[Linear Algebra MAT313 Fall 2022](#)
[Professor Sormani](#)
Lesson 5 Homogeneous Linear Systems

Please be sure to mark down the date and time that you start this lesson. Carefully take notes on pencil and paper while watching the lesson videos. Pause the lesson to try classwork before watching the video going over that classwork. If you work with any classmates, be sure to write their names on the problems you completed together. Please wear masks when meeting with classmates even if you meet off campus.

You will cut and paste the photos of your notes and completed classwork and a selfie taken holding up the first page of your work in a googledoc entitled:

MAT313F22-lesson5-lastname-firstname

and share editing of that document with me sormanic@gmail.com. You will also include your homework and any corrections to your homework in this doc.

If you have a question, type QUESTION in your googledoc next to the point in your notes that has a question and email me with the subject MAT313 QUESTION. I will answer your question by inserting a photo into your googledoc or making an extra video.
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Watch the Playlist 313F21-5-1to9

Once you have completed and submitted the work for this lesson, email Professor Sormani requesting a sample for Exam 1.

Lesson 5

Homogeneous Linear Systems

$$\boxed{\sum_{j=1}^{m} a_{ij}x_j = 0 \text{ for } i = 1 \text{ to } n}$$

Classwork①: Convert to an Augmented Matrix
Do Row Reduction to Reduced Echelon Form
Solve for leader and write Solution Set
If there is one free variable write the solution set as a line with position and direction

Six Variables! x_1 x_2 x_3 x_4 x_5 x_6

$$\boxed{\begin{aligned} 2x_1 + 4x_2 + 6x_6 &= 0 \\ 1x_1 + 2x_2 + 1x_4 &= 0 \\ 4x_1 + 8x_2 + 10x_6 &= 0 \\ 2x_1 + 4x_2 + 2x_5 &= 0 \\ 2x_4 + x_5 &= 0 \\ x_4 + x_5 + 2x_6 &= 0 \end{aligned}}$$

$2x_1 + 4x_2 + \underbrace{0x_3}_{} + \underbrace{0x_4}_{} + \underbrace{0x_5}_{} + 6x_6 = 0$

don't forget missing variables in row

$$\begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \end{matrix} \\ \left[\begin{array}{cccccc|c} \boxed{2} & 4 & 0 & 0 & 0 & 6 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 & 0 \\ 4 & 8 & 0 & 0 & 0 & 10 & 0 \\ 2 & 4 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 2 & 0 \end{array}\right] \end{array}$$

← Our variables

pause + try

← final column is all 0 in a homogeneous system

$\rho_1 \to \frac{1}{2}\rho_1$

Box the first leader
turn it into a 1

$$\left[\begin{array}{cccccc|c} \boxed{1} & 2 & 0 & 0 & 0 & 3 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 & 0 \\ 4 & 8 & 0 & 0 & 0 & 10 & 0 \\ 2 & 4 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 2 & 0 \end{array}\right]$$

$10 - 4 \cdot 3 = 10 - 12 = -2$

$10 - 2 \cdot 3 = 10 - 6 = 4$

must put zeroes under the leader using skew by leader's row $\rho_i \to \rho_i + k\rho_1$

$\rho_2 \to \rho_2 - \rho_1$

$\rho_3 \to \rho_3 - 4\rho_1$

$\rho_4 \to \rho_4 - 2\rho_1$

last two are 0

$$\longrightarrow \left[\begin{array}{cccccc|c} \boxed{1} & 2 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 2 & 0 \end{array}\right]$$

Find 2nd leader: none in the second column (2nd variable is free!)

none in third column (3rd variable is free!)

the leader in the fourth column is a 1 so we do not need to scale and already in row 2 so no switch needed

Next make 0's under the leader
skew by leaders row ρ_2

$$\xrightarrow[\rho_6 \to \rho_6 - \rho_2]{\rho_5 \to \rho_5 - 2\rho_2} \left[\begin{array}{cccccc|c} \boxed{1} & 2 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 6 & 0 \\ 0 & 0 & 0 & 0 & 1 & 5 & 0 \end{array}\right]$$

Find next leader
switch $\rho_5 \leftrightarrow \rho_3$

$$\xrightarrow{\rho_5 \leftrightarrow \rho_3} \left[\begin{array}{cccccc|c} \boxed{1} & 2 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 & 5 & 0 \end{array}\right]$$

Must make all below the leader to a zero

$$\xrightarrow{\rho_6 \to \rho_6 - \rho_3} \left[\begin{array}{cccccc|c} \boxed{1} & 2 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & \boxed{-2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{array}\right]$$

Find next leader
turn it into a 1 scaling it!

