

Normal distribution

The most important continuous probability distribution used in the entire field of statistics is the normal distribution (or Gaussian distribution). This was discovered by De-Moivre (1667 – 1754) in the year 1733. Its graph, called the normal curve, is a bell shaped curve that extends indefinitely in both directions, coming closer and closer to the horizontal axis without touching it. Most of the data relating to economics and business statistics or even in social and physical sciences conform to this distribution.

Probability Density function of Normal Distribution

A continuous random variable X is said to have a normal distribution (N.D) if its probability density function is given by:

$$P(x) = f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}; \quad -\infty < x < +\infty$$

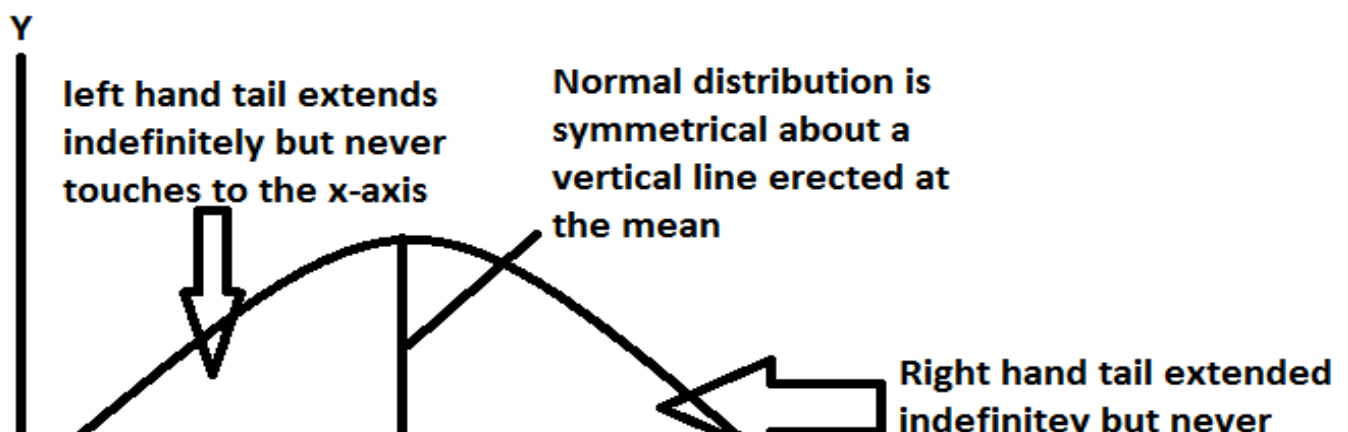
Where X = value of continuous random variable

μ = mean of the distributions

σ = the standard deviation of the distribution.

$$\pi = \frac{22}{7} = 3.142.$$

$e = 2.7183$ (the base of the system of natural logarithms).



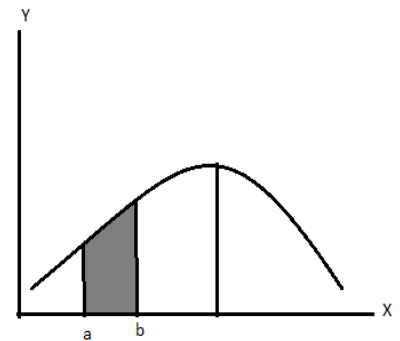
$$\bar{X} = M_d = M_o$$

Here, the mean μ and the standard deviation σ are called the parameters of the normal distribution. Note that $X \sim N(\mu, \sigma^2)$ is used to denote the normal distribution with a mean μ and variance σ^2 .

Properties of normal distributions

The followings are some of the properties of the normal distribution.

i. The area bounded by the normal curve and x -axis is equal to one.



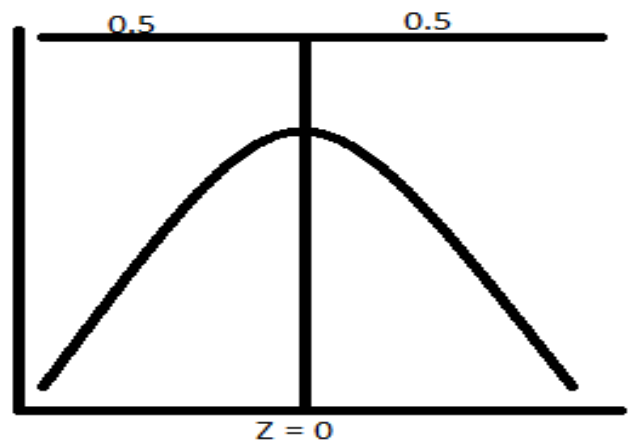
ii. $P(a < x < b)$, i.e. the probability that x lies between a and b is equal to the area bounded by the curve, x -axis and ordinates $X = a$ and b .

iii. The curve is bell-shaped and symmetrical about the line $X = \mu$ (i.e. $Z = 0$)

iv. The mean, median and mode of a normal distribution coincide with each other. Thus,

$$\text{Mean} = \text{Median} = \text{Mode} = \mu$$

v. The curve has a single peak, thus, it is unimodal.



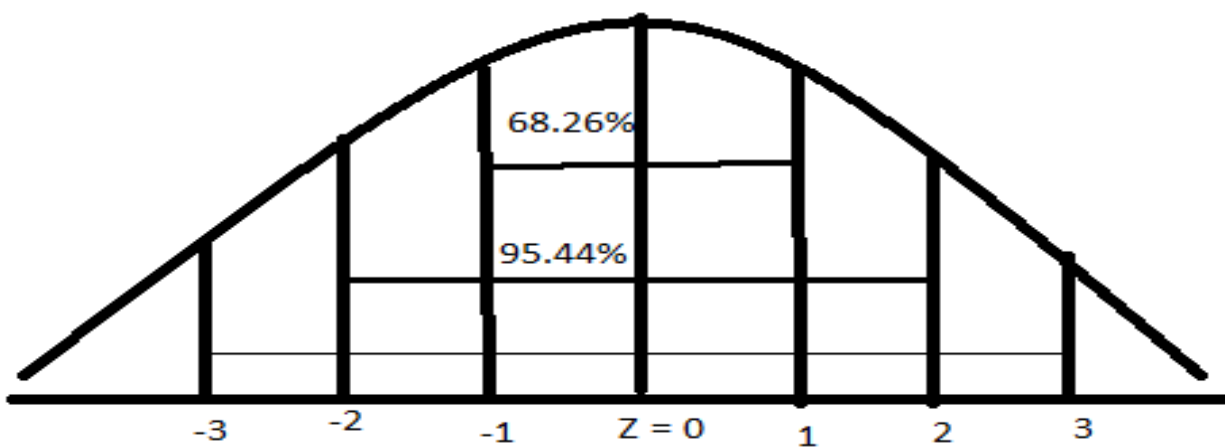
- vi. Since the ordinate at the mean (i.e. $X = \mu$ or $Z = 0$) of the distribution divides the total area under the normal curve into two equal parts and further, the total area under the normal probability curve is 1, the area to the right and left of the ordinate at $X = \mu$ (i.e. $Z = 0$) is 0.5.

vii. Area property

The most important property of the normal probability curve is the area property. If the mean and S.D. of a normal distribution are μ and σ respectively, then area under the normal curve between the ordinates at $X = \mu - \sigma$ and $X = \mu + \sigma$ is 0.6826

Similarly,

$$P(\mu - 2\sigma < X < \mu + 2\sigma) = 0.9544$$



Examples,

1. Findings the areas when the value of Z is known.

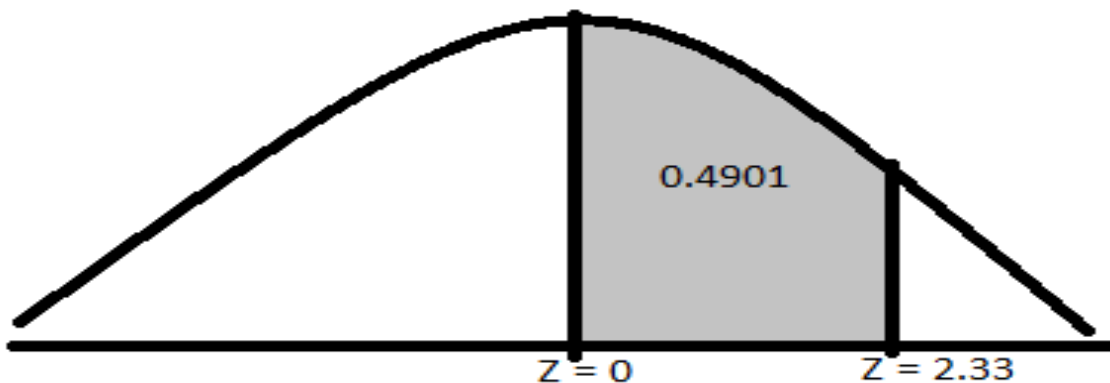
- Find the area under the normal curve between
 - $Z = 0$ and $Z = 2.33$ (i.e. area for $Z = 2.33$)
 - $Z = -2.62$ and $Z = 0$ (i.e. area for $Z = -2.62$)

Solution,

- From the standard normal table, the area between $Z = 0$ and $Z = 2.33$ is 0.4901. The shaded portion in the following figure denotes the area between $Z = 0$ and $Z = 2.33$

Thus, area bounded by the Z – curve, Z – axis and the ordinates $Z = 0$ and $Z = 2.33$ from the normal table is 0.4901.

Symbolically, it can be written as $P(0 < Z < 2.33) = 0.4901$.

