exercises 5 and 6, approximate the coordinates of the points.

5.



6.



In Exercises 7–10, find the coordinates of the point.

7. The point is located three units behind the yz-plane, four units to the right of the xz-plane, and five units above the xy-plane.
8. The point is located seven units in front of the yz-plane, two units to the left of the xz-plane, and one unit below the xy-plane.
9. The point is located on the x-axis, 10 units in front of the yz-plane.
10. The point is located in the yz-plane, three units to the right of the xz-plane, and two units above the xy-plane.
11. *Think About It* What is the z-coordinate of any point in the xy-plane?
12. *Think About It* What is the x-coordinate of any point in the yz-plane?

In Exercises 13–24, determine the location of a point (x, y, z) that satisfies the condition(s).

13. $z = 6$
14. $y = 2$
15. $x = 4$
16. $z = -3$
17. $y < 0$
18. $x < 0$
19. $|y| \le 3$
20. $|x| > 4$
21. $xy > 0, \quad z = -3$
22. $xy < 0, \quad z = 4$
23. $xyz < 0$
24. $xyz > 0$

In Exercises 25–28, find the distance between the points.

25. $(0, 0, 0), \quad (5, 2, 6)$
26. $(-2, 3, 2), \quad (2, -5, -2)$
27. $(1, -2, 4), \quad (6, -2, -2)$
28. $(2, 2, 3), \quad (4, -5, 6)$

In Exercises 29–32, find the lengths of the sides of the triangle with the indicated vertices, and determine whether the triangle is a right triangle, an isosceles triangle, or neither.

29. $(0, 0, 0), (2, 2, 1), (2, -4, 4)$
30. $(5, 3, 4), (7, 1, 3), (3, 5, 3)$
31. $(1, -3, -2), (5, -1, 2), (-1, 1, 2)$
32. $(5, 0, 0), (0, 2, 0), (0, 0, -3)$
33. *Think About It* The triangle in Exercise 29 is translated five units upward along the z-axis. Determine the coordinates of the translated triangle.
34. *Think About It* The triangle in Exercise 30 is translated three units to the right along the y-axis. Determine the coordinates of the translated triangle.

In Exercises 35 and 36, find the coordinates of the midpoint of the line segment joining the points.

35. $(5, -9, 7), (-2, 3, 3)$
36. $(4, 0, -6), (8, 8, 20)$

In Exercises 37–40, find the standard equation of the sphere.

37. Center: $(0, 2, 5)$
 Radius: 2
38. Center: $(4, -1, 1)$
 Radius: 5
39. Endpoints of a diameter: $(2, 0, 0), (0, 6, 0)$
40. Center: $(-3, 2, 4)$, tangent to the yz-plane

In Exercises 41–44, complete the square to write the equation of the sphere in standard form. Find the center and radius.

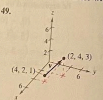
41. $x^2 + y^2 + z^2 - 2x + 6y + 8z + 1 = 0$
42. $x^2 + y^2 + z^2 + 9x - 2y + 10z + 19 = 0$
43. $9x^2 + 9y^2 + 9z^2 - 6x + 18y + 1 = 0$
44. $4x^2 + 4y^2 + 4z^2 - 4x - 32y + 8z + 33 = 0$

In Exercises 45–48, describe the solid satisfying the condition.

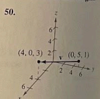
45. $x^2 + y^2 + z^2 \le 36$
46. $x^2 + y^2 + z^2 > 4$
47. $x^2 + y^2 + z^2 < 4x - 6y + 8z - 13$
48. $x^2 + y^2 + z^2 > -4x + 6y - 8z - 13$

In Exercises 49–52, (a) find the component form of the vector v and (b) sketch the vector with its initial point at the origin.

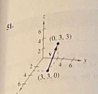
49.



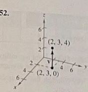
50.



51.



52.



In Exercises 53–56, find the component form and magnitude of the vector u with the given initial and terminal points. Then find a unit vector in the direction of u.

	Initial Point	*Terminal Point*
53.	$(3, 2, 0)$	$(4, 1, 6)$
54.	$(4, -5, 2)$	$(-1, 7, -3)$
55.	$(-4, 3, 1)$	$(-5, 3, 0)$
56.	$(1, -2, 4)$	$(2, 4, -2)$

In Exercises 57 and 58, the initial and terminal points of a vector v are given. (a) Sketch the directed line segment, (b) find the component form of the vector, and (c) sketch the vector with its initial point at the origin.

