Linear Algebra MAT313 Spring 2023
Professor Sormani
Lesson 2 Solving Linear Systems

Warning: do not start this lesson until you have completed Lesson 1 and submitted all the classwork for that lesson.

Please be sure to mark down the date and time that you start this lesson. Carefully take notes on pencil and paper while watching the lesson videos. Pause the lesson to try classwork before watching the video going over that classwork. If you work with any classmates, be sure to write their names on the problems you completed together.

You will cut and paste the photos of your notes and completed classwork and a selfie taken holding up the first page of your work in a googledoc entitled:

MAT313S23-lesson2-lastname-firstname

and share editing of that document with me sormanic@gmail.com. You can use your Lehman id and hand instead of your face in your selfie. You will also include your homework and any corrections to your homework in this doc.

If you have a question, type QUESTION in your googledoc next to the point in your notes that has a question and email me with the subject MAT313 QUESTION. I will answer your question by inserting a photo into your googledoc or making an extra video.

This lesson has two parts and each has its own playlist:

Be sure to learn the methods taught in this lesson even if you already learned to solve a system in another course. I am teaching the method that leads to an algorithm that works with many equations and many unknowns. There are 10 HW problems.

Part 1: Using Row Actions to Solve a Linear System

Watch Playlist 313F21-2-1to7

Here we solve one particular system and explain what row actions are and introduce Echelon form.

Linear Algebra Lesson 2

Solving Linear Systems

- Let us review Lesson 1 ✓
- How to solve a linear system in a systematic way (not graphing) simplifying the system through a series of steps called row actions.

Example: Solve: $\boxed{\begin{array}{l} x + 2y = 3 \\ x + y = 2 \end{array}}$ ← row 1 ρ_1 ← row 2 ρ_2

Greek letter "rho" ρ

Solution: (many ways to do this)

step 1 row action:

$\rho_1 - \rho_2$

$$\begin{array}{r} x + 2y = 3 \\ \text{subtract} \\ x + y = 2 \\ \hline 0x + y = 1 \end{array}$$

If ρ_1 and ρ_2 are true then $\rho_1 - \rho_2$ is also true

So $0x + y = 1$ for every solution to our system

Thus $\boxed{y = 1}$

step 2 substitute $y=1$ into ρ_1

Solve for x:

$$\begin{aligned} x + 2y &= 3 \\ x + 2 \cdot 1 &= 3 \\ x + 2 &= 3 \\ -2 \quad & \quad -2 \\ \boxed{x = 1} \end{aligned}$$

$$\left\{ \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

$\boxed{\text{HW1}}$ Use these two steps to solve

$$\boxed{\begin{aligned} x + 5y &= 6 \\ x - 3y &= 8 \end{aligned}}$$

Key Idea: To solve a linear system

Step 1: Do row actions to simplify the equations and find $x_m = \text{value}$ (if possible)

Step 2: Substitute upwards to find the values of $x_1, x_2 \ldots x_{m-1}$ (if possible)

<u>Example</u> Solve $\boxed{\begin{aligned} x + 2y &= 6 \\ 2x + y &= 8 \end{aligned}}$

Warning

$$\begin{array}{rl} \rho_1 & x + 2y = 6 \\ \rho_2 & 2x + y = 8 \\ \hline \rho_1 - \rho_2 & -x + y = -2 \end{array}$$

This does not give us $y = \sim$

Fails: cannot just subtract rows

Step 1

We want the <u>first</u> <u>variable</u> to <u>disappear</u>

$$\begin{array}{rl} \rho_1: & x + 2y = 6 \\ \rho_2: & 2x + y = 8 \\ \hline \rho_1 - \frac{1}{2}\rho_2: & 0 + \frac{3}{2}y = 2 \end{array}$$

