

Connections to Math

Problem History

River crossing puzzles are a genre of logic puzzles dating back to at least the manuscript *Propositiones ad Acuendos Juvenes* written by Alcuin of York (735-804). Alcuin's manuscript included the following variants, as described in [HS]:

- “Three friends each with a sister needed to cross a river. Each one of them coveted the sister of another. At the river they found only a small boat, in which only two of them could cross at a time. How did they cross the river without any of the women being defiled by the men?”
- “A man had to take a wolf, a goat and a bunch of cabbages across a river. The only boat he could find could only take two of them at a time. But he had been ordered to transfer all of these to the other side in good condition. How could this be done?”
- “A man and woman, each the weight of a cartload, with two children who together weigh as much as a cartload, have to cross a river. They find a boat which can only take one cartload. Make the transfer if you can, without sinking the boat.”

The wolf-goat-cabbage variant has appeared in various different forms. (Another popular variant is stated in terms of a fox, a hen, and a sack of grain, for example.) Versions of this puzzle have been found in many different cultures in Europe and Africa, and the differences in logical structure suggest independent conception between the continents. [A]

The sibling problem was eventually reimagined as a problem about jealous husbands and wives. Evolving from a problem about men traveling with harems, to one concerning masters and servants, and eventually to one in terms of cannibals and missionaries, a new variant was created that removed the bijective relationship between the two classes of people:

- Equal numbers of cannibals and missionaries wish to cross a river. If the cannibals ever outnumber the missionaries on either riverbank, those missionaries will be devoured.

Generalizations of these puzzles were considered by mathematicians such as Tartaglia (1499-1557) and Dudeney (1857-1930), increasing the number of items or people, the space in the boat, or adding islands to inflate the number of riverbanks. [PS]

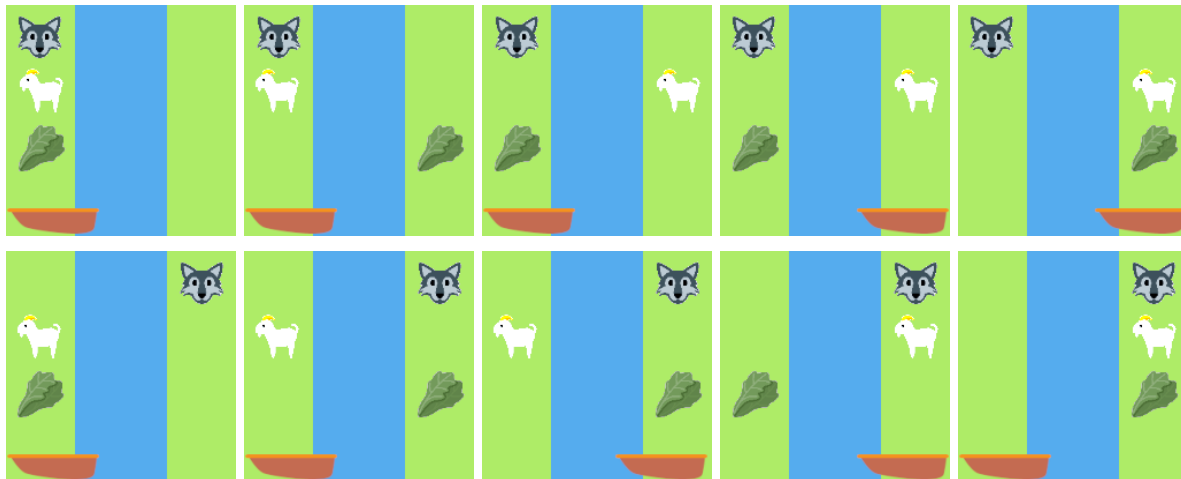
In our sequences, we have retained the classic wolf-goat-cabbage and adult-child puzzles. “Zombies and Humans” is our spin on the dated cannibal-missionary puzzles. Monsters is a fun novel puzzle adding an arithmetic aspect.