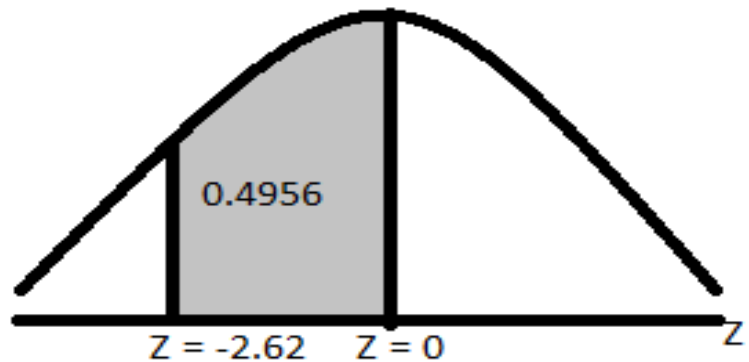


- The table given at the end of the book does not contains entries corresponding to negative value of Z , But, since the curve is symmetrical, we can find the area between $Z = -2.62$ and $Z = 0$ by looking the area corresponding to $Z = 2.62$. The shaded portion shown in the following figure represents the area between $Z = -2.62$ and $Z = 0$

Thus, area between $Z = -2.62$ and $Z = 0$ from normal table is 0.4956.

Symbolically, it can be written as

$$P(-2.62 < Z < 0) = P(0 < Z < 2.62) = 0.4956.$$



2. Find the area under the normal curve to the

a) Right of $Z = 0.35$

b) Left of $Z = 0.35$

Solution,

a. In order to find the area under normal curve to the right of $Z = 0.35$ (i.e. positive value of Z), we should subtract the tabular value from 0.5000. The reason is that the normal curve is symmetrical, the area to the right of the mean is equal to the area to the left of the mean and each area is 0.5000. The following shaded portion represents the area to the right of $Z = 0.35$

$$\text{Area (between } Z = 0 \text{ and } Z = 0.35) \text{ from normal table} = P(0 < Z < 0.35) = 0.1368$$

Since,

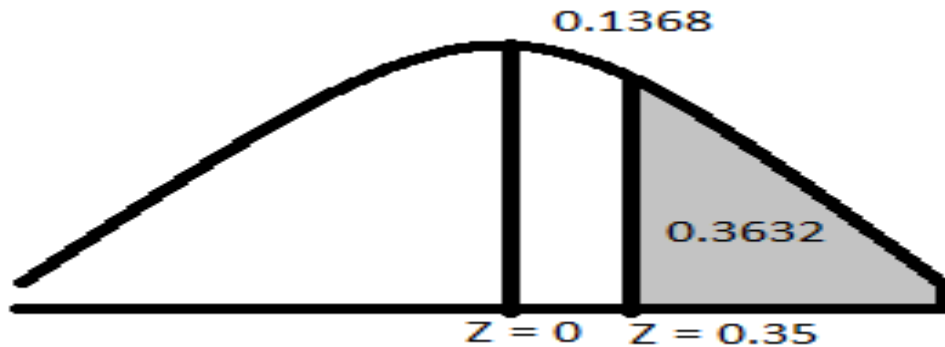
$$\text{Area to the right of mean } (Z = 0) = P(0 < Z < \infty) = 0.5000$$

$$\therefore \text{Area to the right of } (Z = 0.35)$$

$$= P(Z > 0.35)$$

$$= 0.5000 - 0.1368$$

$$= 0.3632$$



Symbolically,

It can be written as $P(Z > 0.35) = P(0 < Z < \infty) - P(0 < Z < 0.35) = 0.5000 - 0.1368 = 0.3632$.

b. Area (between $Z = 0$ and $Z = 0.35$) from normal table = 0.1368

Area to the left of mean (i.e. $Z = 0$) = 0.5000.

Area to the left of ($Z = 0.35$) =

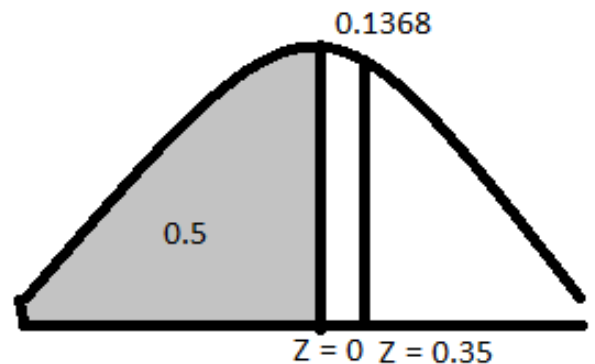
$$0.1368 + 0.5000 = 0.6368$$

Symbolically,

It can be written as $P(Z < 0.35) = P$

$(-\infty < Z < 0) + P(0 < Z < 0.35) =$

$$0.5000 + 0.1368 = 0.6368$$



3. Find the area under the normal curve to the

a. Left to $Z = -1.25$

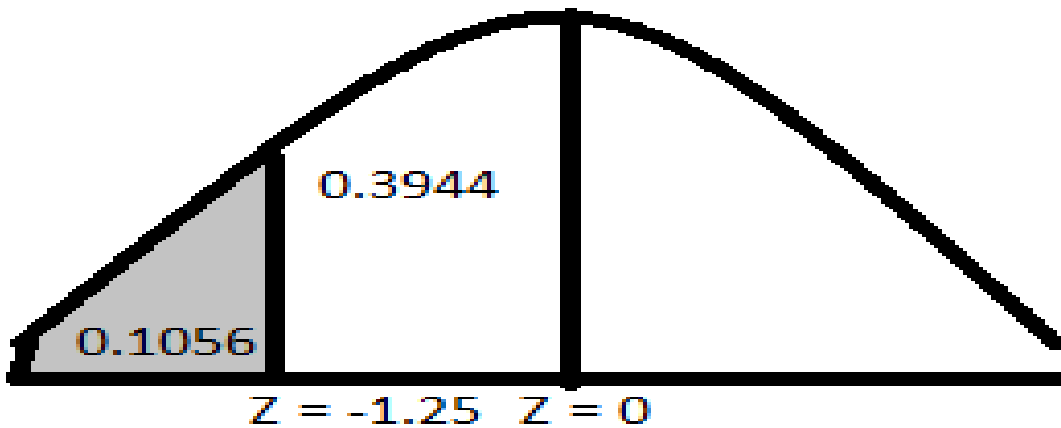
b. The right of $Z = -1.25$

Solution,

a. Area (between $Z = -1.25$ and $Z = 0$) from normal table = 0.3944

Area to the left of mean (i.e. $Z = 0$) = 0.5000

Area to the left of ($Z = -1.25$) = $0.5000 - 0.3944 = 0.1056$



Symbolically, it can be written as

$$P(Z < -1.25) = P(-\infty < Z < 0) - P(-1.25 < Z < 0)$$

$$= 0.5 - P(0 < Z < 1.25)$$

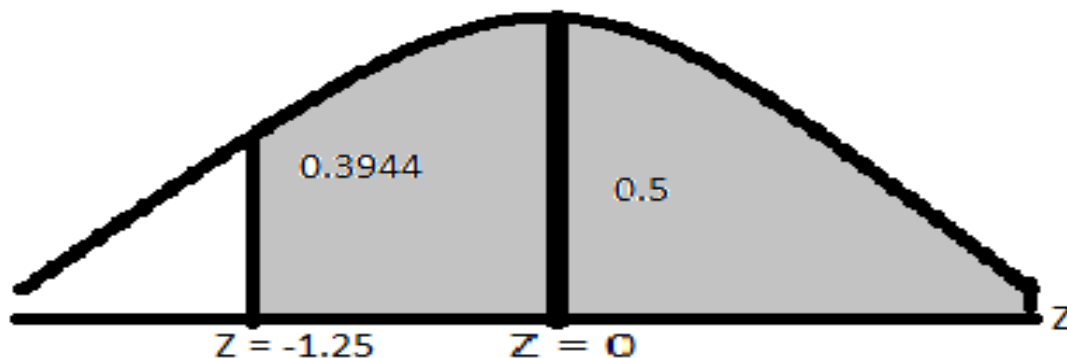
$$= 0.5 - 0.3944$$

$$= 0.1056.$$

b. Area (between $Z = -1.25$ to 0) from normal table = 0.3944 .

Area to the right mean (i.e. $Z = 0$) = 0.5000

Area to the right of ($Z = -1.25$) = $0.3944 + 0.5000 = 0.8944$



Symbolically, it can be written as

$$P(Z > -1.25)$$

$$= P(-1.25 < Z < 0) + P(0 < Z < \infty)$$

$$= P(0 < Z < 1.25) + 0.5$$

$$= 0.5 + 0.3944$$

$$= 0.8944.$$

Standard Normal Distribution

Since the location and shape of normal distribution depends upon the values of its parameter (μ and σ), there are different bell-shaped curves with different means and standard deviations.

In order to avoid the complicated task of preparing for each value of (μ and σ), a special form of normal distributions is used to convert these normal distributions into one standardized form by making a very simple transformation, which is called a standard normal curve.

The normal probability curve with mean μ and standard deviation σ is given by

$$P(x) = f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}; \quad -\infty < x < +\infty$$

The continuous random variable Z, defined as

$$Z = \frac{X - \mu}{\sigma}$$

Is called the normal variate (S.N.V) corresponding to x . The standard normal distribution or Z-distribution is given by

$$P(Z) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{Z^2}{2}}, \quad -\infty < Z < \infty$$

Relation between binomial and normal distributions

Normal distributions is limiting case of the binomial probability distribution under the following conditions

- i. n , the number of trials is indefinite large, i.e. $n \rightarrow \infty$
- ii. Neither p nor q is very small.