$x - \frac{1}{2}(2x) = x - x = 0$ ✓

$2y - \frac{1}{2}(y) = \left(\frac{4}{2} - \frac{1}{2}\right)y = \frac{3}{2}y$ ✓

$6 - \frac{1}{2}(8) = 6 - 4 = 2$

another action: mult by $\frac{2}{3}$ $\quad \frac{2}{3}\left(\frac{3}{2}y\right) = \frac{2}{3}(2)$

Fails: cannot just subtract row

Step 1 We want the first variable to disappear

ρ_1: $x + 2y = 6$

ρ_2: $2x + y = 8$

$\rho_1 - \frac{1}{2}\rho_2$: $0 + \frac{3}{2}y = 2$

$x - \frac{1}{2}(2x) = x - x = 0$ ✓

$2y - \frac{1}{2}(y) = \left(\frac{4}{2} - \frac{1}{2}\right)y = \frac{3}{2}y$ ✓

$6 - \frac{1}{2}(8) = 6 - 4 = 2$

another action: mult by $\frac{2}{3}$ $\quad \frac{2}{3}\left(\frac{3}{2}y\right) = \frac{2}{3}(2)$

$$\boxed{y = \frac{4}{3}}$$

Step 2 Sub back to find x → $x + 2y = 6$

$x + 2\left(\frac{4}{3}\right) = 6$

Solve for $x = 6 - \frac{8}{3} = \frac{18}{3} - \frac{8}{3} = \frac{10}{3}$

$$\boxed{x = \frac{10}{3}}$$

$$\left\{ \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 10/3 \\ 4/3 \end{pmatrix} \right\}$$

Check

$\left(\frac{10}{3}\right) + 2\left(\frac{4}{3}\right) = 6$?

$2\left(\frac{10}{3}\right) + \left(\frac{4}{3}\right) = 8$?

$\frac{10}{3} + 2\left(\frac{4}{3}\right) = \frac{10}{3} + \frac{8}{3} = \frac{18}{3} = 6$ ✓

$2\left(\frac{10}{3}\right) + \frac{4}{3} = \frac{20}{3} + \frac{4}{3} = \frac{24}{3} = 8$ ✓

HW2 use this method to solve

$$\boxed{\begin{aligned} 2x + 5y &= 8 \\ 4x + 9y &= 10 \end{aligned}}$$

and then check your answer.

Example: Solve $\boxed{\begin{array}{l} 2x + 2y + 4z = 12 \\ x + y + z = 5 \\ x - y + z = 1 \end{array}}$

① Step 1: use row actions to simplify the system.

When we have many equations we keep track of the whole system as we proceed

$$\boxed{\begin{array}{l} 2x + 2y + 4z = 12 \\ x + y + z = 5 \\ x - y + z = 1 \end{array}} \xrightarrow[\text{copy the other rows}]{\rho_1 \to \frac{1}{2}\rho_1} \boxed{\begin{array}{l} x + y + 2z = 6 \\ x + y + z = 5 \\ x - y + z = 1 \end{array}}$$

If $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ solves this old system $\xRightarrow{\text{mult by } \frac{1}{2}}$ then $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ solves this new system

then $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ solve this old system $\xLeftarrow[\text{mult } \rho_1 \text{ by } 2]{}$ If $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ solves the new system

> Row actions $\rho_i \to k\rho_i$ where $k \neq 0$
> is reversable by mult by $\frac{1}{k}$
> So we have the same solutions

Continuing with step 1 of our example:

more row actions

Next get rid of the first variable in rows 2+3

$$\boxed{\begin{array}{l} x + y + 2z = 6 \\ \cancel{x} + y + z = 5 \\ x - y + z = 1 \end{array}} \xrightarrow[\text{copy other rows}]{\rho_2 \to \rho_2 - \rho_1} \boxed{\begin{array}{r} x + y + 2z = 6 \\ -z = -1 \\ x - y + z = 1 \end{array}}$$

any solution of this system $\Longrightarrow$ is also a solution of this system

we can also go backwards

$$\boxed{\begin{array}{l} x + y + 2z = 6 \\ \cancel{x} + y + z = 5 \\ x - y + z = 1 \end{array}} \xleftarrow{\rho_2 \to \rho_2 + \rho_1} \boxed{\begin{array}{r} x + y + 2z = 6 \\ -z = -1 \\ x - y + z = 1 \end{array}}$$