57. Initial point: $(-1, 2, 3)$
 Terminal point: $(3, 3, 4)$
58. Initial point: $(2, -1, -2)$
 Terminal point: $(-4, 3, 7)$

In Exercises 59 and 60, the vector v and its initial point are given. Find the terminal point.

59. $\mathbf{v} = \langle 3, -5, 6\rangle$
 Initial point: $(0, 6, 2)$
60. $\mathbf{v} = \left\langle 1, -\frac{2}{3}, \frac{1}{2}\right\rangle$
 Initial point: $\left(0, 2, \frac{5}{2}\right)$

In Exercises 61 and 62, find each scalar multiple of v and sketch its graph.

61. $\mathbf{v} = \langle 1, 2, 2\rangle$
 (a) $2\mathbf{v}$ (b) $-\mathbf{v}$ (c) $\frac{3}{2}\mathbf{v}$ (d) $0\mathbf{v}$
62. $\mathbf{v} = \langle 2, -2, 1\rangle$
 (a) $-\mathbf{v}$ (b) $2\mathbf{v}$ (c) $\frac{1}{2}\mathbf{v}$ (d) $\frac{5}{2}\mathbf{v}$

In Exercises 63–68, find the vector z, given that $\mathbf{u} = \langle 1, 2, 3\rangle$, $\mathbf{v} = \langle 2, 2, -1\rangle$, and $\mathbf{w} = \langle 4, 0, -4\rangle$.

63. $\mathbf{z} = \mathbf{u} - \mathbf{v}$
64. $\mathbf{z} = \mathbf{u} - \mathbf{v} + 2\mathbf{w}$
65. $\mathbf{z} = 2\mathbf{u} + 4\mathbf{v} - \mathbf{w}$
66. $\mathbf{z} = 5\mathbf{u} - 3\mathbf{v} - \frac{1}{2}\mathbf{w}$
67. $2\mathbf{z} - 3\mathbf{u} = \mathbf{w}$
68. $2\mathbf{u} + \mathbf{v} - \mathbf{w} + 3\mathbf{z} = \mathbf{0}$

In Exercises 69–72, determine which of the vectors is (are) parallel to z. Use a graphing utility to confirm your results.

69. $\mathbf{z} = \langle 3, 2, -5\rangle$
 (a) $\langle -6, -4, 10\rangle$
 (b) $\left\langle 2, \frac{4}{3}, -\frac{10}{3}\right\rangle$
 (c) $\langle 6, 4, 10\rangle$
 (d) $\langle 1, -4, 2\rangle$
70. $\mathbf{z} = \frac{1}{2}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{3}{4}\mathbf{k}$
 (a) $6\mathbf{i} - 4\mathbf{j} + 9\mathbf{k}$
 (b) $-\mathbf{i} + \frac{4}{3}\mathbf{j} - \frac{3}{2}\mathbf{k}$
 (c) $12\mathbf{i} + 9\mathbf{k}$
 (d) $\frac{3}{4}\mathbf{i} - \mathbf{j} + \frac{9}{8}\mathbf{k}$
71. $\mathbf{z}$ has initial point $(1, -1, 3)$ and terminal point $(-2, 3, 5)$.
 (a) $-6\mathbf{i} + 8\mathbf{j} + 4\mathbf{k}$ (b) $4\mathbf{j} + 2\mathbf{k}$
72. $\mathbf{z}$ has initial point $(5, 4, 1)$ and terminal point $(-2, -4, 4)$.
 (a) $\langle 7, 6, 2\rangle$ (b) $\langle 14, 16, -6\rangle$

In Exercises 73–76, use vectors to determine whether the points are collinear.

73. $(0, -2, -5), (3, 4, 4), (2, 2, 1)$
74. $(4, -2, 7), (-2, 0, 3), (7, -3, 9)$
75. $(1, 2, 4), (2, 5, 0), (0, 1, 5)$
76. $(0, 0, 0), (1, 3, -2), (2, -6, 4)$

In Exercises 77 and 78, use vectors to show that the points form the vertices of a parallelogram.

