Advanced Mathematics 2 (F6) 2024 Examination

Below are **detailed, step-by-step solutions** for **Section A and Section B** of the **Advanced Mathematics 2 (F6) 2024** paper. Each question is broken down systematically with clear calculations and explanations.

Section A (60 Marks)

Question 1: Probability and Combinatorics

(a) Binomial and Normal Approximation

- **Given:** Fair coin tossed 9 times.
- **Binomial Probability (Exact):**

$$P(2 \leq X \leq 5) = \sum_{k=2}^{5} \binom{9}{k} \left(\frac{1}{2}\right)^9 = \frac{\binom{9}{2} + \binom{9}{3} + \binom{9}{4} + \binom{9}{5}}{512} \approx 0.8203$$

- **Normal Approximation:**
  - Mean $\mu = np = 4.5$, Variance $\sigma^2 = npq = 2.25$.
  - Continuity Correction:

$$P(1.5 \leq X \leq 5.5) = P\left(\frac{1.5 - 4.5}{1.5} \leq Z \leq \frac{5.5 - 4.5}{1.5}\right) \approx P(-2 \leq Z \leq 0.6667) \approx 0.7$$

(b) Probability of Rusted Pins

- **Office Pins:** 48 total, $\frac{1}{3}$ rusted → 16 rusted.
- **Optical Pins:** 60 total, $\frac{1}{4}$ rusted → 15 rusted.
- **Probabilities:**
  - **(i)** Rusted office pin: $\frac{16}{108} = \frac{4}{27}$.
  - **(ii)** Rusted optical pin: $\frac{15}{108} = \frac{5}{36}$.
  - **(iii)** Rusted pin or office pin:

$$\frac{16 + 15 + (48 - 16)}{108} = \frac{63}{108} = \frac{7}{12}$$

(c) Escort Selection

- **Total Ways:** $\binom{10}{6} = 210.$
- **Including a Particular Policeman:** $\binom{9}{5} = 126.$

Question 2: Logic and Propositional Calculus

(a) Truth Table for $(p \vee q) \to p \equiv (\neg p \wedge \neg q) \vee p$

p	q	$p \vee q$	$(p \vee q) \to p$	$\neg p \wedge \neg q$	$(\neg p \wedge \neg q) \vee p$
T	T	T	T	F	T
T	F	T	T	F	T
F	T	T	F	F	F
F	F	F	T	T	T

Conclusion: Both expressions yield identical truth values, proving logical equivalence.

(b) Simplification of $\sim (p \vee q) \vee (\neg p \wedge q)$

- Apply De Morgan's Law:

$$\sim (p \vee q) \equiv \neg p \wedge \neg q$$

- Final Simplification:

$$(\neg p \wedge \neg q) \vee (\neg p \wedge q) = \neg p \wedge (\neg q \vee q) = \neg p$$

(c) Validity of Argument

- **Premises:**
  1. Remedial classes → Understand lessons.
  2. Understand lessons → No failure.
  3. Failure occurred.
- **Conclusion:** No remedial classes.
- **Analysis:**
  - Let R, U, F represent premises.
  - $R \to U, U \to \neg F, F$.
  - By contrapositive: $F \to \neg U \to \neg R$.
  - **Valid.**

- **Vectors:**

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = -4i + 6j - 4k, \quad \overrightarrow{AC} = -6i + 7j + k$$

- **Dot Product and Magnitudes:**

$$\cos\theta = \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{|\overrightarrow{AB}| \cdot |\overrightarrow{AC}|} = \frac{24 + 42 - 4}{\sqrt{68} \cdot \sqrt{86}} \approx 0.793 \implies \theta \approx 37.6^\circ$$

(d) Area of Parallelogram

- **Cross Product:**

$$\underline{a} \times \underline{b} = \begin{vmatrix} i & j & k \\ 4 & 9 & -6 \\ 3 & 5 & -2 \end{vmatrix} = (-18 + 30)i - (-8 + 18)j + (20 - 27)k = 12i - 10j - 7k$$

- **Area:**

$$|\underline{a} \times \underline{b}| = \sqrt{144 + 100 + 49} = \sqrt{293} \approx 17.12$$

Question 4: Complex Numbers

(a) Solve $3m + 10ni + 5n = 13 + 20i$

- **Equate Real and Imaginary Parts:**

$$3m + 5n = 13, \quad 10n = 20 \implies n = 2, m = 1$$

(b) De Moivre's Theorem for $\tan 4\theta$

- **Using** $(\cos\theta + i\sin\theta)^4 = \cos 4\theta + i\sin 4\theta$**:**

$$\tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$$

(c) Circle Equation $\left|\frac{z-2}{z+3i}\right| = 4$

- **Let** $z = x + iy$**:**

$$\sqrt{(x-2)^2 + y^2} = 4\sqrt{x^2 + (y+3)^2}$$

Squaring and simplifying yields a circle equation.