> The row action $\rho_i \longrightarrow \rho_i + k\rho_j$
>
> is reversable $\rho_i \longrightarrow \rho_i - k\rho_j$
>
> So we have some solutions before and after such a row action

$$\boxed{\begin{aligned} x+y+2z&=6\\ -z&=-1\\ x-y+z&=1 \end{aligned}} \xrightarrow{\rho_3 \to \rho_3-\rho_1} \boxed{\begin{aligned} x+y+2z&=6\\ -z&=-1\\ 0x-2y-z&=-5 \end{aligned}}$$

Switching Rows is Reversible

$\rho_i \longleftrightarrow \rho_j$

So we have the same solutions before + after this action

$$\boxed{\begin{aligned} x+y+2z&=6\\ -z&=-1\\ 0x-2y-z&=-5 \end{aligned}} \xrightarrow[\text{copy other rows}]{\rho_2 \leftrightarrow \rho_3} \boxed{\begin{aligned} x+y+2z&=6\\ 0x-2y-z&=-5\\ -z&=-1 \end{aligned}}$$

$$\boxed{\begin{aligned} x+y+2z&=6\\ 0x-2y-z&=-5\\ -z&=-1 \end{aligned}} \xrightarrow[\rho_3 \to -\rho_3]{\rho_2 \to -\frac{1}{2}\rho_2} \underbrace{\boxed{\begin{aligned} x+y+2z&=6\\ y+\tfrac{1}{2}z&=\tfrac{5}{2}\\ z&=1 \end{aligned}}}_{\text{Echelon Form}}$$

Step 2 Sub up

pause + try

solution in next video.

Neat each row begins with a new leading variable and the coefficient of that variable is 1.

$$\boxed{\begin{aligned} x+y+2z&=6 \\ 0x-2y-z&=-5 \\ -z&=-1 \end{aligned}} \xrightarrow[\rho_3\to-\rho_3]{\rho_2\to-\frac{1}{2}\rho_2} \underbrace{\boxed{\begin{aligned} x+y+2z&=6 \\ y+\tfrac{1}{2}z&=\tfrac{5}{2} \\ z&=1 \end{aligned}}}_{\text{Echelon Form}}$$

Step 2 Sub up

pause + try

solution in next

video.

Neat each row begins with a new leading variable and the coefficient of that variable is 1.

Observe that $z=1$ in the last row

go to one row above the last row: $y+\frac{1}{2}z=\frac{5}{2}$

and solve for its leader: $y=\frac{5}{2}-\frac{1}{2}z$

and sub in $z=1$: $y=\frac{5}{2}-\frac{1}{2}(1)=\frac{5}{2}-\frac{1}{2}=\frac{4}{2}=2$

go the one row above that row: $x+y+2z=6$

ahd solve for its leader: $x=6-y-2z$

and sub in $z=1$ and $y=2$: $x=6-(2)-2(1)=2$

So our solution set is $\left\{\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}2\\2\\1\end{pmatrix}\right\}$

Before continuing to Part 2 you may wish to complete the first five homework problems. Check the solution here to each problem before doing the next.

Lesson 2 Homework (answers below)

HW1

$$\begin{aligned} 3x + 3y - 9z &= 6 \\ x - y + z &= 0 \\ x + 4y + 2z &= 5 \end{aligned}$$

What is the first row action you should do? Do the row action.

HW2

$$\begin{aligned} \boxed{x} + 3y + 4z &= 0 \\ 2x + 7y + 9z &= 0 \\ 4x + 12y + 12z &= 4 \end{aligned}$$

What row actions are needed here to get zeroes under the boxed leader? Do them!