Let X is the binomial variable with parameters p and n . then

$$E(X) = np \text{ and } \text{Var}(X) = npq$$

Under the above conditions, the distributions of standard normal variate.

$$Z = \frac{X - E(X)}{\sigma_x} = \frac{X - np}{\sqrt{npq}}, \text{ tends to the distributions of standard normal variate}$$

Relation between Poisson and normal distributions

Let X be the Poisson variate with parameter λ , then

$$E(X) = \text{mean} = \lambda, \text{ and } \text{Var}(X) = \sigma^2 = \lambda$$

It can be proved that standard Poisson

$$\text{variate} = Z = \frac{X - E(X)}{\sigma_x} = \frac{X - \lambda}{\sqrt{\lambda}},$$

Tends to be a standard normal variate $\lambda \rightarrow \infty$. Thus, normal distributions is a limiting case of Poisson distribution as the parameter $\lambda \rightarrow \infty$

Method of measuring Area (distance) Under the normal Curve by Using the Standard Normal Probability Distribution table

The standard normal probability distribution table shows the area under the normal curve between the mean and any value of the normally distributed random variable (X). This normal curve table is given at the end of the book. Notice that in this table the location of the column labeled Z . The value of Z is calculated from the following formula.

$$Z = \frac{X - \mu}{\sigma}$$

where,

X = value of the random variable

μ = mean of the distribution

σ = standard deviation of this distribution

Z = standard normal variate

(or number of standard deviation from X to the mean of this distribution.)

Examples,

4. Find the area between

i. $Z = 0.30$ and 1.50

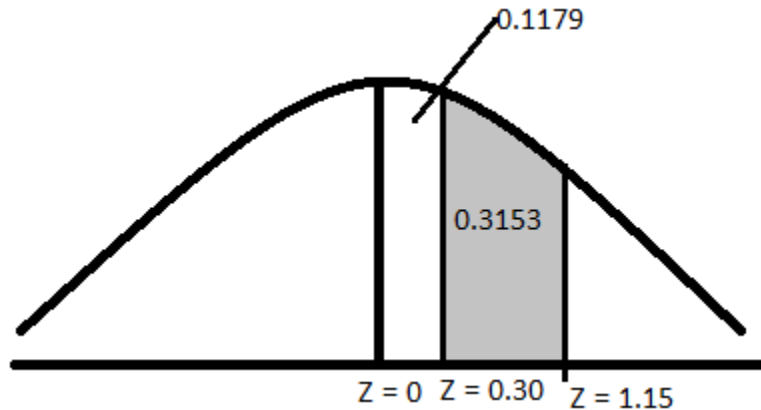
ii. $Z = -1.25$ to -0.25

Solution,

i. Area (between $Z = 0$ to 0.30) from the normal table = 0.1179

Area (between $Z = 0$ to 1.50) from the normal table = 0.4332

Area (between $Z = 0.30$ to 1.50) = $0.4332 - 0.1179 = 0.3153$



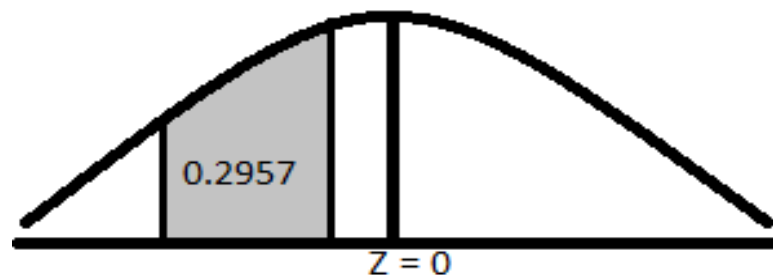
Symbolically,

*It can be written as $P(0.30 < Z < 1.50) = P(0 < Z < 1.50) - P(0 < Z < 0.30)$
 $= 0.4332 - 0.1179 = 0.3153$*

ii. Area (between $Z = 0$ to -1.25) from the normal table = 0.3944

Area (between $Z = 0$ to -0.25) from the normal table = 0.0987

Area (between $Z = -1.25$ to -0.25) = $0.3944 - 0.0987 = 0.2957$



Symbolically,

It can be written as $P(-1.25 < Z < -0.25) = P(0 < Z < -1.25) - P(0 < Z < -0.25) = 0.3944 - 0.0987 = 0.2957$

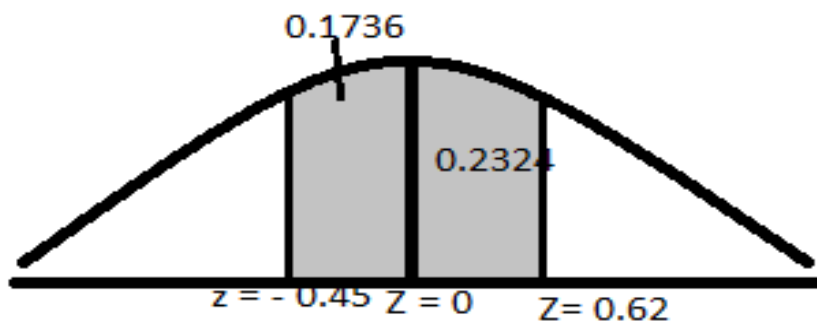
5. Find the area under the normal curve between $Z = -0.45$ and $Z = 0.62$

Solution,

Area (between $Z = 0$ to -0.45) from the normal table = 0.1736

Area (between $Z = 0$ to 0.62) from the normal table = 0.2324

Area (between $Z = -0.45$ to 0.62) = $0.1736 + 0.2324 = 0.4060$



Symbolically,

It can be written as $P(-0.45 < Z < 0.62) = P(0 < Z < -0.45) + P(0 < Z < 0.62) = 0.1736 + 0.2324 = 0.4060$

6. Let Z be a standard normal variety. Find (Homework)

i. $P(0 < Z < 1)$

ii. $P(-0.5 < Z < 0)$

iii. $P(-0.60 < Z < 0.42)$

iv. $P(0.6 < Z < 1.70)$

v. $P(Z < -2)$

vi. $P(Z > 1.32)$

vii. $P(-1.96 < Z < 1.96)$

viii. $P(-2.58 < Z < 2.58)$

ix. $P(-3 < Z < 3)$

x. $P(-1.23 < Z < 0.8)$

Finding the area when the score is known

7. Students of a class are given a test. Their marks were normally distributed with mean 60 and standard deviation 5. What percentage of students scored:

i. More than 60 marks

ii. Less than 50 marks

iii. Between 45 and 65 marks

iv. Between 65 and 75 marks?

Solution,

Let a random variable X denotes the marks obtain by the students. Let Z be the corresponding standard normal variable.

Then,

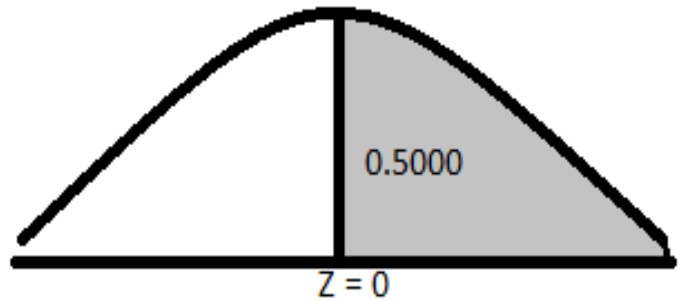
$$Z = \frac{X - \mu}{\sigma} \text{ where, } \mu = 60, \sigma = 5$$

$$\therefore Z = \frac{X - 60}{5}$$

i. $P(\text{more than 60 marks}) = P(X > 60) = ?$

When $X = 60$, then

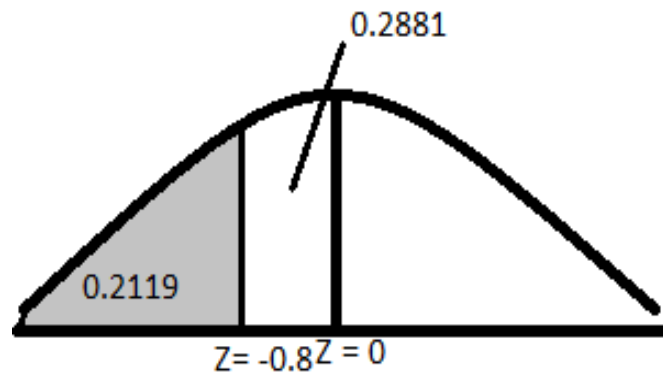
$$Z = \frac{60-60}{5} = \frac{0}{5} = 0$$



$$\therefore P(X > 60) = P(Z > 0) = 0.5 = 50\%.$$

ii. $P(\text{less than 56 marks}) = P(X < 56) = ?$

$$\text{When } X = 56, \text{ then } Z = \frac{56-60}{5} = -\frac{4}{5} = -0.8$$



$$\begin{aligned} &\therefore \text{Area (between } X = 56 \text{ and } \mu = 60) \\ &= P(56 < X < 60) \\ &= P(-0.8 < Z < 0) \\ &= P(0 < Z < 0.8) \\ &= 0.2881 \end{aligned}$$

$$\begin{aligned} \therefore \text{Area to the left of } Z > -0.8 &= P(X < 56) = P(Z < -0.8) \\ &= P(Z > 0.8) \text{ (by symmetry of normal curve)} \\ &= 0.5 - P(0 < Z < 0.8) = 0.5 - 0.2881 = 0.2119 = 21.19\% \end{aligned}$$