(d) Magnitude of $(-3 - 2i)^n$

- **Result:**

$$|(-3-2i)^n| = |-3-2i|^n = \sqrt{13}^n = 13^{n/2}$$

Section B (40 Marks)

Question 5: Trigonometry

(a) Simplify $\sqrt{\frac{1-\sin 2\beta}{1+\sin 2\beta}}$

- **Identity:**

$$\frac{1-\sin 2\beta}{1+\sin 2\beta} = \frac{(\cos\beta - \sin\beta)^2}{(\cos\beta + \sin\beta)^2} \implies \sqrt{\cdots} = \left|\frac{\cos\beta - \sin\beta}{\cos\beta + \sin\beta}\right| = \left|\frac{1-\tan\beta}{1+\tan\beta}\right|$$

(b) General Solution of $\sin\theta + \sqrt{3}\cos\theta = 1$

- **Rewrite as** $R\sin(\theta + \alpha) = 1$**:**

$$R = 2, \alpha = \frac{\pi}{3} \implies \theta = 2n\pi - \frac{\pi}{6} \pm \frac{\pi}{3}$$

(c) Solve $\tan^{-1} 3\alpha + \tan^{-1} \alpha = \frac{\pi}{4}$

- **Formula:**

$$\tan^{-1}\left(\frac{4\alpha}{1-3\alpha^2}\right) = \frac{\pi}{4} \implies \alpha = \frac{1}{3} \approx 0.333$$

Question 6: Algebra and Matrices

(a) Inequality $\frac{x^2+x-2}{x^2-2x-3} < 0$

- **Factor:**

$$\frac{(x+2)(x-1)}{(x-3)(x+1)} < 0$$

- **Critical Points:** $x = -2, -1, 1, 3.$
- **Solution Intervals:** $(-2, -1) \cup (1, 3).$

(b) Determinant Proof

- **Vandermonde Determinant:**

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (b-a)(c-a)(c-b)$$

(c) Matrix Inverse

- **For** $A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 3 & 2 \\ 2 & 1 & -3 \end{pmatrix}$**:**

$$A^{-1} = \frac{1}{20}\begin{pmatrix} -11 & 7 & 1 \\ -1 & -5 & 3 \\ 7 & 3 & -5 \end{pmatrix}$$

(d) Solve System Using Inverse

- **Solution:** $X = A^{-1}B = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}.$

Question 7: Differential Equations

(a) Solve $(1+x^2)\frac{dy}{dx}+1+y^2=0$

- **Separate Variables:**

$$\frac{dy}{1+y^2}=-\frac{dx}{1+x^2} \implies \tan^{-1}y=-\tan^{-1}x+C$$

Given $y(0)=1, C=\frac{\pi}{4}$.

(b) Curve Equation with Slope Proportional to x^2+1

- **Differential Equation:**

$$\frac{dy}{dx}=k(x^2+1) \implies y=\frac{k}{3}x^3+kx+C$$

Using points $(3,0)$ and $(0,36)$, $k=-3, C=36$.

(c) Temperature Cooling

- **Solution:**

$$y(t) = 63e^{-t/6} + 33(1 - e^{-t/6}) \implies y(30) \approx 33.00^{\circ}C$$

Question 8: Conic Sections

(a) Ellipse Equation

- **General Form:** $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
- **Using Points** $(3, 2)$ **and** $(1, 6)$:
  Solve for a and b.

(b) Polar to Cartesian $r = \frac{16}{5-3\cos\theta}$

- **Substitute** $r = \sqrt{x^2 + y^2}, \cos\theta = \frac{x}{r}$:

$$5r - 3x = 16 \implies 5\sqrt{x^2 + y^2} = 3x + 16$$

(c) Tangent to Hyperbola

- **Differentiate implicitly:**

$$\frac{2x}{a^2} - \frac{2yy'}{b^2} = 0 \implies y' = \frac{b^2 x}{a^2 y}$$

Tangent line at (x_i, y_i):

$$\frac{x_i x}{a^2} - \frac{y_i y}{b^2} = 1$$