$\rho_4 \xrightarrow{\quad} -\frac{1}{2}\rho_4$

$$\left[\begin{array}{cccccc|c} \boxed{1} & 2 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{array}\right]$$

change to 0 skew by ρ_4

$\rho_5 \to \rho_5 - 4\rho_4$
$\rho_6 \to \rho_5 + \rho_4$

$$\xrightarrow{\quad} \left[\begin{array}{cccccc|c} \boxed{1} & 2 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Echelon Form!

Next want Reduced Echelon Form with zeroes above the leaders starting with bottom leader

skew by leaders row ρ_4

$\rho_1 \to \rho_1 - 3\rho_4$
$\rho_2 \to \rho_2 + 3\rho_4$
$\rho_3 \to \rho_3 - 6\rho_4$

$$\xrightarrow{\quad} \left[\begin{array}{cccccc|c} \boxed{1} & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Check third leader ✓
second leader ✓

Reduced Echelon Form.

Write as a Linear System

$\boxed{1x_1} + 2x_2 = 0$ $\qquad$ $x_1 = -2x_2$

$\boxed{1x_4} = 0$ $\qquad$ solve for leaders $\qquad$ $x_4 = 0$

$\boxed{1x_5} = 0$ $\qquad$ $x_5 = 0$

$\boxed{1x_6} = 0$ $\qquad$ $x_6 = 0$

free variables $\quad x_2 = x_2$

$x_3 = x_3$

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} -2x_2 \\ x_2 \\ x_3 \\ 0 \\ 0 \\ 0 \end{pmatrix} : x_2, x_3 \in \mathbb{R} \right\}$$

two free variables!

cannot write in line form!

$$\left\{ \begin{pmatrix} x_1 \\ x_3 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 0 - 2x_2 + 0x_3 \\ 0 + 1x_2 + 0x_3 \\ 0 + 0x_2 + 1x_3 \\ 0 + 0x_2 + 0x_3 \\ 0 + 0x_2 + 0x_3 \\ 0 + 0x_2 + 0x_3 \end{pmatrix} : x_2, x_3 \in \mathbb{R} \right\}$$

position with two directions

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} ; x_2, x_3 \in \mathbb{R} \right\}$$

Our Algorithm:

Step 1: Write Augmented Matrix ← Lesson 3

Step 2: Row Actions to Echelon Form ← Lessons 1-2

reversible row action
- scale
- skew
- switch

- Make upper left leader into 1 using scaling (or switch if 0)
- Make zeroes beneath leader using skew by leader's row
- Move down to the next row and make sure next column has a leader which is 1 by repeating red step. Then repeat blue step.

Move down again until Echelon Form
Each leader in red box is a 1.
Below each leader are 0's.

Step 3: Row actions to Reduced Echelon Form ← Lesson 3 Reduced Echelon Form

- start with bottom leader
Make sure 0's above it using skew by the leader's row.
Repeat with next to last leader until 0's above all leaders

Step 4: Write as equations and solve for leaders and set free variables equal to themselves ← No need to sub up

be careful "$0+0\cdots+0=$ not 0" no solution

Write as $\left\{\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} \\ \\ \end{pmatrix} ; \text{free} \in \mathbb{R}\right\}$

Step 5: write in position directions form ← Lesson 5

$$\left\{\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \underbrace{\begin{pmatrix} \\ \\ \end{pmatrix}}_{\text{position}} + \underbrace{x_i}_{\text{free variable}} \underbrace{\begin{pmatrix} \\ \\ \end{pmatrix}}_{\text{direction for that free variable}} + \cdots \;\middle|\; \text{free} \in \mathbb{R}\right\}$$

position free variable variable

When the system is homogeneous the position vector is $\begin{pmatrix}0\\0\\\vdots\\0\end{pmatrix}$

If a homogeneous system has 1 free variable, t

$$\left\{\vec{x} = \vec{0} + t\vec{a} \mid t \in \mathbb{R}\right\}$$ a line through the origin.