Alternative method

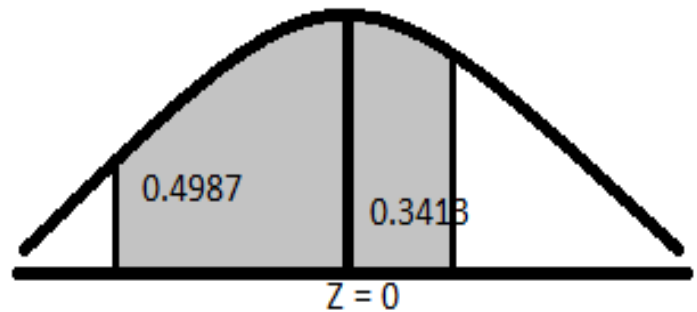
$$P(\text{less than 56 marks}) = P(X < 56) = P(Z < -0.8) = 0.5 - (-0.8 < Z < 0) = 0.5 - 0.2881 = 0.2119 = 21.19\%.$$

iii. $P(\text{between 45 and 65 marks}) = P(45 < X < 65) = ?$

$$\text{When } X = 45, Z_1 = \frac{45-60}{5} = -3 \text{ and When } X = 65,$$

$$Z_2 = \frac{65-60}{5} = 1$$

$$\begin{aligned} \therefore P(45 < X < 65) &= P(-3 < Z < 1) \\ &= P(-3 < Z < 0) \\ &+ P(0 < Z < 1) \\ &= 0.4987 + 0.3413 \\ &= 0.84 = 84\% \end{aligned}$$



Hence, required percentage of students getting marks between 45 and 65 is 84%.

iv. $P(\text{between 65 and 75 marks}) = P(65 < X < 75) = ?$

$$\text{When } X = 65, Z_1 = \frac{65-60}{5} = 1, \text{ When } X = 75, Z_2 = \frac{75-60}{5} = 3$$

$$\therefore P(65 < X < 75)$$

$$\begin{aligned}
&= P(1 < Z < 3) \\
&= P(3 < Z < 0) \\
&- P(0 < Z < 1) \\
&= 0.4987 - 0.3413 \\
&= 0.1574 = 15.74\%
\end{aligned}$$

Hence, required percentage of students getting marks between 65 and 75 is 15.74%.

8. The IQ score of students is normally distributions with mean of 120 and standard deviation of 20.

a) Find the score so that 20% of the students have an IQ above the score.

b) Find the score so that 70% of the students have an IQ below the score.

c) What are the limits within which the central (middle) 50% of the score lie?

Solution,

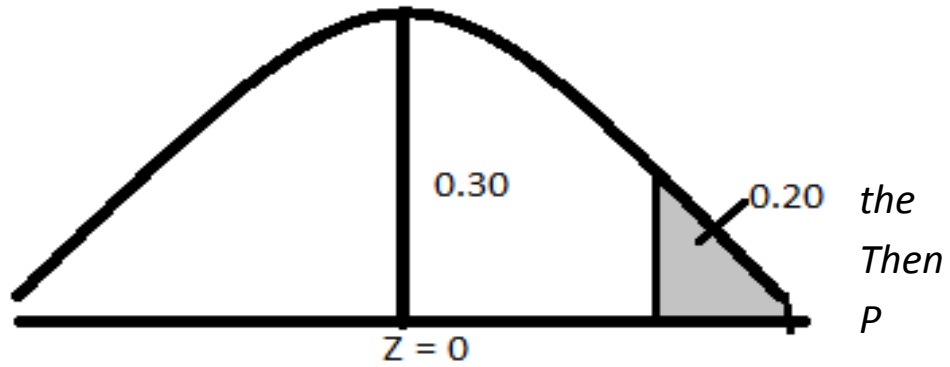
Let X be the normal variable score of students. Let Z be the S.N.V of X.

then, $Z = \frac{X - \mu}{\sigma}$, here $\mu = 120$, $\sigma = 20$

$$\therefore Z = \frac{X - 120}{20}$$

a) Let X_1 be the normal variable score so that 20% of the students have an IQ above this score. Then, $P(X > X_1) = 20\% = 0.2$

Let Z_1 be
S.N.V. of X_1 .
 $P(X > X_1) =$
 $(Z > Z_1) = 0.2$



Therefore, $P(0 < Z < Z_1) = 0.5 - P(Z > Z_1) = 0.5 - 0.2 = 0.3$

Since the area of normal table are only between the mean and any given value of X , and such area between mean score 120 and score X_1 is $P(120 < X < X_1) = P(0 < Z < Z_1) = 0.30$

Since we are looking at the 20%, it is clear that the score must be above the mean. Hence, Z-score is positive.

Therefore, $Z_1 = 0.84$

We get, $Z_1 = \frac{X_1 - 120}{20}$

Or, $0.84 = \frac{X_1 - 120}{20}$

Or, $0.84 \times 20 = X_1 - 120$

Or, $X_1 = 120 + 16.8 = 136.8$

This means that 20% of the students have an IQ above 136.8 and 80% of the students have an IQ below 136.8

9. Find Z , if the area between 0 and Z is

i. 0.4887

ii. 0.4549

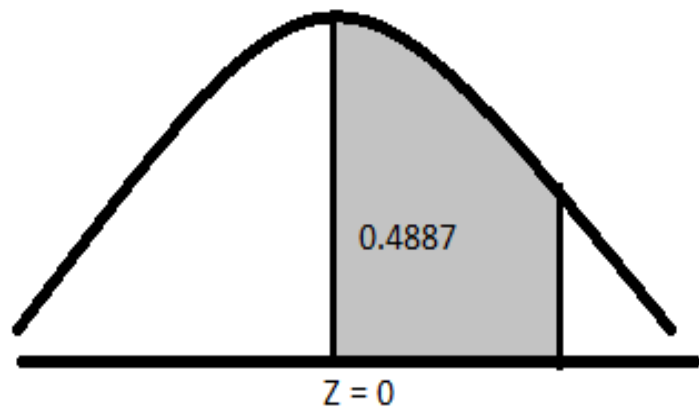
iii. 0.8621 to the left of Z .

Solution,

Area (between $Z = 0$ and $Z = Z_1$,

So, the corresponding value of Z_1 from the normal table is 2.28.

Therefore, $Z_1 = 2.28$



10. A normal curve has $\bar{X} = 20$ and $\sigma = 10$.

find the area between $X_1 = 15$ and $X_2 = 40$

Solution,

$Z_1 = \frac{X_1 - \mu}{\sigma}$ and $Z_2 = \frac{X_2 - \mu}{\sigma}$ and find the value of $Z_1 < Z < Z_2$

11. Assume that the mean height of soldiers to be 68.52 inches with a variance of 10.8 inches. How

many soldiers in a regiment of 1000 would you expect to be over six feet tall?

Solution,

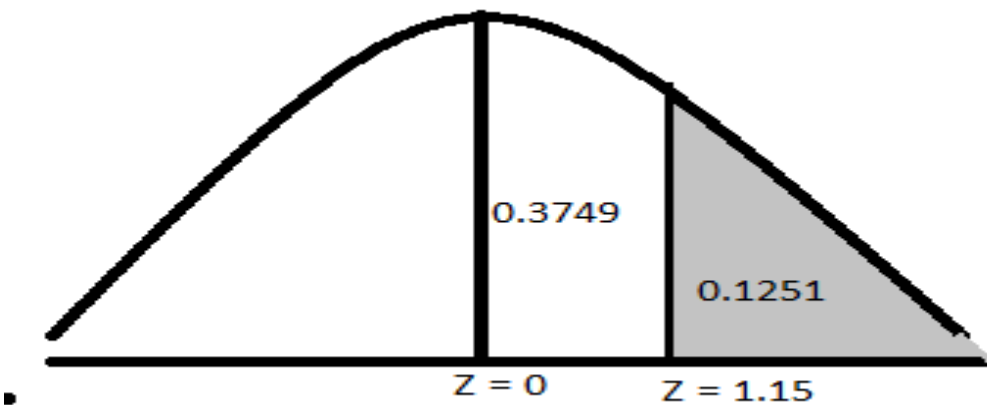
Assume that the distribution of height is normal. Let X be the height of soldiers and Z be the S.N.V. of X .

$$\text{Then, } Z = \frac{X - \mu}{\sigma},$$

$$\text{Here } \mu = 68.52, \sigma = \sqrt{10.8} = 3.286$$

$$X = 6 \text{ feet} = 6 \times 12 \text{ inches} = 72$$

$$\therefore Z = \frac{72 - 68.52}{3.286} = 1.15$$



Area (between $Z = 0$ and $Z = 1.15$) from the normal table = $P(0 < Z < 1.15)$

$$= P(Z > 0) - P(Z < 1.15) = 0.5 - 0.3749 = 0.1251$$

Now, $N = 1000$

The expected number of soldiers over six feet tall = $N \times P(X > 72) = 1000 \times 0.1251 = 125.1$ or 125

12. The average test marks in particular class is 79. The standard deviation is 5. If the marks are distributed normally, how many students in a class of 200 did not receive marks between 75 and 82?