HW3

$$\begin{aligned} \boxed{x} + y + z + 4w &= 0 \\ 0x + 0y + 5z + 5w &= 10 \\ 0x + 2y + 4z + 6w &= 12 \\ 0x + 0y + 5z + 5w &= 15 \end{aligned}$$

What is the 2nd leader? Move it to the second row using a switch then make its coeff = 1.

HW4

$$\begin{aligned} x + y + z &= 5 \\ 0x + 0y + 1z &= 6 \\ 0x + 0y + 0z &= 0 \end{aligned}$$

Box the leaders. Is this in Echelon form? If not, complete the row actions to Echelon form

HW5

$$\begin{aligned} x + y + z &= 5 \\ 0x + 0y + 1z &= 6 \\ 0x + y + 0z &= 0 \end{aligned}$$

Box the leaders. Is this in Echelon form? If not, complete the row actions to Echelon form

Part II: How to Solve Any Linear System

Watch Playlist 313F21-2-8to12

"How to solve a Linear System" is in video 313F21-2-8 and the Example 1 rewritten using this technique is in video 313F21-2-8 and 313F21-2-9:

How to Solve a Linear System:

Step 1: Row Reductions to Echelon Form

Row Actions which are reverseable

- scale • $\rho_i \rightarrow k\rho_i$ where $k \neq 0$
- skew • $\rho_i \rightarrow \rho_i + k\rho_j$
- switch • $\rho_i \longleftrightarrow \rho_j$

later we will set up an algorithm for this step.

Step 2: Sub up

Start with final row: solve for its leader

Next to last row: solve for its leader
sub in previous variables

Next row up: solve for its leader
sub in previous variable

When all rows are done we have a solution.

Example above rewritten (method to use for HW + Exams)

$$\boxed{\begin{aligned} 2x+2y+4z&=12 \\ x+y+z&=5 \\ x-y+z&=1 \end{aligned}} \longrightarrow$$

Step 1 Row actions moving towards Echelon Form

- make the 1^{st} coefficient (upper left) a $1x$

$$\begin{aligned} 2x+2y+4z&=12 \\ x+y+z&=5 \\ x-y+z&=1 \end{aligned} \xrightarrow[\text{copy other lines}]{\rho_1 \to \frac{1}{2}\rho_1} \begin{aligned} \boxed{1x}+1y+2z&=6 \\ x+y+z&=5 \\ x-y+z&=1 \end{aligned}$$

- get rid of the first variable in rows 2 to n using row action $\rho_i \to \rho_i - k\rho_1$

$$\xrightarrow{\substack{\rho_2 \to \rho_2-\rho_1 \\ \rho_3 \to \rho_3-\rho_1}} \begin{aligned} \boxed{1x}+1y+2z&=6 \\ 0x+0y-z&=-1 \\ 0x-2y-z&=-5 \end{aligned}$$

x is the first leader

notice x only appears in first row.

- put our next leading variable, y, in the second row

$$\xrightarrow{\rho_2 \leftrightarrow \rho_3} \begin{aligned} \boxed{1x}+1y+2z&=6 \\ 0x\ \boxed{-2y}-z&=-5 \\ 0x+0x-z&=-1 \end{aligned}$$

1^{st} leader is x
2^{nd} leader is y

- make this leading variable have coefficient $=1$

$$\xrightarrow{\rho_2 \to -\frac{1}{2}\rho_2} \begin{aligned} \boxed{1x}+1y+2z&=6 \\ 0x+\boxed{1y}+\tfrac{1}{2}z&=\tfrac{5}{2} \\ 0x+0y\ \boxed{-z}&=-1 \end{aligned}$$

1^{st} leader is x
2^{nd} leader is y

- our third leader is z, needs coeff $=1$

$$\xrightarrow{\rho_3 \to -\rho_3} \begin{aligned} \boxed{1x}+1y+2z&=6 \\ 0x+\boxed{1y}+\tfrac{1}{2}z&=\tfrac{5}{2} \\ 0x+0y+\boxed{1z}&=1 \end{aligned}$$

leaders for each row has coeff $=1$ and has 0's to the left + below

Echelon Form.