If it has two free variables, s, t

$$\left\{\vec{x} = \vec{0} + t\underset{\substack{\uparrow\\\text{direction}\\\text{of } t}}{\vec{a}} + s\underset{\substack{\uparrow\\\text{direction}\\\text{of } s}}{\vec{b}} \;\middle|\; s, t \in \mathbb{R}\right\}$$

notice that when $s, t = 0$
then $\vec{x} = 0$
So $x_1 = 0$ $x_2 = 0$ ... $x_m = 0$
is one of the solutions to system.

If there are no free variables the only solution is $\{\vec{x} = \vec{0}\}$

It is never an $\emptyset$.

Classwork (2)

Solve and write solution for

$$\left[\begin{array}{ccccc|c} \boxed{1} & 0 & 4 & 2 & 5 & 0 \\ 0 & 1 & 2 & 0 & 6 & 0 \\ 1 & 1 & 6 & 2 & 11 & 0 \\ 2 & 0 & 8 & 5 & 11 & 0 \end{array}\right]$$

box 1^{st} leader
is it a 1? yes
change numbers below it
$\rho_i \to \rho_i + k\rho_1$

$$\xrightarrow[\rho_4 \to \rho_3 - 2\rho_1]{\rho_3 \to \rho_3 - \rho_1} \left[\begin{array}{ccccc|c} \boxed{1} & 0 & 4 & 2 & 5 & 0 \\ 0 & \boxed{1} & 2 & 0 & 6 & 0 \\ 0 & 1 & 2 & 0 & 6 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

box 2^{nd} leader
is it a 1? ✓
chang below to 0
$\rho_i \to \rho_i + k\rho_2$

$$\xrightarrow{\rho_3 \to \rho_3 - \rho_2} \left[\begin{array}{ccccc|c} \boxed{1} & 0 & 4 & 2 & 5 & 0 \\ 0 & \boxed{1} & 2 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 1 & 0 \end{array}\right]$$

box 3^{rd} leader
none in 3^{rd} column
(x_3 is free)
in 4^{th} column

move that leader to the third row

$$\xrightarrow{\rho_4 \leftrightarrow \rho_3} \left[\begin{array}{ccccc|c} \boxed{1} & 0 & 4 & 2 & 5 & 0 \\ 0 & \boxed{1} & 2 & 0 & 6 & 0 \\ 0 & 0 & 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

zeroes below 3^{rd} leader ✓
4^{th} leader?
none in 5^{th} column
(x_5 is free)

(leaders are x_1, x_2 & x_4)

This is in Echelon form.

Go to Reduced Echelon Form: check for zeroes above bottom

7:16 PM Sun Aug 22 28%

Linear Algebra 0

$$\rho_1 \to \rho_1 - 2\rho_3 \quad \longrightarrow \quad \left[\begin{array}{ccccc|c} \boxed{1} & 0 & 4 & 0 & 1 & 0 \\ 0 & \boxed{1} & 2 & 0 & 6 & 0 \\ 0 & 0 & 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

leader skew by row 3

check above second leader ✓

Is Reduced Echelon Form

Write the solution Set:

write as a system + Solve for the leaders

$$\begin{aligned} \boxed{1x_1} + 0x_2 + 4x_3 + 0x_4 + 1x_5 &= 0 \\ 0x_1 + \boxed{1x_2} + 2x_3 + 0x_4 + 6x_5 &= 0 \\ 0x_1 + 0x_2 + 0x_3 + \boxed{1x_4} + 1x_5 &= 0 \\ 0 \quad - \quad - \quad - \quad - \quad - &= 0 \end{aligned}$$

$$\begin{aligned} \boxed{x_1} &= 0 - 4x_3 - 1x_5 \\ \boxed{x_2} &= 0 - 2x_3 - 6x_5 \\ x_3 &= x_3 \text{ (free)} \\ \boxed{x_4} &= 0 - 1x_5 \\ x_5 &= x_5 \text{ (free)} \end{aligned}$$

homogeneous

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 - 4x_3 - 1x_5 \\ 0 - 2x_3 - 6x_5 \\ x_3 \\ 0 - 1x_5 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 - 4x_3 - 1x_5 \\ 0 - 2x_3 - 6x_5 \\ 0 + 1x_3 + 0x_5 \\ 0 + 0x_3 - 1x_5 \\ 0 + 0x_3 + 1x_5 \end{pmatrix} : x_3, x_5 \in \mathbb{R} \right\}$$