Given $P(0 \leq Z \leq 0.7) = 0.2580$ $P(0 \leq Z \leq 0.8) = 0.2881$ and $P(0 \leq Z \leq 0.6) = 0.2257$

Solution,

If the random variable X denotes the marks obtain by the students in the given test we are given

$$X \sim N(\mu, \sigma^2), \text{ where } \mu = 79 \text{ and } \sigma = 5$$

$$\text{Let } Z \text{ be the S. N. V. of } X. \text{ then } Z = \frac{X - \mu}{\sigma} = \frac{X - 79}{5}$$

$$\text{When } X_1 = 75, \text{ then } Z = \frac{75 - 79}{5} = -\frac{4}{5} = -0.8$$

$$\text{When } X_2 = 82, \text{ then } Z = \frac{82 - 79}{5} = \frac{3}{5} = 0.6$$

Now, the area (between $X_1 = 75$ and $X_2 = 82$)

$$= P(75 \leq X \leq 82) = P(-0.8 \leq Z \leq 0.6)$$

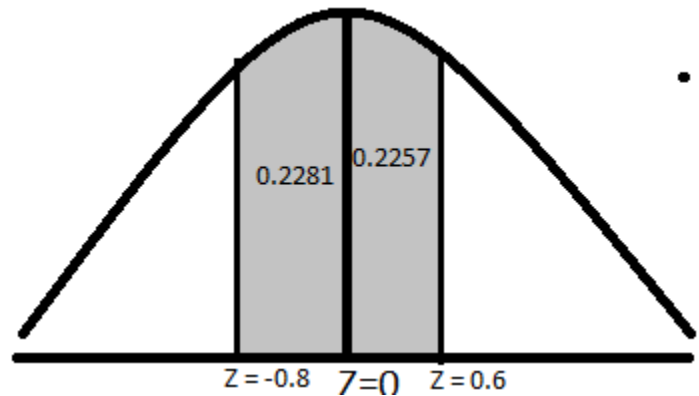
$$= P(-0.8 \leq Z \leq 0) + P(0 \leq Z \leq 0.6)$$

$$= 0.2881 + 0.2257 = 0.5138$$

Hence, the probability that a student gets marks between 75 and 82 is given by $P(75 \leq X \leq 82) = 0.5138$

The probability that a student does not get a marks between 75 and 82 is given by $(q) = 1 - P(75 \leq X \leq 82)$

$$= 1 - 0.5138 = 0.4862.$$



Hence, in a class of 200 students, the number of students who did not receive marks between 75 and 82 is given by

$$N \times q = 200 \times 0.4862 = 97.24 \text{ or } 97.$$

13. The income of group of 10000 persons was found to be normally distributed with mean Rs. 750 per month and standard deviation is Rs. 50. Of this group, show that

i. About 95% had income exceeding Rs. 668

ii. Only 5% had income exceeding Rs. 832.

iii. What was the lowest income among the richest 100?

Solution,

Let X denotes the variable income of persons. Let Z be the corresponding standard normal variate

Then,

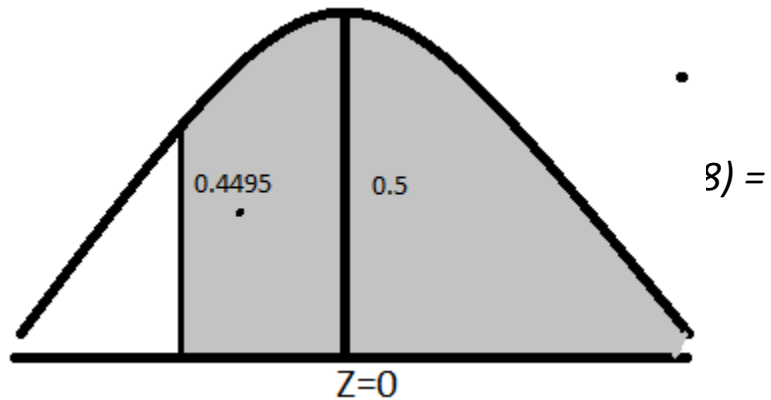
$$\text{Let } Z \text{ be the S. N. V. of } X. \text{ then } Z = \frac{X - \mu}{\sigma} = \frac{X - 750}{50}$$

$$\text{When } X_1 = 668, \text{ then } Z = \frac{668 - 750}{50} = -\frac{82}{50} = -1.64$$

When $X_2 = 832$ then $Z = \frac{832 - 750}{82} = 1.64$

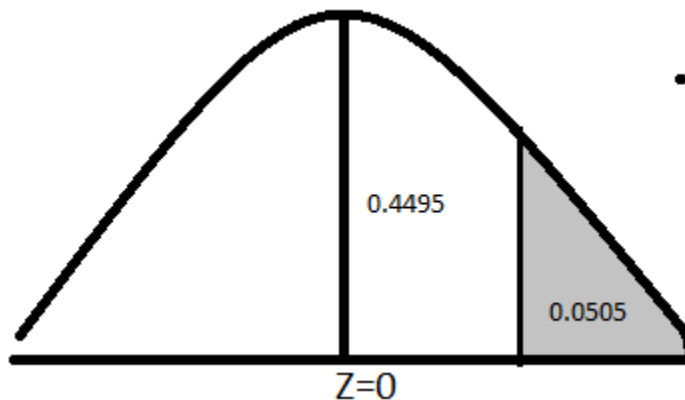
We have to show that i) P

- i. Probability of a pers
 $P(Z > -1.64) = P(-1.64 < Z < \infty)$
 $0.9495 = 0.95$



\therefore Percentage of persons getting more than Rs. 668 = $N \times P(X > 668) = 10000 \times 0.95 = 9500$

- ii. Probability of a person's income exceeding Rs. 832 = $P(X > 832) = P(Z > 1.64) = 0.5000 - P(0 < Z < 1.64) = 0.5000 - 0.4495 = 0.0505 = 0.05$



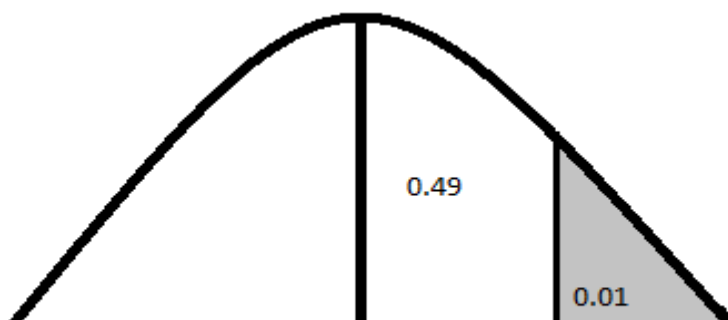
\therefore Percentage of persons getting more than Rs. 832 = $N \times P(X > 832) = 10000 \times 0.0505 = 505$

- iii. Probability of person being among the richest 100 = $100/10000 = 0.01$

Let Rs. X_1 be the lowest income among richest 100.

Then, $P(X > X_1) = 0.01$

Let Z_1 be the S.N.V. of X_1 . Then



$$Z_1 = \frac{X_1 - \mu}{\sigma} = \frac{X_1 - 750}{50}$$

Now, $P(X > X_1) = 0.01$

Or, $P(Z > Z_1) = 0.01$

Or, $0.5 - P(0 < Z < Z_1) = 0.01$

Or, $P(0 < Z < Z_1) = 0.05 - 0.01 = 0.49$

The value of Z_1 at the area 0.49 from normal table is 2.33

Therefore, $Z_1 = 2.33$

So, we get

$$Z_1 = \frac{X_1 - 750}{50}$$

$$\text{Or, } 2.33 = \frac{X_1 - 750}{50}$$

$$\therefore X_1 = 866.50$$

Hence, the lowest income among richest 100 is Rs. 866.50

14. Incomes of a group of 1000 persons were found to be normally distributed with mean Rs. 520 and standard deviation Rs. 60. Find

i. The highest income of poorest 10% of the persons.

ii. Lowest income of richest 20% of the persons.

Solutions,

Let X denotes the normal variable income of persons. Let Z be the corresponding S.N.V. then,

$$Z = \frac{X - \mu}{\sigma}$$

Here, $\mu = 520$ and $\sigma = 60$

$$\therefore Z = \frac{X - 520}{60}$$

i. Let X_1 be the highest income of poorest 10% of the persons. Then

$$P(X < X_1) = 0.10$$

Let Z_1 be the corresponding S.N.V. of X_1 . Then

$$Z_1 = \frac{X_1 - 520}{60}$$

$$\text{Now, } P(X < X_1) = 0.10$$

$$\text{Or, } P(Z < Z_1) = 0.10$$

$$\text{Or, } 0.5000 - P(-Z_1 < Z < 0) = 0.10$$

$$\text{Or, } P(-Z_1 < Z < 0) = 0.5000 - 0.10 = 0.40.$$

Therefore, the value of Z_1 at probability 0.40 is 1.28

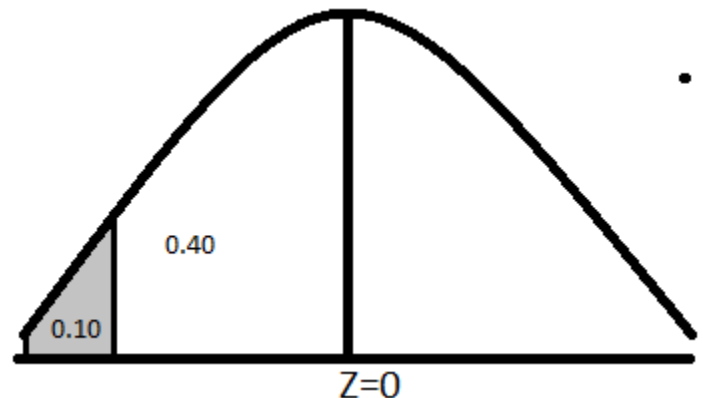
But, it lies below mean

$$\text{So, } Z_1 = -1.28$$

And,

$$Z_1 = \frac{X_1 - 520}{60}$$

$$\text{Or, } -1.28 = \frac{X_1 - 520}{60}$$



$$\therefore X_1 = 443.2$$

Hence, the highest income of poorest 10% of the person is Rs. 443.20

ii. Let X_2 be the lowest income of richest 20% of the persons. Then
 $P(X > X_2) = 0.20$

Let Z_2 be the corresponding S.N.V. of X_1 . Then

$$Z_2 = \frac{X_2 - 520}{60}$$

Now, $P(X > X_2) = 0.20$

Or, $P(Z > Z_2) = 0.20$

Or, $0.5000 - P(0 < Z < Z_2) = 0.20$

Or, $P(0 < Z < Z_2) = 0.5000 - 0.20 = 0.30$.

