Article

Non-local Constitutive Relations with Hidden Parameter for the Vector Potential in Maxwell Equations

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Abstract

The main goal of this paper is to establish a new general nonlocality constitutive relationship for electromagnetic field $\mathbf{E}$, $\mathbf{B}$, $\mathbf{D}$, and $\mathbf{H}$ in frequency-time domain setting. The four basic assumptions of the medium are linearity, invariance to time translations, causality and continuity. A short review of the Born, Born-Fedorov and Engheta-Jaggard formalism of media in chiral media, respectively, is made; also expressions for scalar and vector potentials are derived which satisfies a velocity gauge. The chiral parameter T appears as a hidden factor for the vector potential $\mathbf{A}$. This velocity gauge condition is not satisfied by other chiral formalisms, so the proposed approach is useful to numerical calculations and applications of scalar and vector potentials. In this context, a major role could be played by the electromagnetic (EM) simulators, which are usually employed to solve very complex and challenging EM field problems. In connection with the Dirac equation for the graphene systems, we find analytically the Fermi velocity in terms of the velocity gauge and the fine-structure constant.

Keywords: Maxwell equations, velocity gauge, non-local, constitutive relations.

1. Introduction

Chiral media are interesting from a homogenization point of view, since they are difficult to treat rigorously with classical methods. In 1914, Lindman created an artificial medium composed of small metal coils, all wound the same way, embedded in a background material [1, 2]. The resulting material exhibited different propagation speed for left and right hand circularly polarized waves, respectively. Many others have followed, and a similar effect exists for visible light in sugar solutions, where the different refractive indices for left and right circular polarized light is explained by the fact that the sugar molecules found in nature have a handedness, which introduces a preference for left or right. For more information on chiral media we refer to the book [3] and references therein. An interesting fact about chiral materials is that there is some arbitrariness in how to model the chirality. Essentially, four different models can be identified: the Lindell**,** Engheta, Tellegen and Drude–Born–Fedorov model, respectively. For time-harmonic fields, all these models can be transformed into each other, and one may ask if any model is more

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natural than the others. In the contribution of [4], he shows that within a particular framework of homogenization, the Drude–Born–Fedorov model seems more natural than the others.

Homogenization is the science of calculating effective material parameters for composite materials, based on knowledge of the material parameters of the components and the microscopic structure in which the components are mixed. We refer to the books [5, 6] and references therein for a review of this huge field. The problem with classical homogenization is that in order to formulate rigorous mathematical results, it is assumed that the applied wavelength is infinitely large compared with the unit cell. In this limit, the electric and magnetic fields decouple, and it seems there can be no chiral effects. In recent years, new formalisms which provide a framework of finite scale homogenization have been developed [7]; these provide a window of opportunity to rigorously calculate chiral effects in composite media. If the permittivity and permeability of the component materials involved have so small losses that they can be considered as real valued parameters, only the Drude–Born–Fedorov model can be considered as a reasonable choice. The argument is surprisingly simple, and rests on a simple symmetry relation for Maxwell's equations.

The Maxwell equations for the macroscopic free electromagnetic fields, (without charge and current) are well known:

$$\nabla\times\mathbf{E}=\partial_t\mathbf{B},\quad \nabla\times\mathbf{H}=\partial_t\mathbf{D},\quad \nabla\cdot\mathbf{B}=0,\quad \nabla\cdot\mathbf{D}=0,$$

(1.1)

these equations, however, are not complete. Six more equations, the constitutive relations, have to be added relating the electric field $\mathbf{E}$, the magnetic induction $\mathbf{B}$, the displacement field $\mathbf{D}$ and the magnetic field $\mathbf{H}$ to each other; these constitutive relations are completely independent of the Maxwell equations, which involve only the fields and their sources. The constitutive relations are concerned with the equations of motion of the constituents of the medium in an electromagnetic field [8-14]. Traditionally, these constitutive relations are described as a relation at fixed frequency. The intensified interest in transient phenomena, however, especially wave propagation properties in more complex media, motivates a fresh look at these problems from a different starting point. The constitutive relations in its most general form are usually given as a relationship between the pairs of fields $\{\mathbf{D},\mathbf{H}\}$ and $\{\mathbf{E},\mathbf{B}\}$. Other combinations between different pairs of fields are also frequently used [13-15]. The constitutive relations employed in this paper can formally be written as a general functional dependence:

$$\mathbf{E}=\mathbf{E}\{\mathbf{D},\mathbf{H}\},\quad \mathbf{B}=\mathbf{B}\{\mathbf{D},\mathbf{H}\}.$$

(1.2)

If space is empty the vacuum relations between the fields hold, i.e.

$$\mathbf{E} = \mathbf{D}/\varepsilon_0 \quad , \quad \mathbf{B} = \mu_0 \mathbf{H} \quad ,$$

(1.3)

where ε_0 and μ_0 are the vacuum permittivity and permeability, respectively.