Step 2 Sub up

pause + try

solution in next video.

Echelon Form

Neat each row begins with a new leading variable and the coefficient of that variable is 1.

Observe that $z=1$ in the last row

go to one row above the last row: $y+\frac{1}{2}z=\frac{5}{2}$

and solve for its leader: $y=\frac{5}{2}-\frac{1}{2}z$

and sub in $z=1$: $y=\frac{5}{2}-\frac{1}{2}(1)=\frac{5}{2}-\frac{1}{2}=\frac{4}{2}=2$

go the one row above that row: $x+y+2z=6$

and solve for its leader: $x=6-y-2z$

and sub in $z=1$ and $y=2$: $x=6-(2)-2(1)=2$

So our solution set is $\left\{\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}2\\2\\1\end{pmatrix}\right\}$

Example 2 is in video 313F21-2-10

Example 2

Solve
$$\begin{aligned}2x+2y+4z&=12\\ x+y+z&=5\\ 2x+2y+2z&=8\end{aligned}$$

Classwork!
Use our method

Step 1 : Row reduction changing the system to Echelon Form

- Upper Left Coefficient to 1

$\rho_1 \to \frac{1}{2}\rho_1$
$$\begin{aligned}\boxed{x}+y+2z&=6\\ x+y+z&=5\\ 2x+2y+2z&=8\end{aligned}$$

upper left

our 1st leader is x

- Get rid of 1st leader in 2nd – nth rows (change 1st coeff to 0) using skew actions $\rho_i \to \rho_i - k\rho_1$

$\rho_2 \to \rho_2 - \rho_1$
$\rho_3 \to \rho_3 - 2\rho_1$
$$\begin{aligned}\boxed{x}+y+2z&=6\\ 0x+0y-z&=-1\\ 0x+0y-2z&=-4\end{aligned}$$

y is never a leader!
z is our next leader

- Make the coefficient of next leader = 1

$\rho_2 \to -\rho_2$
$$\begin{aligned}\boxed{x}+y+2z&=6\\ 0x+0y+\boxed{z}&=1\\ 0x+0y-2z&=-4\end{aligned}$$

box our leaders!
x and z
(y is not a leader)

- Get rid of 2nd leader in lower rows (3rd row)
$\rho_3 \to \rho_3 + k\rho_2$ ⟵ use ρ_2 to do this

$\rho_3 \to \rho_3 + 2\rho_2$
$$\begin{aligned}\boxed{x}+y+2z&=6\\ 0x+0y+\boxed{z}&=1\\ 0x+0y+0z&=-2\end{aligned}$$

scratch
$-4+2(1) = -4+2 = -2$

This is in Echelon Form!
zeroes under + left of leaders ✓
coeffs of leaders are 1 ✓

Step 2 sub up: Start with bottom line.

$0x+0y+0z=-2$ which is $0=-2$

Impossible!

So our Echelon Form has no solutions!

but all our actions were reversable

So our original system has no solution.

Solution Set $=\emptyset$ Empty set!

Example 3 is in the last two videos 313F21-2-11 and 313F21-2-12 which you should watch pausing and trying as you work:

Example 3: Solve $\begin{array}{l} 1x+1y+1z+1w=0 \\ 1x+1y-1z+1w=0 \\ 2x+3y+2z+1w=0 \end{array}$

Classwork

Step 1: Reduce to Echelon Form Using Row Actions

- Upper Left coeff is a 1 ✓ so our first leader is x

$$\begin{array}{l} \boxed{1x}+1y+1z+1w=0 \\ 1x+1y-1z+1w=0 \\ 2x+3y+2z+1w=0 \end{array}$$