Therefore, the value of Z_2 at probability 0.30 is 0.84

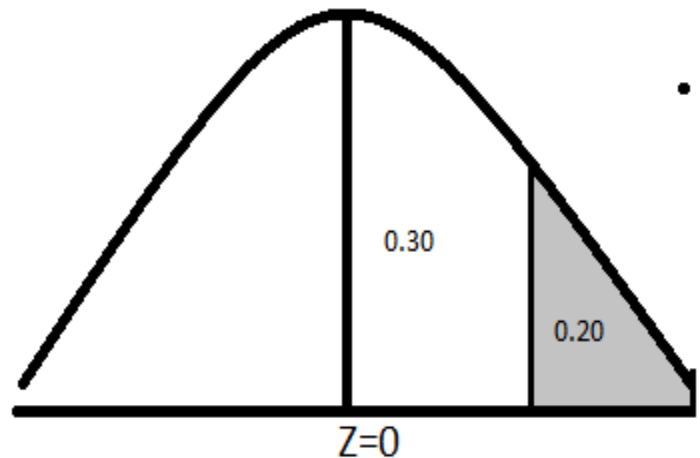
But, it lies above the mean

So, $Z_2 = 0.84$

And,

$$Z_2 = \frac{X_2 - 520}{60}$$

$$\text{Or, } 0.84 = \frac{X_2 - 520}{60}$$



$$\therefore X_2 = 570.40$$

Hence, the highest income of poorest 20% of the person is Rs. 570.40

15. A sales tax officer has reported that the average sales of the 500 business that he has to deal with during a year amount to Rs. 36000 with standard deviation of Rs. 10000. Assuming that the sales in this business are normally distributed, find

- i. The number of business, the sales of which are over Rs. 40000.
- ii. The percentage of business, the sales of which are likely to range between Rs. 30000 and Rs. 40000. Ans: 38.11%
- iii. The probability that the sales of a business selected at random will be over Rs. 30000. Proportions of area under the normal curve. Ans: 0.7257

Z	0.25	0.40	0.50	0.60
Area	0.0987	0.1554	0.1915	0.2257

Solution,

Let the random variable X denote the sales of the business during a year. Let Z be the corresponding S.N.V. then

$$Z = \frac{X - \mu}{\sigma}$$

Here, $\mu = 36000$ and $\sigma = 10000$

$$\therefore Z = \frac{X - 36000}{10000}$$

i. When $X = 40000$ then, $Z = \frac{40000 - 36000}{10000} = 0.4$

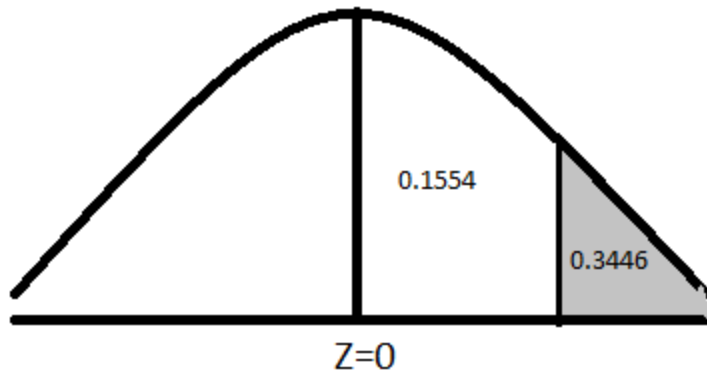
P (the sales of a business in over RS. 400000) = $P(X > 40000) = P(Z > 0.4) = 0.5 - P(0 \leq Z \leq 0.4) = 0.5 - 0.1554 = 0.3446$

Hence in a group of 500 businesses, the expected number of business with annual sales over Rs. 40000 is

$$\begin{aligned}
 N \times P(X > 40000) &= \\
 500 \times 0.3446 &= 172.3 \\
 &= 172.
 \end{aligned}$$

ii. Find $P(30000 < X < 40000) = ?$

iii. The probability that the annual sales of business selected at random will be over Rs. 30000 is given by $P(X > 30000) = P(Z > -0.6)$



16. The arithmetic mean of purchases per day by a customer in a large store is Rs.25 with a standard deviation of Rs. 10. If on a particular day, 100 customers purchased for Rs. 37.80 or more, estimate the total number of customers who purchased from the store that day.

Given that the normal area between $Z = 0$ and $Z = 1.28$ is 0.4000 , where Z is a standard normal variate.

Solution,

Let X denotes the purchases per day by a customer in a large store. Let Z be the corresponding S.N.V. then

$$Z = \frac{X - \mu}{\sigma}$$

Here, $\mu = 25$ and $\sigma = 10$

$$\therefore Z = \frac{X - 25}{10}$$

The probability of purchasing of Rs. 37.80 or more = $P(X \geq 37.80)$

Let the total no. of customers be N .

$$\therefore P(X \geq 37.80) = \frac{100}{N}$$

$$\text{When } X = 37.80, Z = \frac{37.80 - 25}{10} = 1.28$$

$$\therefore P(X \geq 37.80) = P(Z \geq 1.28)$$

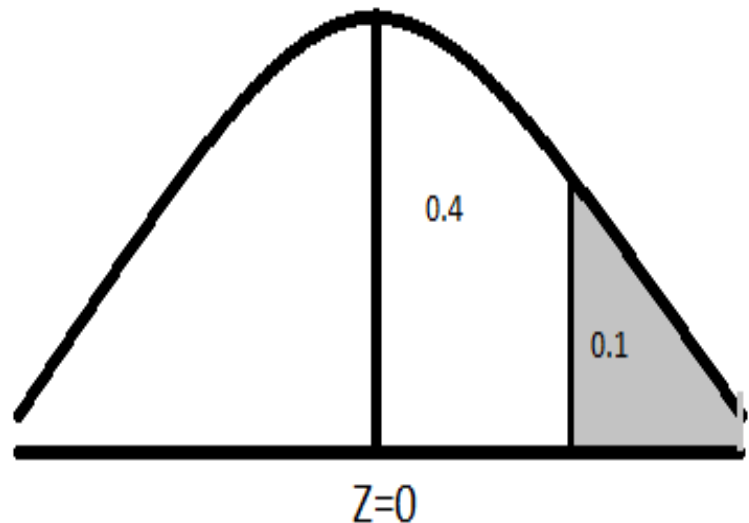
$$= 0.5 - P(0 < Z < 1.28)$$

$$= 0.5 - 0.4000 = 0.1$$

$$\text{So, } P(X \geq 37.80) = \frac{100}{N}$$

$$\text{Or, } 0.1 = \frac{100}{N}$$

$$\therefore N = \frac{100}{0.1} = 1000$$



Hence, the total number of customers who purchased from the store that day is 1000.

17. Given a normal distribution with mean is 50 and standard deviation is 10. Find the value of X that has

i. 5% of the area to its left. Ans: 33.6

ii. 14% of the area to its right. Ans: 60.8

Same above the question no. 14

18. The marks obtained by the students in a degree examination are normally distributed. The mean marks and S.D. of the distribution are 500 and 100 respectively, if 674 appeared in the examination and out of those, 550 are to be declared passed, what should be the minimum pass marks?

Solution,

Let X be the normal variable score of students. Let Z be the S.N.V of X .

$$\text{then, } Z = \frac{X - \mu}{\sigma},$$

$$\text{Here } \mu = 500, \sigma = 100$$

$$\therefore Z = \frac{X - 500}{100}$$

Let the minimum pass marks be X_1 .

$$\text{Then, } P(X > X_1) = \frac{550}{674} = 0.816$$

Let Z_1 be the corresponding S.N.V. of X_1 .

$$\text{Then, } Z_1 = \frac{X_1 - 500}{100}$$

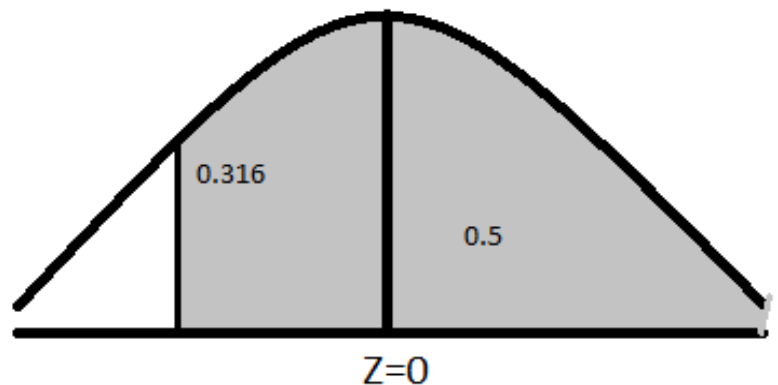
$$\text{Now, } P(X > X_1) = 0.816$$

$$\text{Or, } P(Z > -Z_1) = 0.816$$

$$\text{Or, } 0.5 + P(-Z_1 < Z < 0) = 0.816$$

$$\text{Or, } P(-Z_1 < Z < 0) = 0.816 - 0.5$$

$$\text{Or, } P(0 < Z < Z_1) = 0.316$$



The value of Z_1 at probability 0.316 is -0.9

Therefore, $Z_1 = -0.9$

$$Z_1 = \frac{X_1 - 500}{100}$$

$$\text{Or, } -0.9 = \frac{X_1 - 500}{100}$$

$$\therefore X_1 = 410$$

Hence, minimum pass marks is 410.