77. $(2, 9, 1), (3, 11, 4), (0, 10, 2), (1, 12, 5)$
78. $(1, 1, 3), (9, -1, -2), (11, 2, -9), (3, 4, -4)$

In Exercises 79–84, find the magnitude of v.

79. $\mathbf{v} = \langle 0, 0, 0\rangle$
80. $\mathbf{v} = \langle 1, 0, 3\rangle$
81. $\mathbf{v} = \mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$
82. $\mathbf{v} = -4\mathbf{i} + 3\mathbf{j} + 7\mathbf{k}$
83. Initial point of $\mathbf{v}$: $(1, -3, 4)$
 Terminal point of $\mathbf{v}$: $(1, 0, -1)$
84. Initial point of $\mathbf{v}$: $(0, -1, 0)$
 Terminal point of $\mathbf{v}$: $(1, 2, -2)$

In Exercises 85–88, find a unit vector (a) in the direction of u and (b) in the direction opposite of u.

85. $\mathbf{u} = \langle 2, -1, 2\rangle$
86. $\mathbf{u} = \langle 6, 0, 8\rangle$
87. $\mathbf{u} = \langle 3, 2, -5\rangle$
88. $\mathbf{u} = \langle 8, 0, 0\rangle$
89. *Programming* You are given the component forms of the vectors **u** and **v**. Write a program for a graphing utility in which the output is (a) the component form of $\mathbf{u} + \mathbf{v}$, (b) $\|\mathbf{u} + \mathbf{v}\|$, (c) $\|\mathbf{u}\|$, and (d) $\|\mathbf{v}\|$.
90. *Programming* Run the program you wrote in Exercise 89 for the vectors $\mathbf{u} = \langle -1, 3, 4\rangle$ and $\mathbf{v} = \langle 5, 4.5, -6\rangle$.

In Exercises 91 and 92, determine the values of c that satisfy the equation. Let $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

91. $\|c\mathbf{v}\| = 5$
92. $\|c\mathbf{u}\| = 3$

In Exercises 93–96, find the vector v with the given magnitude and direction u.

	Magnitude	*Direction*
93.	10	$\mathbf{u} = \langle 0, 3, 3\rangle$
94.	3	$\mathbf{u} = \langle 1, 1, 1\rangle$
95.	$\frac{3}{2}$	$\mathbf{u} = \langle 2, -2, 1\rangle$
96.	$\sqrt{5}$	$\mathbf{u} = \langle -4, 6, 2\rangle$

In Exercises 97 and 98, sketch the vector v and write its component form.

97. $\mathbf{v}$ lies in the yz-plane, has magnitude 2, and makes an angle of 30° with the positive y-axis.

98. $\mathbf{v}$ lies in the xz-plane, has magnitude 5, and makes an angle of 45° with the positive z-axis.

In Exercises 99 and 100, use vectors to find the point that lies two-thirds of the way from P to Q.

99. $P(4, 3, 0)$, $Q(1, -3, 3)$ **100.** $P(1, 2, 5)$, $Q(6, 8, 2)$

101. Let $\mathbf{u} = \mathbf{i} + \mathbf{j}$, $\mathbf{v} = \mathbf{j} + \mathbf{k}$, and $\mathbf{w} = a\mathbf{u} + b\mathbf{v}$.

(a) Sketch $\mathbf{u}$ and $\mathbf{v}$.

(b) If $\mathbf{w} = \mathbf{0}$, show that a and b must both be zero.

(c) Find a and b such that $\mathbf{w} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$.

(d) Show that no choice of a and b yields $\mathbf{w} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

102. ***Writing*** The initial and terminal points of the vector $\mathbf{v}$ are (x_1, y_1, z_1) and (x, y, z). Describe the set of all points (x, y, z) such that $\|\mathbf{v}\| = 4$.

Writing About Concepts

103. A point in the three-dimensional coordinate system has coordinates (x_0, y_0, z_0). Describe what each coordinate measures.

104. Give the formula for the distance between the points (x_1, y_1, z_1) and (x_2, y_2, z_2).

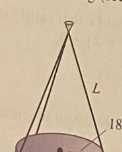
105. Give the standard equation of a sphere of radius r, centered at (x_0, y_0, z_0).