Write in position direction form:

pause + try

position, direction for x_3, direction for x_5

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -4 \\ -2 \\ +1 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -1 \\ -6 \\ 0 \\ -1 \\ +1 \end{pmatrix} : x_3, x_5 \in \mathbb{R} \right\}$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 4 & 2 & 5 & 0 \\ 0 & 1 & 2 & 0 & 6 & 0 \\ 1 & 1 & 6 & 2 & 11 & 0 \\ 2 & 0 & 8 & 5 & 11 & 0 \end{array}\right] \leftarrow$$

Check we have a solution

$1x_1 + 0x_2 + 4x_3 + 2x_4 + 5x_5 = 0$

...

when the free variables are zero
the position is a solution
so plugging it into the rows should give $= d_i$
in homogeneous case $0+0+0 \cdots +0 = 0$ ✓

In fact the i^{th} row of the augmented matrix dot product with position is

$$\begin{pmatrix} 1 \\ 0 \\ 4 \\ 2 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_5 \end{pmatrix} = d_i$$

Check your position using dot products.

each direction is a solution to the homogeneous system
taking its free variable $= 1$
and other free variables $= 0$.

Take a dot product $\text{row}_i \cdot \text{direction} = 0$

$$\begin{pmatrix} 1 \\ 0 \\ 4 \\ 2 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} -4 \\ -2 \\ +1 \\ 0 \\ 0 \end{pmatrix} = -4+0+4+0+0 = 0 \checkmark$$

$$\begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} -4 \\ -2 \\ +1 \\ 0 \\ 0 \end{pmatrix} = 0(-4)+1(-2)+2(1)+0\cdot 0+6\cdot 0 = 0 \checkmark$$

$$\begin{pmatrix} 1 \\ 1 \\ 6 \\ 2 \\ 11 \end{pmatrix} \cdot \begin{pmatrix} -4 \\ -2 \\ +1 \\ 0 \\ 0 \end{pmatrix} \overset{?}{=} \text{\_\_\_\_\_\_} = 0 \checkmark$$

$$\begin{pmatrix} 2 \\ 0 \\ 8 \\ 5 \\ 11 \end{pmatrix} \cdot \begin{pmatrix} -4 \\ -2 \\ +1 \\ 0 \\ 0 \end{pmatrix} = \text{\_\_\_\_\_\_} = 0 \checkmark$$

and then check the other direction with all four rows.

When you do this check, you will discover it is not zero!!!! There is an error in the row reduction! Below we show how to trace back and discover the error in the last row action! See Video 313F21-5-8b.

$\xrightarrow{\rho_4 \leftrightarrow \rho_3}$ $\left[\begin{array}{ccccc|c} \boxed{1} & 0 & 4 & 2 & 5 & 0 \\ 0 & \boxed{1} & 2 & 0 & 6 & 0 \\ 0 & 0 & 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$ zeroes below 3rd leader ✓ 4th leader? none in 5th column (x_5 is free)

(leaders are x_1, x_2 & x_4)

This is in Echelon form.

Go to Reduced Echelon Form: check for zeroes above bottom

pause + try

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{\text{position}} + x_3 \underbrace{\begin{pmatrix} -4 \\ -2 \\ +1 \\ 0 \\ 0 \end{pmatrix}}_{\text{direction for } x_3} + x_5 \underbrace{\begin{pmatrix} -1 \\ -6 \\ 0 \\ -1 \\ +1 \end{pmatrix}}_{\text{direction for } x_5} : x_3, x_5 \in \mathbb{R} \right\}$$

$\left[\begin{array}{ccccc|c} 1 & 0 & 4 & 2 & 5 & 0 \\ 0 & 1 & 2 & 0 & 6 & 0 \\ 1 & 1 & 6 & 2 & 11 & 0 \\ 2 & 0 & 8 & 5 & 11 & 0 \end{array}\right]$ ← check we have a solution

$1x_1 + 0x_2 + 4x_3 + 2x_4 + 5x_5 = 0$

...