19. The life of electronic tubes of a certain type may be assumed to be normally distributed with mean 155 hours and standard deviation 19 hours. What is the probability that the life of a randomly chosen tube is

i. Less than 117 hours, Ans: 0.0228

ii. Between 136 hours and 194 hours, Ans: 0.8211

iii. More than 395 hours Ans: 0

Solution,

i. Find $P(X < 117)$

ii. Find $P(136 < X < 194)$

iii. Find $P(X > 395)$

20. In a certain examination, the percent of passes and distinction were 45 and 9 respectively, Estimate the mean and standard deviation obtained by the candidates, the minimum pass and distinction marks being 40 and 75. (Assume the distribution of marks to be normal)

Solution,

Let X denotes the normal variable marks. Let μ and σ be the mean and standard deviation of X respectively, Let Z be the S.N.V. of X . then

$$Z = \frac{X - \mu}{\sigma}$$

Now, percent of passes = 45%

Therefore, $P(X > 40) = 0.45$

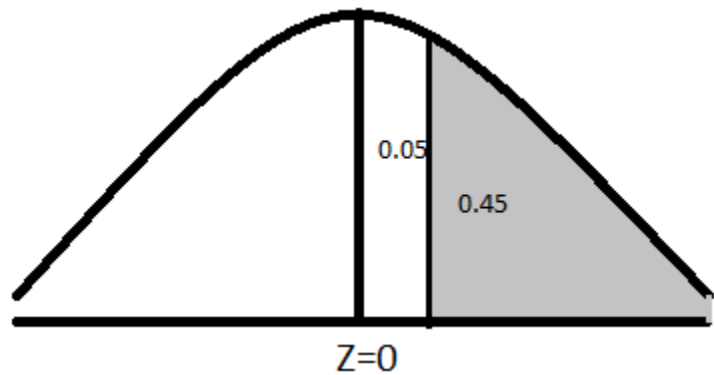
When $X = 40$, then $Z = \frac{40 - \mu}{\sigma} = Z_1$ (say)

Now, $P(X > 40) = 0.45$

Or, $P(Z > Z_1) = 0.45$

Or, $0.5 - P(0 < Z < Z_1) = 0.45$

Or, $P(0 < Z < Z_1) = 0.5 - 0.45$
 $= 0.05$



The value of Z_1 at probability 0.05 is 0.12

$$\therefore Z_1 = 0.12$$

$$\text{Now, } Z_1 = \frac{40 - \mu}{\sigma}$$

$$\text{Or, } 0.12 = \frac{40 - \mu}{\sigma}$$

$$\text{Or, } 0.12 \times \sigma = 40 - \mu$$

$$\therefore \mu = 40 - 0.12 \times \sigma \dots (1)$$

Also, percentage of distinction = 9%

$$\therefore P(X > 75) = 0.09$$

When $X = 75$ then

$$Z = \frac{75 - \mu}{\sigma} = Z_2(\text{say})$$

$$\text{Now, } P(X > 75) = 0.09$$

$$\text{Or, } P(Z > Z_2) = 0.09$$

$$\text{Or, } 0.5 - P(0 < Z < Z_2) = 0.09$$

$$\text{Or, } P(0 < Z < Z_2) = 0.5 - 0.09 = 0.41$$

The value of Z_2 at probability 0.41 is 1.34

$$\therefore Z_2 = 1.34$$

$$\text{Now, } Z_2 = \frac{75 - \mu}{\sigma}$$

$$\text{Or, } 1.34 = \frac{75 - \mu}{\sigma}$$

$$\text{Or, } 1.34 \times \sigma = 75 - \mu$$

$$\therefore \mu = 75 - 1.34 \times \sigma \dots (2)$$

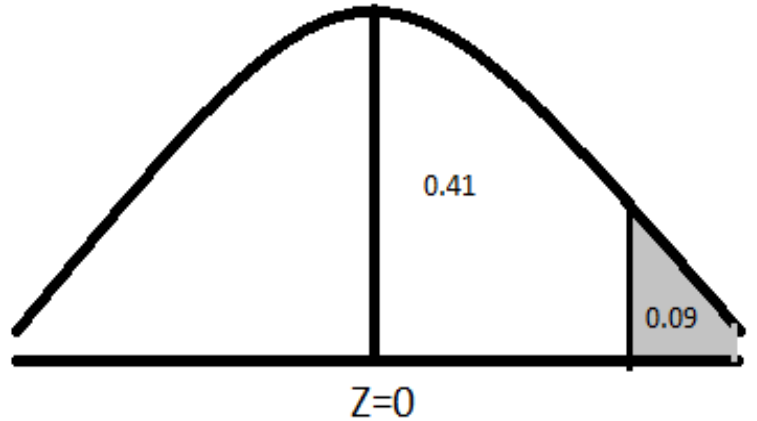
from (1) and (2)

$$\mu = \mu$$

$$\text{Or, } 40 - 0.12 \times \sigma = 75 - 1.34 \times \sigma$$

$$\text{Or, } -0.12\sigma + 1.34\sigma = 75 - 40$$

$$\text{Or, } 1.22\sigma = 35$$



$$\therefore \sigma = \frac{35}{1.22} = 28.69 = 29$$

And taking (1)

$$\therefore \mu = 40 - 0.12 \times 29 = 40 - 3.48 = 36.52 = 37$$

Hence, mean is 37 and standard deviation is 29.

21. In a distribution exactly normal, 7% of the item are under 35 and 89% are under 63. What are the mean and standard deviation of the distribution? (Given $Z_{0.43} = 1.48$ and $Z_{0.39} = 1.23$)

Ans: mean = 50.3 and standard deviation = 10.33

Solution,

$$P(X < 35) = 0.07 \text{ and } P(X < 63) = 0.89$$

22. In an examination, it was observed that 10% of the candidates got first class and 30% failed. The pass marks is 40% and first division marks is 60%. Assuming that the marks follow normal distribution, find the mean and standard deviation.

Ans: mean = 11.11 and standard deviation = 45.75

Solution,

$$P(X < 40) = 0.30 \text{ when } X = 40$$

$$P(X > 60) = 0.10 \text{ When } X = 60$$

23. A normal distribution has 77 as mean. Find the standard deviation if 20% of the area under the normal curve lies to the right of 90.

Solution,

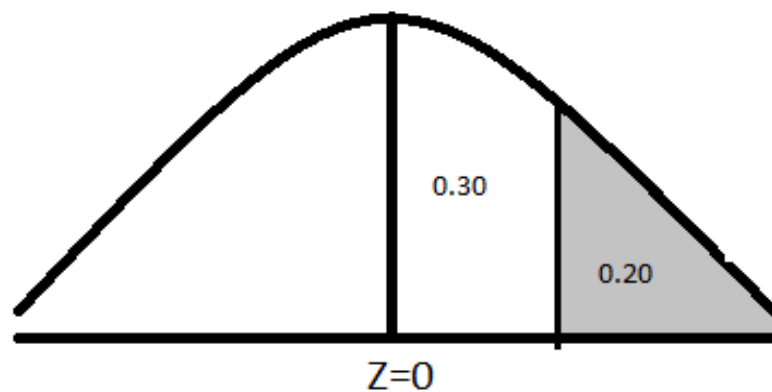
Let X be the normal variable and Z be its S.N.V. of X . then

$$Z = \frac{X - \mu}{\sigma}$$

Here, $\mu = 77$, $\sigma = ?$

$$\therefore Z = \frac{X - 77}{\sigma}$$

By the given condition, $P(X > 90) = 0.20$



When $X = 90$ then $Z = \frac{90 - 77}{\sigma} = \frac{13}{\sigma} = Z_1$ (say) ... (1)

$$\therefore P(X > 90) = 0.20$$

$$\text{Or, } 0.5 - P(0 < Z < Z_1) = 0.20$$

$$\text{Or, } P(0 < Z < Z_1) = 0.5 - 0.2 = 0.3$$

\therefore The value of Z_1 corresponding to 0.30 from normal table = 0.84

$$\therefore Z_1 = 0.84$$

From (1), we get

$$0.84 = \frac{13}{\sigma}$$

$$\text{Or, } \sigma \times 0.84 = 13$$

$$\therefore \sigma = \frac{13}{0.84} = 15.48$$

24. A random variable has a normal distribution with 10 as standard deviation. Find its mean if the probability that the random variable takes on a value less than 80.5 is 0.3264. ans: 85

Solution,

$$P(X < 80.5) = 0.3264$$

25. The mean height and variance of height of 500 students were found to be 165 cm and 25 cm^2 respectively. Find the range of height of middle 80% of the students.