106. State the definition of parallel vectors.

107. Let A, B, and C be vertices of a triangle. Find $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}$.

108. Let $\mathbf{r} = \langle x, y, z\rangle$ and $\mathbf{r}_0 = \langle 1, 1, 1\rangle$. Describe the set of all points (x, y, z) such that $\|\mathbf{r} - \mathbf{r}_0\| = 2$.

109. ***Numerical, Graphical, and Analytic Analysis*** The lights in an auditorium are 24-pound discs of radius 18 inches. Each disc is supported by three equally spaced cables that are L inches long (see figure).



(a) Write the tension T in each cable as a function of L. Determine the domain of the function.

(b) Use a graphing utility and the function in part (a) to complete the table.

L	20	25	30	35	40	45	50
T							

(c) Use a graphing utility to graph the function in part (a). Determine the asymptotes of the graph.

(d) Confirm the asymptotes of the graph in part (c) analytically.

(e) Determine the minimum length of each cable if a cable is designed to carry a maximum load of 10 pounds.

110. ***Think About It*** Suppose the length of each cable in Exercise 109 has a fixed length $L = a$, and the radius of each disc is r_0 inches. Make a conjecture about the limit $\lim_{r_0 \to a^-} T$ and give a reason for your answer.

111. ***Diagonal of a Cube*** Find the component form of the unit vector $\mathbf{v}$ in the direction of the diagonal of the cube shown in the figure.

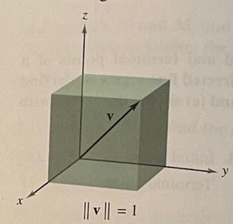


Figure for 111

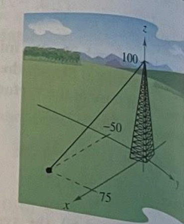


Figure for 112

112. ***Tower Guy Wire*** The guy wire to a 100-foot tower has a tension of 550 pounds. Using the distances shown in the figure, write the component form of the vector $\mathbf{F}$ representing the tension in the wire.

113. ***Load Supports*** Find the tension in each of the supporting cables in the figure if the weight of the crate is 500 newtons.

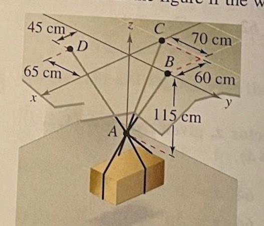


Figure for 113

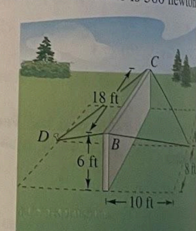


Figure for 114

114. ***Construction*** A precast concrete wall is temporarily kept in its vertical position by ropes (see figure). Find the total force exerted on the pin at position A. The tensions in AB and AC are 420 pounds and 650 pounds.

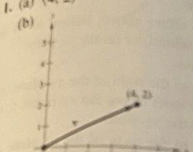
115. Write an equation whose graph consists of the set of points $P(x, y, z)$ that are twice as far from $A(0, -1, 1)$ as from $B(1, 2, 0)$.

Solutions from back of text:

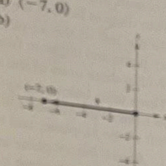
Chapter 11

Section 11.1 (page 769)

1. (a) $\langle 4, 2\rangle$
(b)

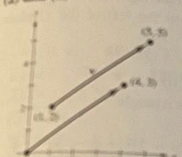


3. (a) $\langle -7, 0\rangle$
(b)



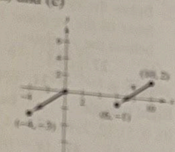
5. $\mathbf{u} = \mathbf{v} = \langle 2, 4\rangle$ 7. $\mathbf{u} = \mathbf{v} = \langle 6, -5\rangle$

9. (a) and (c)



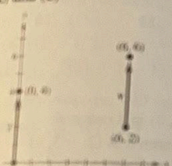
(b) $\langle 4, 3\rangle$

11. (a) and (c)



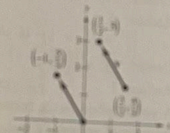
(b) $\langle -4, -3\rangle$

13. (a) and (c)



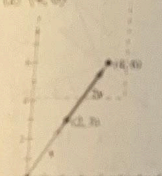
(b) $\langle 0, 4\rangle$

15. (a) and (c)