$\begin{pmatrix} 1 \\ 0 \\ 4 \\ 2 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} -4 \\ -2 \\ +1 \\ 0 \\ 0 \end{pmatrix} = -4+0+4+0+0=0$ ✓

$\begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} -4 \\ -2 \\ +1 \\ 0 \\ 0 \end{pmatrix} = 0(-4)+1(-2)+2(1)+0\cdot 0+6\cdot 0 = 0$ ✓

$\begin{pmatrix} 1 \\ 1 \\ 6 \\ 2 \\ 11 \end{pmatrix} \cdot \begin{pmatrix} -4 \\ -2 \\ +1 \\ 0 \\ 0 \end{pmatrix} = \_\_\_\_\_\_ = 0$ ✓ ✓

$\begin{pmatrix} 2 \\ 0 \\ 8 \\ 5 \\ 11 \end{pmatrix} \cdot \begin{pmatrix} -4 \\ -2 \\ +1 \\ 0 \\ 0 \end{pmatrix} = \_\_\_\_\_\_ = 0$ ✓ ✓

and then check the <u>other direction</u> with all four rows.

Fails!

Deliberate Error in this problem that you catch!

pause + try

$$\begin{pmatrix} 1 \\ 0 \\ 4 \\ 2 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -6 \\ 0 \\ -1 \\ +1 \end{pmatrix} = 1(-1)+0(-6)+4\cdot 0+2(-1)+5(1) = -1-2+5=3$$

Not zero!

So there is an error in the second direction!

Go up + find source of the error and fix it!

$$\left[\begin{array}{ccccc|c} \boxed{1} & 0 & 4 & 0 & 1 & 0 \\ 0 & \boxed{1} & 2 & 0 & 6 & 0 \\ 0 & 0 & 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

check above second leader ✓

Is Reduced Echelon Form

Write the solution Set:
write as a system + Solve for the leaders

$$\begin{aligned} \boxed{1x_1} + 0x_2 + 4x_3 + 0x_4 + 1x_5 &= 0 \\ 0x_1 + \boxed{1x_2} + 2x_3 + 0x_4 + 6x_5 &= 0 \\ 0x_1 + 0x_2 + 0x_3 + \boxed{1x_4} + 1x_5 &= 0 \\ 0 \quad - \quad - \quad - \quad - \quad - &= 0 \end{aligned}$$

homogeneous

$$\begin{aligned} \boxed{x_1} &= 0 - 4x_3 - 1x_5 \\ \boxed{x_2} &= 0 - 2x_3 - 6x_5 \\ x_3 &= x_3 \text{ (free)} \\ \boxed{x_4} &= 0 - 1x_5 \\ x_5 &= x_5 \text{ (free)} \end{aligned}$$

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 - 4x_3 - 1x_5 \\ 0 - 2x_3 - 6x_5 \\ x_3 \\ 0 - 1x_5 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 - 4x_3 - 1x_5 \\ 0 - 2x_3 - 6x_5 \\ 0 + 1x_3 + 0x_5 \\ 0 + 0x_3 - 1x_5 \\ 0 + 0x_3 + 1x_5 \end{pmatrix} : x_3, x_5 \in \mathbb{R} \right\}$$

Write in position direction form:

pause + try

position — direction for x_3 — direction for x_5

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -4 \\ -2 \\ +1 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -1 \\ -6 \\ 0 \\ -1 \\ +1 \end{pmatrix} : x_3, x_5 \in \mathbb{R} \right\}$$

error in direction for x_5

$$\left[\begin{array}{ccccc|c} 1 & 0 & 4 & 2 & 5 & 0 \\ 0 & 1 & 2 & 0 & 6 & 0 \\ 1 & 1 & 6 & 2 & 11 & 0 \\ 2 & 0 & 8 & 5 & 11 & 0 \end{array}\right]$$

Check we have a solution

$1x_1 + 0x_2 + 4x_3 + 2x_4 + 5x_5 = 0$
...

pause + try

Classwork ②
Solve and Write Solution for

$$\left[\begin{array}{ccccc|c} \boxed{1} & 0 & 4 & 2 & 5 & 0 \\ 0 & 1 & 2 & 0 & 6 & 0 \\ 1 & 1 & 6 & 2 & 11 & 0 \\ 2 & 0 & 8 & 5 & 11 & 0 \end{array}\right]$$

box 1st leader
is it a 1? yes
change numbers below it
$\rho_i \to \rho_i + k\rho_1$

$$\xrightarrow[\rho_4 \to \rho_3 - 2\rho_1]{\rho_3 \to \rho_3 - \rho_1} \left[\begin{array}{ccccc|c} \boxed{1} & 0 & 4 & 2 & 5 & 0 \\ 0 & \boxed{1} & 2 & 0 & 6 & 0 \\ 0 & 1 & 2 & 0 & 6 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