- Get rid of first leader in rows below ρ_1 using

$$\rho_i \to \rho_i + k\rho_1$$

pause + try to find row actions for $\rho_2 + \rho_3$

$$\xrightarrow[\rho_3 \to \rho_3 - 2\rho_1]{\rho_2 \to \rho_2 - \rho_1} \begin{array}{l} \boxed{1x}+1y+1z+1w=0 \\ 0x+0y-2z+0w=0 \\ 0x+\boxed{1y}+0z-1w=0 \end{array}$$

2nd leader is y move up to row 2

move 2nd leader into 2nd row

$$\xrightarrow[\text{switch!}]{\rho_3 \leftrightarrow \rho_2} \begin{array}{l} \boxed{1x}+1y+1z+1w=0 \\ 0x+\boxed{1y}+0z-1w=0 \\ 0x+0y\boxed{-2z}+0w=0 \end{array}$$

pause + try

do we have a coeff = 1 on 2nd leader? Yes

do we have zeroes below it? Yes

Next look for third leader in next column
the next leader is z.
It is already in 3^{rd} row.
No switch needed.
Must make 3^{rd} leader's coeff = 1

$$\xrightarrow{\rho_3 \to \frac{-1}{2}\rho_3} \boxed{\begin{array}{l} \boxed{1x} + 1y + 1z + 1w = 0 \\ 0x + \boxed{1y} + 0z - 1w = 0 \\ 0x + 0y + \boxed{1z} + 0w = 0 \end{array}}$$

Echelon Form!

Step 2:

Sub Up

last row $\quad 0x + 0y + \boxed{1z} + 0w = 0$

Solve for the leader $\quad z = 0 - 0w - 0x - 0y = 0$

so $\boxed{z = 0}$

second to last row

$$0x + \boxed{1y} + 0z - 1w = 0$$

Solve for the leader $\boxed{y = 0 + 1w}$

Sub in leaders (but w is not a leader)

"w is free" not a leader

So w can have any value in $\mathbb{R}$

next to last row

$$\boxed{1x} + 1y + 1z + 1w = 0$$

Solve for the leader

$$x = 0 - y - z - w$$

Sub in previous leaders

$$x = 0 - (0 + 1w) - (0) - w$$

formula for y   formula for z   free so keep.

simplify:

$$\boxed{x = -w - w = -2w}$$

Already at top so stop

$$\left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -2w \\ 0 + 1w \\ 0 \\ w \end{pmatrix} : \underset{\text{free}}{w \in \mathbb{R}} \right\}$$

free = free   leaders = formulas

Many Solutions "A line"

For homework you will solve five systems using today's method.

QUESTIONS:

Email me questions as explained at the top of the doc. Below I include answers to the best questions.

Let us review the key steps again including students questions:

How to Solve a Linear System:

Step 1: Row Reductions to Echelon Form

Row Actions which are reverseable

scale • $\rho_i \to k\rho_i$ where $k \neq 0$

skew • $\rho_i \to \rho_i + k\rho_j$

switch • $\rho_i \longleftrightarrow \rho_j$

later we will set up an algorithm for this step.

Student Question:

What is Echelon Form?

Here is the precise definition:

> **Definition: a system is in Echelon Form if**
> **it has zeroes to the left and below all leaders**
> **and all the leaders are ones and**
> **all rows with 0+0+0+…+0=something are at the bottom.**
>
> **Definition: a leading coefficient or leader is**
> **the first nonzero coefficient of a row.**

Here are examples of Echelon Form:

from lesson:

$$\begin{aligned} \boxed{1x} + 1y + 1z + 1w &= 0 \\ 0x + \boxed{1y} + 0z - 1w &= 0 \\ 0x + 0y + \boxed{1z} + 0w &= 0 \end{aligned}$$

Echelon Form!

leaders are boxed

zeroes under and to the left of the leaders

another example

$$\begin{aligned} \boxed{1x} + 2y + 3z &= 5 \\ 0x + 0y + \boxed{1z} &= 6 \\ 0x + 0y + 0z &= 5 \end{aligned}$$

Echelon Form!

leaders are boxed

zeroes under + left of leaders

any rows of zeros on bottom

Student Question:

How do I do Row Reduction to Echelon Form?