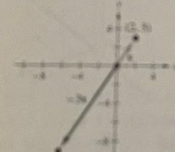


(b) $\langle -1, \frac{5}{3}\rangle$

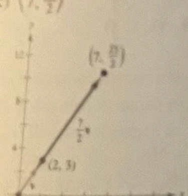
17. (a) $\langle 4, 6\rangle$



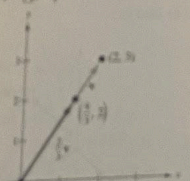
(b) $\langle -6, -9\rangle$



(c) $\langle 7, \frac{21}{2}\rangle$

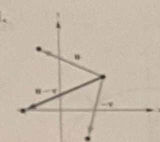


(d) $\langle \frac{8}{3}, 2\rangle$



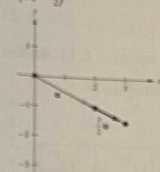
19.

21.

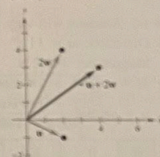


23. (a) $\langle \frac{8}{3}, 6\rangle$ (b) $\langle -2, -14\rangle$ (c) $\langle 18, -7\rangle$

25. $\langle 3, -\frac{3}{2}\rangle$



27. $\langle 4, 3\rangle$



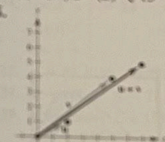
29. $\langle 3, 5\rangle$ 31. 5 33. $\sqrt{61}$ 35. 4

37. $\langle \sqrt{17}/17, 4\sqrt{17}/17\rangle$ 39. $\langle 3\sqrt{34}/34, 5\sqrt{34}/34\rangle$

41. (a) $\sqrt{2}$ (b) $\sqrt{5}$ (c) 1 (d) 1 (e) 1 (f) 1

43. (a) $\sqrt{5}/2$ (b) $\sqrt{13}$ (c) $\sqrt{85}/2$ (d) 1 (e) 1 (f) 1

45.



$\|\mathbf{u}\| + \|\mathbf{v}\| = \sqrt{5} + \sqrt{41}$ and $\|\mathbf{u} + \mathbf{v}\| = \sqrt{74}$

$\sqrt{74} < \sqrt{5} + \sqrt{41}$

47. $\langle 2\sqrt{2}, 2\sqrt{2}\rangle$ 49. $\langle 1, \sqrt{3}\rangle$ 51. $\langle 3, 0\rangle$ 53. $\langle -\sqrt{3}, 1\rangle$

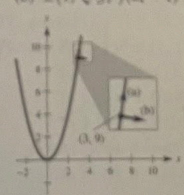
55. $\left\langle \dfrac{2 + 3\sqrt{2}}{2}, \dfrac{3\sqrt{2}}{2}\right\rangle$ 57. $\langle 2\cos 4 + \cos 2, 2\sin 4 + \sin 2\rangle$

59. Answers will vary. Example: A scalar is a single real number such as 2. A vector is a line segment having both direction and magnitude. The vector $\langle \sqrt{3}, 1\rangle$, given in component form, has a direction of $\pi/6$ and a magnitude of 2.

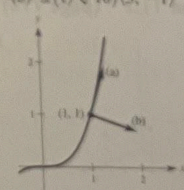
61. (a) Vector; has magnitude and direction
(b) Scalar; has only magnitude

63. $a = 1, b = 1$ 65. $a = 1, b = 2$ 67. $a = \frac{2}{3}, b = \frac{1}{3}$

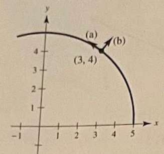
69. (a) $\pm(1/\sqrt{37})\langle 1, 6\rangle$
(b) $\pm(1/\sqrt{37})\langle 6, -1\rangle$



71. (a) $\pm(1/\sqrt{10})\langle 1, 3\rangle$
(b) $\pm(1/\sqrt{10})\langle 3, -1\rangle$



73. (a) $\pm\frac{1}{5}\langle -4, 3\rangle$
(b) $\pm\frac{1}{5}\langle 3, 4\rangle$