Solution,

Let X denotes a normal variable of height of students and Z be its S.N.V. of X . Then,

$$Z = \frac{X - \mu}{\sigma}$$

$$\text{Here, } \mu = 165, \sigma^2 = 25$$

$$\therefore \sigma = \sqrt{25} = 5$$

$$\text{Or, } Z = \frac{X - 165}{5} \dots (1)$$

Let X_1 and X_2 be the lower and upper limits of height of middle 80% of the students. Then

$$P(X_1 < X < X_2) = 0.80$$

Let Z_1 and Z_2 be the S.N.V. of X_1 and X_2 respectively then

$$Z = \frac{X_1 - 165}{5} = -Z_1 \text{ (say) ... (2)}$$

$$\text{And } Z = \frac{X_2 - 165}{5} = Z_2 \text{ (say) ... (3)}$$

Now, $P(X_1 < X < X_2) = 0.80$

Or, $P(Z_1 < Z < Z_2) = 0.80$

$$\therefore P(-Z_1 < Z < 0) = P(0 < Z < Z_2) = \frac{0.80}{2} = 0.40$$

$$\text{and } P(0 < Z < Z_2) = 0.40$$

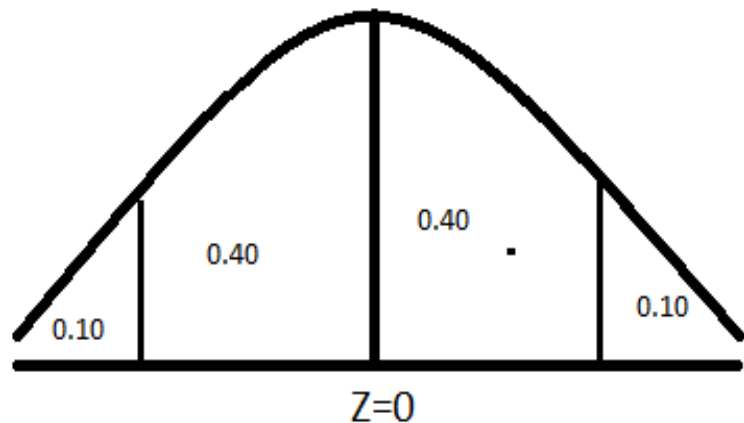
The value of Z_1 at the probability 0.40 is 1.28

Or, $Z_1 = 1.28$

From (2)

$$\frac{X_1 - 165}{5} = -Z_1$$

$$\text{Or, } -1.28 = \frac{X_1 - 165}{5}$$



$$\text{or, } -1.28 \times 5 = X_1 - 165$$

$$\therefore X_1 = 165 - 6.4 = 158.6 \text{ cm}$$

The value of Z_2 at the probability 0.40 is 1.28

Or, $Z_2 = 1.28$

From (2)

$$\frac{X_2 - 165}{5} = Z_2$$

$$\text{Or, } 1.28 = \frac{X_2 - 165}{5}$$

$$\text{or, } 1.28 \times 5 = X_2 - 165$$

$$\therefore X_2 = 165 + 6.4 = 171.4 \text{ cm}$$

Required range of height of middle 80% students = $X_2 - X_1 = 171.4 - 158.6 = 12.8 \text{ cm}$

26. The final examination score in statistics are normally distributed with an average score of 70 and variance of 25.

- i. If the lowest passing grade is 58, what percentage of the class is failing?
- ii. If the professor gives the grades on a curve and everybody getting 82 or above gets a grade of A, then what percent of students get A grade?
- iii. If the highest 80% of the class are to pass the course, what is the lowest passing score?
- iv. What should be the score be so only 15% of the students get a score higher than this?

Solution,

Let X be the normal variable of marks obtained by students and Z be its S.N.V. of X . then

$$Z = \frac{X - \mu}{\sigma}$$

$$\text{Here, } \mu = 70, \sigma^2 = 25$$

$$\therefore \sigma = \sqrt{25} = 5$$

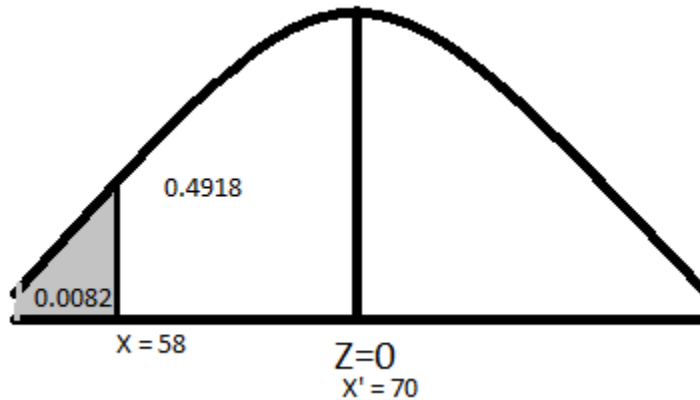
$$\text{Or, } Z = \frac{X - 70}{5} \dots (1)$$

i. When $X = 58$ then

$$Z = \frac{58 - 70}{5} = -\frac{12}{5} = -2.4$$

Let Z_1 be the corresponding S.N.V.

Then, Area between ($X = 58$ and mean = 70) = $P(58 < X < 70) = P(-Z_1 < Z < 0) = P(0 < Z < Z_1) = 0.4918$ from the normal table.



$$\therefore \text{Area to the left of } (X = 58) \text{ is } P(X < 58) = P(Z < -Z_1)$$

$$= 0.5 - P(-Z_1 < Z < 0)$$

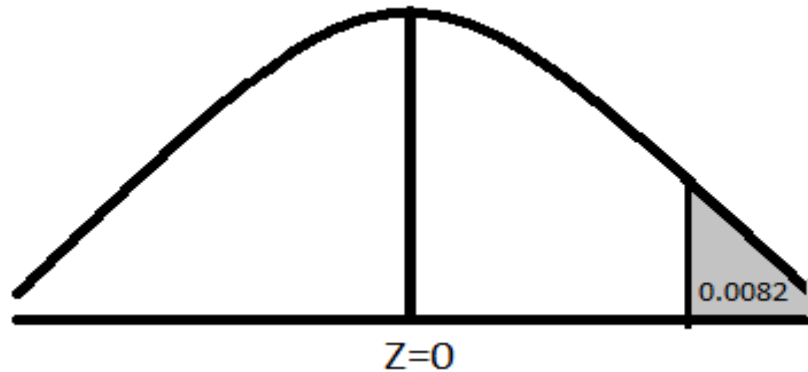
$$= 0.5 - 0.4918 = 0.0082 = 0.82\%$$

ii. When $X = 82$ then

$$Z = \frac{82 - 70}{5} = \frac{12}{5} = 2.4$$

Let Z_2 be the corresponding S.N.V.

Then, Area between ($X = 82$ and mean = 70) = $P(70 < X < 82) = P(0 < Z < Z_2) = P(0 < Z < 2.4) = 0.4918$ from the normal table.



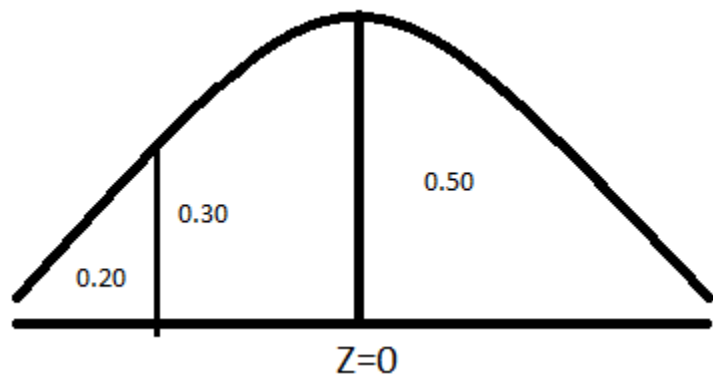
$$\begin{aligned} \therefore \text{Area to the right of } (X = 82) &= P(X > 82) = P(Z > 2.4) \\ &= 0.5 - P(0 < Z < 2.4) \\ &= 0.5 - 0.4918 = 0.0082 = 0.82\% \end{aligned}$$

- iii. Let X_1 be the lowest passing score and Z_1 be its S.N.V. then
 $P(X > X_1) = P(Z > -Z_1) = 0.80$
 Or, $P(-Z_1 < Z < 0) + 0.5 = 0.80$
 Or, $P(0 < Z < Z_1) = 0.80 - 0.50 = 0.30$
 The value of Z_1 at probability 0.30 is 0.84
 Or, $Z_1 = -0.84$

Now,

$$\frac{X_1 - 70}{5} = -Z_1$$

$$\text{Or, } -0.84 = \frac{X_1 - 70}{5}$$



$$\text{Or, } -0.84 \times 5 = X_1 - 70$$

$$\therefore X_1 = 70 - 4.2 = 65.8$$

Hence, a student must get at least 65.8 points to pass the course.

iv. Let X_2 be the lowest score where only 15% of the students get a score higher than this and Z_2 be its S.N.V. then

$$P(X > X_2) = P(Z > Z_2) = 0.15$$

$$\text{Or, } 0.5 - P(0 < Z < Z_2) = 0.15$$

$$\text{Or, } P(0 < Z < Z_2) = 0.50 - 0.15 = 0.35$$

The value of Z_2 at probability 0.35 is 1.04

$$\text{Or, } Z_2 = 1.04$$

Now,

$$\frac{X_2 - 70}{5} = Z_2$$

$$\text{Or, } 1.04 = \frac{X_2 - 70}{5}$$

$$\text{Or, } 1.04 \times 5 = X_2 - 70$$

$$\therefore X_2 = 70 + 4.2 = 75.2$$

Hence, only 15% of the students get a score higher than 75.2