Mathematical Theory

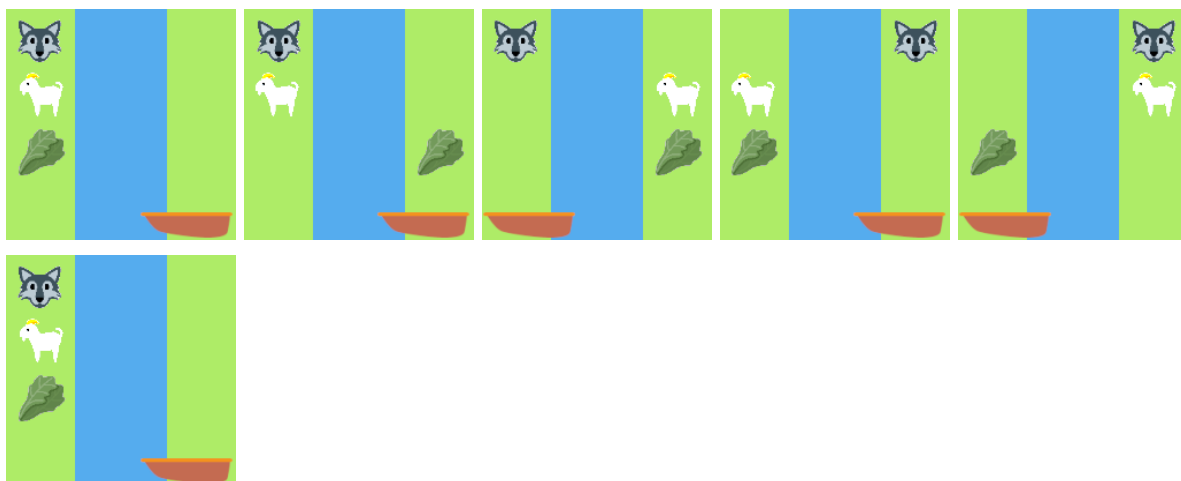
State Transition Graphs

At the heart of every river crossing puzzle is a collection of legal arrangements (or states) and transitions from one state to another. A natural way to represent a puzzle is as a graph, taking these states as vertices and transitions as edges. In the classic wolf-goat-cabbage puzzle, there are 16 possible states, divided below into legal and illegal:

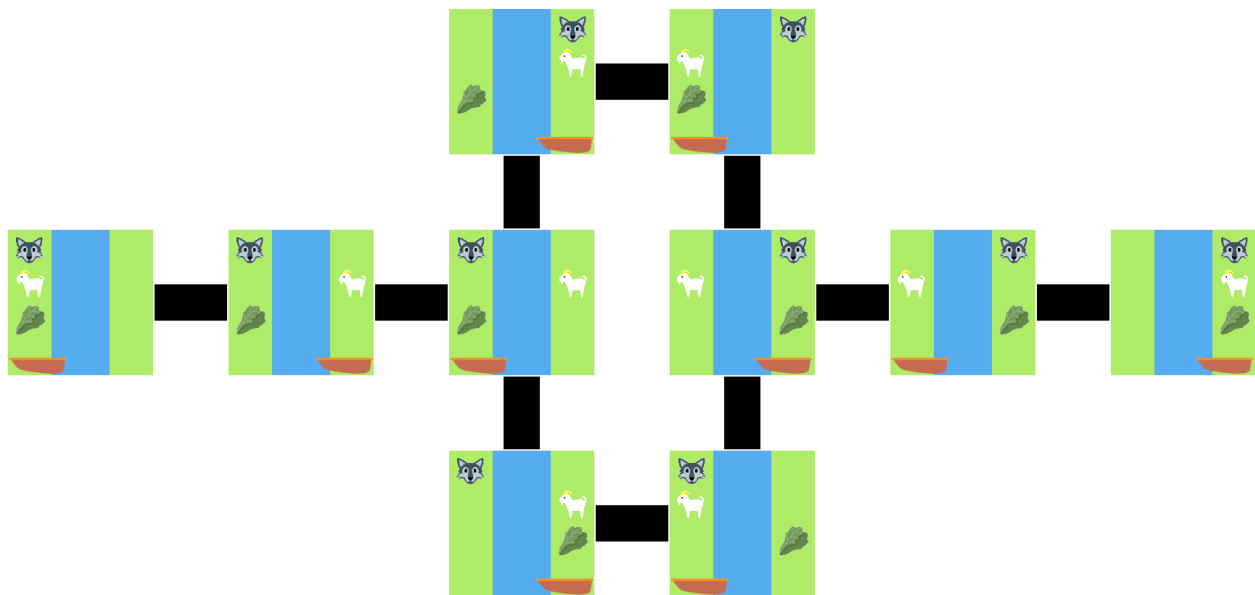
Legal



Illegal



We can then examine the 10 legal states and diagram the possible transitions:



The far left vertex is the starting state and the far right is the desired end state, so solving the puzzle now amounts to finding a path from the start to end following the edges. We can see there are two shortest paths (the upper path and the lower path), each requiring 7 crossings.