75. $\langle -\sqrt{2}/2, \sqrt{2}/2\rangle$

77. (a)–(c) Answers will vary. **79.** 1.33, 132.5°

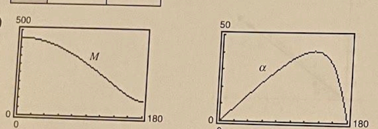
81. (a) Direction: $\alpha = 11.8°$
Magnitude: 440.2 N
(b) $M = \sqrt{(275 + 180\cos\theta)^2 + (180\sin\theta)^2}$
$\alpha = \arctan\left(\dfrac{180\sin\theta}{275 + 180\cos\theta}\right)$

(c)

θ	0°	30°	60°	90°	120°
M	455.0	440.2	396.9	328.7	241.9
α	0°	11.8°	23.1°	33.2°	40.1°

θ	150°	180°
M	149.3	95.0
α	37.1°	0°

(d)



(e) M decreases because the forces change from acting in the same direction to acting in opposite directions as θ increases from 0° to 180°.

83. 71.3°, 228.5 lb

85. (a) $\theta = 0°$ (b) $\theta = 180°$
(c) No, the resultant can only be less than or equal to the sum.

87. $(-4, -1)$, $(6, 5)$, $(10, 3)$

89. Tension in cable $AC \approx 1758.8$ lb
Tension in cable $BC \approx 1305.4$ lb

91. Horizontal: 1193.43 ft/sec
Vertical: 125.43 ft/sec

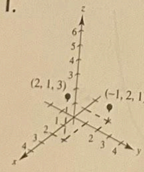
93. 38.3° north of west
882.9 kph

95. True **97.** True **99.** False. $\|a\mathbf{i} + b\mathbf{j}\| = \sqrt{2}|a|$

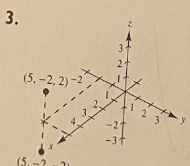
101–103. Proofs **105.** $x^2 + y^2 = 25$

Section 11.2 (page 778)

1.



3.



5. $A(2, 3, 4)$
$B(-1, -2, 2)$

7. $(-3, 4, 5)$ **9.** $(10, 0, 0)$ **11.** 0

13. Six units above the xy-plane

15. Four units in front of the yz-plane

17. To the left of the xz-plane and either above, below, or on the xy-plane and either in front of, behind, or on the yz-plane

19. Within three units of the xz-plane

21. Three units below the xy-plane, to the right of the xz-plane, and in front of the yz-plane, *or* three units below the xy-plane, to the left of the xz-plane, and behind the yz-plane

23. 1. Above the xy-plane and (a) to the right of the xz-plane and behind the yz-plane or (b) to the left of the xz-plane and in front of the yz-plane, *or*
2. Below the xy-plane and (a) to the right of the xz-plane and in front of the yz-plane or (b) to the left of the xz-plane and behind the yz-plane

25. $\sqrt{65}$ **27.** $\sqrt{61}$

29. $3, 3\sqrt{5}, 6$
Right triangle

31. $6, 6, 2\sqrt{10}$
Isosceles triangle

33. $(0, 0, 5)$, $(2, 2, 6)$, $(2, -4, 9)$

35. $\left(\frac{3}{2}, -3, 5\right)$ **37.** $(x - 0)^2 + (y - 2)^2 + (z - 5)^2 = 4$

39. $(x - 1)^2 + (y - 3)^2 + (z - 0)^2 = 10$

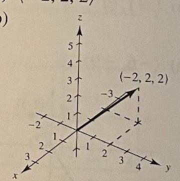
41. $(x - 1)^2 + (y + 3)^2 + (z + 4)^2 = 25$
Center: $(1, -3, -4)$
Radius: 5

43. $\left(x - \frac{1}{3}\right)^2 + (y + 1)^2 + z^2 = 1$
Center: $\left(\frac{1}{3}, -1, 0\right)$
Radius: 1

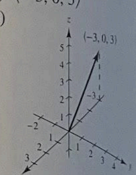
45. A solid sphere with center $(0, 0, 0)$ and radius 6

47. Interior of sphere of radius 4 centered at $(2, -3, 4)$

49. (a) $\langle -2, 2, 2\rangle$
(b)



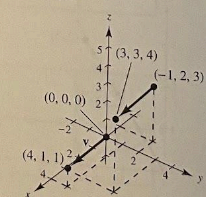
51. (a) $\langle -3, 0, 3\rangle$
(b)