box 2nd leader
is it a 1? ✓
chang below to 0
$\rho_i \to \rho_i + k\rho_2$

$$\xrightarrow{\rho_3 \to \rho_3 - \rho_2} \left[\begin{array}{ccccc|c} \boxed{1} & 0 & 4 & 2 & 5 & 0 \\ 0 & \boxed{1} & 2 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 1 & 0 \end{array}\right]$$

box 3rd leader
none in 3rd column
(x_3 is free)
in 4th column

move that leader to the third row

$$\xrightarrow{\rho_4 \leftrightarrow \rho_3} \left[\begin{array}{ccccc|c} \boxed{1} & 0 & 4 & 2 & 5 & 0 \\ 0 & \boxed{1} & 2 & 0 & 6 & 0 \\ 0 & 0 & 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

zeroes below 3rd leader ✓
4th leader?
none in 5th column
(x_5 is free)

(leaders are x_1, x_2 + x_4)

This is in Echelon form

Go to Reduced Echelon Form: check for zeroes above bottom

last row action error!

$5 - 2(1) = 5 - 2 = 3$

$$\xrightarrow{\rho_1 \to \rho_1 - 2\rho_3} \left[\begin{array}{ccccc|c} \boxed{1} & 0 & 4 & 0 & 1 & 0 \\ 0 & \boxed{1} & 2 & 0 & 6 & 0 \\ 0 & 0 & 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

leader skew by row 3

'error is in x_5 column
" column 5"

check above second leader ✓

Is Reduced Echelon Form

Write the solution Set:
write as a system + Solve for the leaders

$$\begin{aligned} \boxed{1x_1} + 0x_2 + 4x_3 + 0x_4 + 1x_5 &= 0 \\ 0x_1 + \boxed{1x_2} + 2x_3 + 0x_4 + 6x_5 &= 0 \\ 0x_1 + 0x_2 + 0x_3 + \boxed{1x_4} + 1x_5 &= 0 \\ 0 \quad - \quad - \quad - \quad - \quad - &= 0 \end{aligned}$$

homogeneous

$$\begin{aligned} \boxed{x_1} &= 0 - 4x_3 - 1x_5 \\ \boxed{x_2} &= 0 - 2x_3 - 6x_5 \\ x_3 &= x_3 \text{ (free)} \\ \boxed{x_4} &= 0 - 1x_5 \\ x_5 &= x_5 \text{ (free)} \end{aligned}$$

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 - 4x_3 - 1x_5 \\ 0 - 2x_3 - 6x_5 \\ x_3 \\ 0 - 1x_5 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 - 4x_3 - 1x_5 \\ 0 - 2x_3 - 6x_5 \\ 0 + 1x_3 + 0x_5 \\ 0 + 0x_3 - 1x_5 \\ 0 + 0x_3 + 1x_5 \end{pmatrix} : x_3, x_5 \in \mathbb{R} \right\}$$

Write in position direction form:

pause + try

position — direction for x_3 — direction for x_5

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -4 \\ -2 \\ +1 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -1 \\ -6 \\ 0 \\ -1 \\ +1 \end{pmatrix} : x_3, x_5 \in \mathbb{R} \right\}$$

error in direction for x_5

$$\left[\begin{array}{ccccc|c} 1 & 0 & 4 & 2 & 5 & 0 \\ 0 & 1 & 2 & 0 & 6 & 0 \\ 1 & 1 & 6 & 2 & 11 & 0 \\ 2 & 0 & 8 & 5 & 11 & 0 \end{array}\right]$$

Check we have a solution

$1x_1 + 0x_2 + 4x_3 + 2x_4 + 5x_5 = 0$
...

$5-2(1)=5-2=3$

fix

$\rho_1 \to \rho_1 - 2\rho_3 \longrightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 4 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 & 6 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

leader skew by row 3

check above second leader ✓

Is Reduced Echelon Form

Write the solution Set:

write as a system + Solve for the leaders

$1x_1 + 0x_2 + 4x_3 + 0x_4 + 3x_5 = 0$

$0x_1 + 1x_2 + 2x_3 + 0x_4 + 6x_5 = 0$

$0x_1 + 0x_2 + 0x_3 + 1x_4 + 1x_5 = 0$

$0 - - - - - = 0$

$x_1 = 0 - 4x_3 - 3x_5$

$x_2 = 0 - 2x_3 - 6x_5$

$x_3 = x_3$ (free)