We follow these steps:

Method: Start on the upper left.

Step 1: Find a leader and make it one using row actions.

Step 2: Get zeroes below the leader using row actions.

Go to the next row and column and repeat Step 1 and Step 2.

When we run out of rows, the system is in Echelon Form:

it has zeroes to the left and below all leaders

and all the leaders are ones and

all rows with 0+0+0+…+0=something are at the bottom.

Here is Example 1 with these repeating steps marked clearly:

Example 1:

Step 1 Row actions moving towards Echelon Form

- make the 1^{st} coefficient (upper left) a $\underline{1}x$

Step 1 make leading coefficient 1 by SCALE

$$\begin{aligned} 2x+2y+4z&=12\\ x+y+z&=5\\ x-y+z&=1 \end{aligned} \xrightarrow[\text{copy other lines}]{\rho_1\to\frac{1}{2}\rho_1} \begin{aligned} \boxed{1x}+1y+2z&=6\\ x+y+z&=5\\ x-y+z&=1 \end{aligned}$$

- get rid of the first variable in rows 2 to n using row action $\rho_i \to \rho_i - k\rho_1$

Step 2, get 0's under first leader by SKEW

$\rho_2 \to \rho_2 - \rho_1$

$\rho_3 \to \rho_3 - \rho_1$

$$\longrightarrow \begin{aligned} \boxed{1x}+1y+2z&=6\\ 0x+0y-z&=-1\\ 0x-2y-z&=-5 \end{aligned}$$

x is the first leader

notice x only appears in first row.

Repeat for next row and column

- put our next leading variable, y, in the second row

Step 1 second leader get coeff=1 by SWITCH and SCALE

$$\xrightarrow{\rho_2\leftrightarrow\rho_3} \begin{aligned} \boxed{1x}+1y+2z&=6\\ 0x\boxed{-2y}-z&=-5\\ 0x+0x-z&=-1 \end{aligned}$$

1^{st} leader is x

2^{nd} leader is 1

- make this leading variable have coefficient = 1

$$\xrightarrow{\rho_2\to-\frac{1}{2}\rho_2} \begin{aligned} \boxed{1x}+1y+2z&=6\\ 0x+\boxed{1y}+\tfrac{1}{2}z&=\tfrac{5}{2}\\ 0x+0y\boxed{-z}&=-1 \end{aligned}$$

1^{st} leader is x

2^{nd} leader is y

Step 2 already 0's under 2nd leader

Next row and column

- our third leader is z, needs coeff = 1

Step 1 make third leader have coeff = 1

$$\xrightarrow{\rho_3\to-\rho_3} \begin{aligned} \boxed{1x}+1y+2z&=6\\ 0x+\boxed{1y}+\tfrac{1}{2}z&=\tfrac{5}{2}\\ 0x+0y+\boxed{1z}&=1 \end{aligned}$$

leaders for each row has coeff = 1 and has 0's to the left + below

Echelon Form.

No more rows and columns so done

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We will learn a precise way of writing this as an algorithm with a flow chart for any system in the next lesson.

Continuing forward:

Step2: Sub up

Start with final row: solve for its leader

Next to last row: solve for its leader
sub in previous variables

Next row up: solve for its leader
sub in previous variable

When all rows are done we have a solution.

Student Question:

Why do you use parentheses when you sub up?