53. $\mathbf{u} = \langle 1, -1, 6\rangle$
$\|\mathbf{u}\| = \sqrt{38}$
$\dfrac{\mathbf{u}}{\|\mathbf{u}\|} = \dfrac{1}{\sqrt{38}}\langle 1, -1, 6\rangle$

55. $\mathbf{u} = \langle -1, 0, -1\rangle$
$\|\mathbf{u}\| = \sqrt{2}$
$\dfrac{\mathbf{u}}{\|\mathbf{u}\|} = \dfrac{1}{\sqrt{2}}\langle -1, 0, -1\rangle$

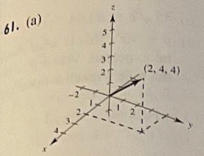
57. (a) and (c)



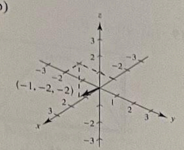
(b) $\langle 4, 1, 1\rangle$

59. $(3, 1, 8)$

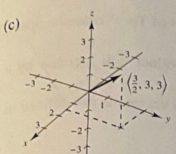
61. (a)



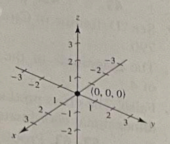
(b)



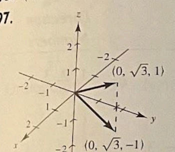
(c)



(d)



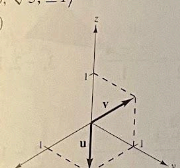
63. $\langle -1, 0, 4\rangle$ **65.** $\langle 6, 12, 6\rangle$ **67.** $\left\langle \frac{7}{2}, 3, \frac{5}{2}\right\rangle$
69. a and b **71.** a **73.** Collinear **75.** Not collinear
77. $\overrightarrow{AB} = \langle 1, 2, 3\rangle$
$\overrightarrow{CD} = \langle 1, 2, 3\rangle$
$\overrightarrow{BD} = \langle -2, 1, 1\rangle$
$\overrightarrow{AC} = \langle -2, 1, 1\rangle$
Since $\overrightarrow{AB} = \overrightarrow{CD}$ and $\overrightarrow{BD} = \overrightarrow{AC}$, the given points form the vertices of a parallelogram.
79. 0 **81.** $\sqrt{14}$ **83.** $\sqrt{34}$
85. (a) $\frac{1}{3}\langle 2, -1, 2\rangle$ (b) $-\frac{1}{3}\langle 2, -1, 2\rangle$
87. (a) $\left(1/\sqrt{38}\right)\langle 3, 2, -5\rangle$ (b) $-\left(1/\sqrt{38}\right)\langle 3, 2, -5\rangle$
89. (a)–(d) Answers will vary. **91.** $\pm\frac{5}{3}$
93. $\langle 0, 10/\sqrt{2}, 10/\sqrt{2}\rangle$ **95.** $\left\langle 1, -1, \frac{1}{2}\right\rangle$
97.



$\langle 0, \sqrt{3}, \pm 1\rangle$

99. $(2, -1, 2)$

101. (a)

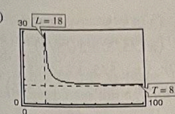


(b) $a = 0, a + b = 0, b = 0$
(c) $a = 1, a + b = 2, b = 1$
(d) Not possible

103. x_0 is directed distance to yz-plane.
y_0 is directed distance to xz-plane.
z_0 is directed distance to xy-plane.
105. $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$ **107.** 0
109. (a) $T = 8L/\sqrt{L^2 - 18^2}, L > 18$
(b)

L	20	25	30	35	40	45	50
T	18.4	11.5	10	9.3	9.0	8.7	8.6

(c)



(d) Proof (e) 30 in.