$x_4 = 0 - 1x_5$

$x_5 = x_5$ (free)

homogeneous

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 - 4x_3 - 3x_5 \\ 0 - 2x_3 - 6x_5 \\ x_3 \\ 0 - 1x_5 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 - 4x_3 - 3x_5 \\ 0 - 2x_3 - 6x_5 \\ 0 + 1x_3 + 0x_5 \\ 0 + 0x_3 - 1x_5 \\ 0 + 0x_3 + 1x_5 \end{pmatrix} : x_3, x_5 \in \mathbb{R} \right\}$$

Write in position direction form:

pause + try

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{\text{position}} + x_3 \underbrace{\begin{pmatrix} -4 \\ -2 \\ +1 \\ 0 \\ 0 \end{pmatrix}}_{\text{direction for } x_3} + x_5 \underbrace{\begin{pmatrix} -3 \\ -6 \\ 0 \\ -1 \\ +1 \end{pmatrix}}_{\text{direction for } x_5} : x_3, x_5 \in \mathbb{R} \right\}$$

$$\begin{pmatrix}1\\0\\4\\2\\5\end{pmatrix}\cdot\begin{pmatrix}-3\\-6\\0\\-1\\+1\end{pmatrix} = 1(-3)+0(-6)+4\cdot 0+2(-1)+5(1) = -3-2+5=0$$

Fixed! ~~Not zero~~!

Now check the other rows with the fixed direction.

$$\begin{pmatrix}0\\1\\2\\0\\6\end{pmatrix}\cdot\begin{pmatrix}-3\\-6\\0\\-1\\1\end{pmatrix} = 0(-3)+1(-6)+2(0)+0(-1)+6(1)$$

$$= 0-6+0+0+6 = 0 \checkmark$$

check last two rows as well!

The following notes are explained in 313F21-5-9a 313F21-5-9b 313F21-5-9c at the end of the Playlist 313F21-5-1to9.

313F21-5-9a

Every Linear System:

$$\begin{aligned} a_{11}x_1 + \dots + a_{1m}x_m &= d_1 \\ a_{21}x_1 + \dots + a_{2m}x_m &= d_2 \\ &\dots \\ a_{n1}x_1 + \dots + a_{nm}x_m &= d_n \end{aligned}$$

has a homogenous system:

$$\begin{aligned} a_{11}x_1 + \dots + a_{1m}x_m &= 0 \\ a_{21}x_1 + \dots + a_{2m}x_m &= 0 \\ &\dots \\ a_{n1}x_1 + \dots + a_{nm}x_m &= 0 \end{aligned}$$

that has the same coefficients $a_{i,j}$.

When we solve them:

$$\left[\begin{array}{cccc|c} \boxed{a_{11}} & a_{12} & \dots & a_{1m} & d_1 \\ a_{21} & a_{22} & \dots & a_{2m} & d_2 \\ & & \dots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} & d_n \end{array}\right] \qquad \left[\begin{array}{cccc|c} \boxed{a_{11}} & a_{12} & \dots & a_{1m} & 0 \\ a_{21} & a_{22} & \dots & a_{2m} & 0 \\ & & \dots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} & 0 \end{array}\right]$$

Same row actions are determined by the coefficients

Reduced Echelon Form

$$\left[\begin{array}{c|c} E & \begin{matrix} \bullet \\ \bullet \\ \bullet \\ \bullet \end{matrix} \end{array}\right] \qquad \left[\begin{array}{c|c} E & \begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \end{matrix} \end{array}\right]$$

So when we solve for leaders

leader = position + (free)(direction) + ... + (free)(dir)

for the homogeneous system the position is zero

same free variables + directions

Sometimes the original system has no solution due to a row

$0 \quad 0 \quad 0 \quad 0 \mid \bullet$ ← nonzero entry here

but the homogeneous system always

$0 \quad 0 \quad 0 \quad 0 \mid 0$ has a solution

Every Linear System: has a homogenous system:

$$\boxed{\begin{aligned} a_{11}x_1 + \dots + a_{1m}x_m &= d_1 \\ a_{21}x_1 + \dots + a_{2m}x_m &= d_2 \\ &\dots \\ a_{n1}x_1 + \dots + a_{nm}x_m &= d_n \end{aligned}} \qquad \boxed{\begin{aligned} a_{11}x_1 + \dots + a_{1m}x_m &= 0 \\ a_{21}x_1 + \dots + a_{2m}x_m &= 0 \\ &\dots \\ a_{n1}x_1 + \dots + a_{nm}x_m &= 0 \end{aligned}}$$

that has the same coefficients $a_{i,j}$.