Any of the river crossing puzzles can be reformulated in terms of such a state transition graph. Whether the puzzle has a solution then amounts to determining whether the beginning state and end state reside in the same component of the graph and the length of a shortest solution is given as the length of a shortest path.

Vertex Covers

In a generalized version of the wolf-goat-cabbage problem, imagine that you have a large collection of objects where each pair can either be left together unsupervised or not. If there are n objects, then the puzzle can always be solved if there are n spaces in the boat, since the ferry person can keep an eye on all of them with a single trip. However, the puzzle may not be solvable if the boat only has room for a single item. There is a smallest boat that can be used to solve the puzzle, and the number of items this boat can ferry is called the Alcuin number. By the above discussion, the Alcuin number is always between 1 and n , inclusive.

Graph theoretic ideas can also be used to get better estimates of this value. Given n objects, you can construct a graph with each object as a vertex and edge between

any two objects that cannot be left together unsupervised. For a graph G , we follow [CHW] in defining:

- We denote its vertex set as V and edge set as E
- We denote its Alcuin number $\text{Alcuin}(G)$, so that

$$1 \leq \text{Alcuin}(G) \leq |V|$$

- A subset S of the vertices V of G is stable if no vertices in S share an edge in G
- The stability number of G (denoted $\alpha(G)$) is the size of a largest stable set
- A subset W of vertices of G is a vertex cover for G if $S = V - W$ is stable.
- The vertex cover number of G (denoted $\tau(G)$) is the size of a smallest vertex cover

Then it can be shown that

$$\tau(G) \leq \text{Alcuin}(G) \leq \tau(G) + 1$$

Since

$$\tau(G) = |V| - \alpha(G)$$

We can find $\tau(G)$ by first finding $\alpha(G)$.

Consider the puzzle consisting of w wolves, g goats, and c cabbages. Then no stable set can contain both a wolf and a goat or both a goat and a cabbage. This means the largest a stable set can be is either g or $w+c$, i.e.,

$$\alpha(G) = \max(g, w + c)$$

Then

$$\tau(G) = |V| - \alpha(G) = (w + g + c) - \max(g, w + c) = \min(g, w + c)$$

And we can conclude the boat needs a capacity of either $\min(g, w + c)$ or $\min(g, w + c) + 1$ to solve the puzzle.

More colloquially, this means that if you find the largest collection of objects that can be left alone, which will either be all of the goats or all of the wolves and cabbage, then you will need at least the rest of the objects in your boat or on another shore to keep an eye on them. You might be able to get away with exactly that much space in your boat, but with one extra space in the boat you can keep that small group in your boat and ferry one of the other objects across at a time.

While this is not hard to stumble upon for the wolf-goat-cabbage problem, the relation $\tau(G) \leq \text{Alcuin}(G) \leq \tau(G) + 1$ may not be as clear with more exotic objects.

Work has also been done on the problem of estimating the minimum boat capacity required to solve the puzzle in at most t crossings and the computational complexity of determining these values exactly. [CHW, LM]

AMS Subject Classification

- Operations research (90BXX)
 - Transportation logistics (90B06)
 - Deterministic scheduling (90B35)
- Mathematical Programming (90CXX)
 - Combinatorial optimization (90C27)
- Graph theory (05CXX)
 - Extremal problems (05C35)
 - Vertex subsets (05C69)

References

[A] M. Ascher, [*A river-crossing problem in cross-cultural perspective*](#), Math. Mag., 63 (1990), pp. 26–29.

[CHW] P. Csorba, C. A. J. Hurkens, and G. J. Woeginger, [*The Alcuin number of a graph and its connections to the vertex cover number*](#), SIAM J. Discrete Math. Vol. 24, No. 3, pp. 757–769.

[HS] J. Hadley and D. Singmaster, [*Problems to sharpen the young*](#), The Mathematical Gazette, Vol. 76, No. 475, The Use of the History of Mathematics in the Teaching of Mathematics (Mar., 1992), pp. 102–126.

[LM] M. Lampis and V. Mitsou, [*The ferry cover problem*](#), in Proceedings of the 4th International Conference on Fun with Algorithms (FUN 2007), Lecture Notes in Comput. Sci. 4475, Springer, Berlin, 2007, pp. 227–239.

[PS] I. Pressman and D. Singmaster, [*“The Jealous Husbands” and “The Missionaries and Cannibals”*](#), The Mathematical Gazette, vol. 73, no. 464, Mathematical Association, 1989, pp. 73–81.