111. $\left(\sqrt{3}/3\right)\langle 1, 1, 1\rangle$
113. Tension in cable AB: 202.919 N
Tension in cable AC: 157.909 N
Tension in cable AD: 226.521 N
115. $\left(x - \frac{4}{3}\right)^2 + (y - 3)^2 + \left(z + \frac{1}{3}\right)^2 = \frac{44}{9}$

Section 11.3 (page 787)

1. (a) -6 (b) 25 (c) 25 (d) $\langle -12, 18\rangle$ (e) -12
3. (a) -17 (b) 26 (c) 26 (d) $\langle 51, -34\rangle$ (e) -34
5. (a) 2 (b) 29 (c) 29 (d) $\langle 0, 12, 10\rangle$ (e) 4
7. (a) 1 (b) 6 (c) 6 (d) $\mathbf{i} - \mathbf{k}$ (e) 2
9. 20 **11.** $\pi/2$ **13.** $\arccos\left(-1/5\sqrt{2}\right) \approx 98.1°$
15. $\arccos\left(\sqrt{2}/3\right) \approx 61.9°$ **17.** $\arccos\left(-8\sqrt{13}/65\right) \approx 116.3°$
19. Neither **21.** Orthogonal **23.** Neither
25. Orthogonal **27.** Right triangle; answers will vary.
29. Acute triangle; answers will vary.
31. $\cos\alpha = \frac{1}{3}$, $\cos\beta = \frac{2}{3}$, $\cos\gamma = \frac{2}{3}$
33. $\cos\alpha = 0$, $\cos\beta = 3/\sqrt{13}$, $\cos\gamma = -2/\sqrt{13}$
35. $\alpha \approx 43.3°, \beta \approx 61.0°, \gamma \approx 119.0°$
37. $\alpha \approx 100.5°, \beta \approx 24.1°, \gamma \approx 68.6°$
39. Magnitude: 124.310 lb
$\alpha \approx 29.48°, \beta \approx 61.39°, \gamma \approx 96.53°$
41. $\alpha = 90°, \beta = 45°, \gamma = 45°$ **43.** $\langle 4, -1\rangle$ **45.** $\langle 2, 1, 1\rangle$
47. (a) $\left\langle \frac{5}{2}, \frac{1}{2}\right\rangle$ (b) $\left\langle -\frac{1}{2}, \frac{5}{2}\right\rangle$
49. (a) $\left\langle 0, \frac{33}{25}, \frac{44}{25}\right\rangle$ (b) $\left\langle 2, -\frac{8}{25}, \frac{6}{25}\right\rangle$
51. See "Definition of Dot Product," page 781.
53. (a) $\theta = \pi/2$ (b) $0 < \theta < \pi/2$ (c) $\pi/2 < \theta < \pi$
55. See the definitions of direction cosines and direction angles on page 784.
57. (a) The vectors are parallel. (b) The vectors are orthogonal.
59. \$12,351.25; Total revenue **61.** (a)–(c) Answers will vary.
63. Answers will vary. **65.** $\langle 0, 0\rangle$
67. Answers will vary. Example: $\langle 4, 3\rangle$ and $\langle -4, -3\rangle$
69. Answers will vary. Example: $\langle 2, 0, 3\rangle$ and $\langle -2, 0, -3\rangle$
71. (a) 8335.1 lb (b) 47,270.8 lb **73.** 425 ft-lb
75. False. For example, $\langle 1, 1\rangle \cdot \langle 2, 3\rangle = 5$ and $\langle 1, 1\rangle \cdot \langle 1, 4\rangle = 5$, but $\langle 2, 3\rangle \neq \langle 1, 4\rangle$.
77. $\arccos\left(1/\sqrt{3}\right) \approx 54.7°$
79. (a) To $y = x^2$ at $(1, 1)$: $\left\langle \pm\sqrt{5}/5, \pm 2\sqrt{5}/5\right\rangle$
To $y = x^{1/3}$ at $(1, 1)$: $\left\langle \pm 3\sqrt{10}/10, \pm\sqrt{10}/10\right\rangle$
To $y = x^2$ at $(0, 0)$: $\langle \pm 1, 0\rangle$
To $y = x^{1/3}$ at $(0, 0)$: $\langle 0, \pm 1\rangle$
(b) At $(1, 1)$, $\theta = 45°$
At $(0, 0)$, $\theta = 90°$

You do not need to buy this textbook (although it costs less than $20 used. If you like how it explains things, then it is a good idea to buy it. If you prefer to continue forward using a calculus book you already own that is also ok.

You will cut and paste the photos of your notes and completed classwork and a selfie taken holding up the first page of your work in a googledoc entitled:

MAT226S25-lesson1-lastname-firstname

and share editing of that document with me sormanic@gmail.com. You will also put photos of your homework in this googledoc.