$$\boxed{\sum_{j=1}^{m} a_{ij}x_j = d_i \text{ for } i=1 \text{ to } n} \qquad \boxed{\sum_{j=1}^{m} a_{ij}x_j = 0 \text{ for } i=1 \text{ to } n}$$

$$\boxed{\begin{gathered} row_i \cdot \vec{x} = d_i \\ \text{for } i=1 \text{ to } n \end{gathered}} \qquad \boxed{row_i \cdot \vec{x} = 0}$$

Remember $row_i = \rho_i = (a_{i,1}, \dots, a_{i,m})$

Theorem: If $\vec{p}$ is a particular solution to the linear system and if $\vec{y}$ is a solution to the homogeneous system with the same coefficients then $\vec{p}+\vec{y}$ is a solution to the original linear system.

Given: $\vec{p}$ is a soln to $\sum_{j=1}^{m} a_{ij}x_j = d_i$

$\vec{y}$ is a soln to $\sum_{j=1}^{m} a_{ij}x_j = 0$

Show: $\vec{x} = \vec{p}+\vec{y}$ is a soln to $\sum_{j=1}^{m} a_{ij}x_j = d_i$

313F21-5-9c

Rewrote this with formula

Given: $\sum_{j=1}^{m} a_{ij} p_j = d_i$ and $\sum_{j=1}^{m} a_{ij} y_j = 0$

Show: $\underbrace{\sum_{j=1}^{m} a_{ij} (p_j + y_j)}_{LHS} = \underbrace{d_i}_{\text{RHS already simplified}}$ because $x_j = p_j + y_j$ by defn of vector addition

Proof:

① $LHS = \sum_{j=1}^{m} a_{ij} (p_j + y_j) = \sum_{j=1}^{m} (a_{ij} p_j + a_{ij} y_j)$ ① by distribution of reals

② $= \sum_{j=1}^{m} a_{ij} p_j + \sum_{j=1}^{m} a_{ij} y_j$ ② by reordering sums

③ $= d_i + 0$ ③ by given $\sum_{j=1}^{m} a_{ij} p_j = d_i$; by given $\sum_{j=1}^{m} a_{ij} y_j = 0$

④ $= d_i$ ← RHS ④ $x + 0 = x$ for any real number x.

QED

Complete HW-HW4 below checking your answers using the dot products as explained above. HW5 is almost the same as the last classwork.

$\boxed{\text{HW1}}$ Solve

$$\boxed{\begin{aligned} x_1 + x_8 &= 2 \\ x_3 + x_5 &= 4 \\ x_2 + x_7 &= 6 \\ x_4 + x_6 &= 8 \end{aligned}}$$

a

$\boxed{\text{HW2}}$ Solve

using work from HW1

and show it has the same direction vectors as HW1.

$$\boxed{\begin{aligned} x_1 + x_8 &= 0 \\ x_3 + x_5 &= 0 \\ x_2 + x_7 &= 0 \\ x_4 + x_6 &= 0 \end{aligned}}$$

$\boxed{\text{HW3}}$ Solve $\left[\begin{array}{ccccccc|c} 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 4 \end{array}\right]$

$\boxed{\text{HW4}}$ Solve the homog systems for HW3 and show you get the same direction vectors

$\boxed{\text{HW5}}$ Prove that if $\vec{v}$ and $\vec{w}$ solve a homogenous system and $\vec{z} = \vec{v} + \vec{w}$ then $\vec{z}$ solves it too.

$\boxed{\text{Extra Credit}}$ Prove that if $\vec{y}$ solves a homogeneous system and $\vec{z} = r\vec{y}$ then $\vec{z}$ solves it too.

There is a proof on Exam 1. Students who want more practice with proofs may try the proofs in the following Vector Calc videos, pausing and trying each step yourself: 226F21-1-7, 226F21-1-9, and 226F21-2-3.

Once you have completed the classwork and homework for this lesson, submit it and email me to request access to the Sample for Exam 1 and to schedule your exam (please tell me all times you are available):

Th Sept 22 3-4:30 pm
Mon Sept 26 8-9:30 pm
Wed Sept 28 8-9:30 pm