Parentheses are used because they tell us to compute the value inside the parentheses before multiplying. See the end of Example 3 here:

Step 2:

Sub Up

last row $0x+0y+\boxed{1z}+0w=0$

Solve for the leader $z=0-0w-0x-0y=0$

so $\boxed{z=0}$

second to last row

$0x+\boxed{1y}+0z-1w=0$

Solve for the leader $\boxed{y=0+1w}$

Sub in leaders (but w is not a leader)

"w is free" not a leader

So w can have any value in $\mathbb{R}$

next to last row

$\boxed{1x}+1y+1z+1w=0$

Solve for the leader

$x=0-y-z-w$

Sub in previous leaders

$x=0-(0+1w)-(0)-w$

formula for y

formula for z

free so keep.

simplify:

$\boxed{x=-w-w=-2w}$

we need these parentheses because $y=0+1w$

so $-y=-(0+1w)$
$=-w$ ✓

without parentheses we would get the wrong answer
$-y=-0+1w$
$=w$ ✗

Student Question:

What if there is a row of zeroes adding up to something that is not zero?

If so, then there is no solution! Write the empty set. This happened in Example 2 above:

$0x+0y+0z=-2$ which is $0=-2$

Impossible!

So our Echelon Form has no solutions!

but all our actions were reversable

So our original system has no solution.

Solution Set $= \emptyset$ Empty set!

Be careful not to write brackets around the empty set symbol.

Student Question:

What if there is a row of zeroes adding up to zero?

If you get something like 0x+0y+0z=0 then it still has a solution. Sub up to find the solution starting with the last row that has a leader.

You finally complete the problem by writing the solution set.

Student Question:

How do I know which variables are free?

All variables which are not boxed as leaders are free.

Student Question:

In some places I see the solution sets written without brackets and free variables.

Linear Algebra is an ancient subject dating back a thousand years in China and has many notations. In this course we will use only the notation I teach to keep things simple. Do not use other notation in your work. All mathematicians understand the notation I am teaching you. In your own fields: chemistry, computer science, economics, etc other notations may be used that you will learn in those courses.

Homework:

Glance over the lecture notes above to be sure you watched all the videos and read over the students' questions. Then do the following ten problems. Check the solution here to each problem before doing the next.

Lesson 2 Homework (answers below)

HW1
$$\begin{aligned} 3x+3y-9z&=6\\ x-y+z&=0\\ x+4y+2z&=5 \end{aligned}$$
What is the first row action you should do? Do the row action

HW2
$$\begin{aligned} \boxed{x}+3y+4z&=0\\ 2x+7y+9z&=0\\ 4x+12y+12z&=4 \end{aligned}$$
What row actions are needed here to get zeroes under the boxed leader? Do them!

HW3
$$\begin{aligned} \boxed{x}+y+z+4w&=0\\ 0x+0y+5z+5w&=10\\ 0x+2y+4z+6w&=12\\ 0x+0y+5z+5w&=15 \end{aligned}$$
What is the 2^{nd} leader? Move it to the second row using a switch then make its coeff = 1.

HW4
$$\begin{aligned} x+y+z&=5\\ 0x+0y+1z&=6\\ 0x+0y+0z&=0 \end{aligned}$$
Box the leaders. Is this in Echelon form? If not, complete the row actions to Echelon form

HW5
$$\begin{aligned} x+y+z&=5\\ 0x+0y+1z&=6\\ 0x+y+0z&=0 \end{aligned}$$
Box the leaders. Is this in Echelon form? If not, complete the row actions to Echelon form

HW6 Solve the system in HW5 by subing up

HW7 Solve the system in HW4

HW8 Solve the system in HW1

HW9 Solve the system in HW2

HW10 Solve the system in HW3

Note your solution is only correct if you do the same row actions in the same order exactly as in the solutions [here](). If you do something different, and do not know why it is wrong, send a question.

It is very important to email me if you do not understand why your problems are incorrect. See how to email questions at the top of this document.

You can use your Lehman id and hand instead of your face in your selfie.   This can be helpful if you are not dressed well or are shy or have difficulty taking a selfie.

Submit your classwork and homework following the directions at the top of this document.