The local wave equation for $\mathbf{B}$ is given by:

$$\nabla^2 \mathbf{B} + \omega^2 \mu_0 \varepsilon_0 \mathbf{B} = 0 \quad .$$

(1.4a)

The difference between the non-vacuum relations and the vacuum ones reflects the presence of a medium. The most frequently used constitutive relations in the literature deal with the case of no coupling between the electric and the magnetic fields. Electric polarization and magnetization are then two separate phenomena and the constitutive relations separate into two functional; one relating the electric fields $\mathbf{E}$ and $\mathbf{D}$, to each other and a separate one relating the magnetic fields, $\mathbf{B}$ and $\mathbf{H}$. There are, however, several classes of materials that do show magneto-electric behavior, and these are modeled by a coupling between the electric and magnetic fields in the constitutive relations. The general name for these constitutive relations having a coupling between the electric and the magnetic fields is bi-anisotropic media [16, 17]. All models lead to partial differential equations where the coefficient multiplying the principal part of the equations is small in some sense. This leads to drastic changes in the wave propagation properties as this constant varies.

The constitutive relations in the electromagnetic case are usually stated as relations between the appropriate fields for a fixed frequency. A Fourier transformation then transforms the fixed frequency result to the time domain. Causality and time invariance are naturally built into the formulations, whereas in the fixed frequency formulation, these properties have to be added to the constitutive relations at a later stage. Some of the mathematical notations used in this paper are introduced in Sec. 2. Also, the different constitutive relations of chiral media founded in literature are presented in Sec. 2. The new approach for chiral media and the velocity gauge are introduced in Secs. 3 and 4, respectively, discussing this gauge in graphene systems in Sec. 5.

2. Constitutive relations for bi-isotropic media in frequency formulation

Symmetry plays an important role in the electromagnetic (EM) response of matter. It is revealed in the form of susceptibilities relating electric and magnetic polarization ($\mathbf{P}$ $\mathbf{M}$) with source

EM field. In high symmetry case, $\mathbf{P}$ and $\mathbf{M}$ consist of independent groups of excitations belonging to different irreducible representations of the symmetry group in consideration. This allows us to treat electric and magnetic properties of matter independently. When a medium lacks in certain mirror symmetry, i.e., the case of chiral symmetry, however, some (or all the) components of $\mathbf{P}$ and $\mathbf{M}$ cannot be distinguished, so that they can be induced by both electric and magnetic source fields. The constitutive relations that are needed to fully describe general bi-isotropic media require four scalar material parameters. There are different constitutive equations due to the various possibilities to link the electric ($\mathbf{E}$ and $\mathbf{D}$) and the magnetic ($\mathbf{H}$ and $\mathbf{B}$) field and flux quantities. There are four major models that have been used to model isotropic chiral materials. For time harmonic fields in source free regions, the different models can be transformed into each other. If sources are present they may need to be transformed too, the Born–Fedorov model appears as the natural choice for describing chiral materials. Since it involves spatial derivatives in the constitutive relation, it is a nonlocal relation. This means special care has to be taken when using the formulation in a scattering problem, where the proper boundary conditions must be applied. Some of these problems are treated in, e.g. [8]. Considering susceptibilities relating macroscopic electric and magnetic polarization ($\mathbf{P}$ $\mathbf{M}$) it is possible to resume the different chiral formalisms.

2.1 Engheta-Jaggard formalism

Here we assume sinusoidal time dependence $\exp(i\omega t)$:

$$\mathbf{D} = \varepsilon_{EJ}\mathbf{E} + i\gamma\mathbf{B}, \qquad \mathbf{H} = \frac{1}{\mu_{EJ}}\mathbf{B} + i\gamma\mathbf{E},$$

(2.1)

the wave propagation is:

$$\nabla^2\mathbf{E}\text{-}2\gamma\mu_{EJ}\omega\nabla\times\mathbf{E}\text{-}\omega^2(\mu_{EJ}\varepsilon_{EJ})\mathbf{E}=0,$$

(2.2)

and the propagation constant of the RCP and LCP waves are [13, 14]:

$$k_{RCP}=\omega(\mu_{EJ}\gamma+\sqrt{\mu_{EJ}\varepsilon_{EJ}+\mu_{EJ}^2\gamma^2}), \qquad k_{LCP}=\omega(-\mu_{EJ}\gamma+\sqrt{\mu_{EJ}\varepsilon_{EJ}+\mu_{EJ}^2\gamma^2}). \qquad (2.3)$$

This formalism was obtained by Jaggard, in a phenomenological form by considering a chiral medium composed of a dilute suspension of perfectly conducting single turn helices embedded in an otherwise achiral host medium.

2.2 Born formalism

Here the relation between $\mathbf{B}$ and $\mathbf{H}$ is local:

$$\mathbf{D} = \epsilon_{BF}\left[\mathbf{E}+\beta(\nabla\times\mathbf{E})\right],\quad \mathbf{B}=\mu_{BF}\mathbf{H},$$

(2.4)

the pseudoscalar β is the chirality parameter [10]; a media following the relations (2.4) is non-reciprocal. Besides, the use of (2.4) fails to satisfy the standard boundary conditions and results in physically unacceptable amplitudes around the critical angles of incidence. By these reasons, the analysis of Uckun [15] comparing formalism 2.1 with 2.2 is wrong. The correct comparison is with the Born-Fedorov formalism [12] which is explained below in subsection 2.3.

The wave propagation is:

$$\nabla\times\nabla\times\mathbf{E}\text{-}\beta\mu_B\varepsilon_B\omega^2\nabla\times\mathbf{E}\text{-}\omega^2(\mu_B\varepsilon_B)\mathbf{E}=0,$$

(2.5)

and the propagation constant of the RCP and LCP waves are [10, 12]:

$$k_{RCP}=\omega(-\omega\mu_B\varepsilon_B\beta/2+\sqrt{\mu_B\varepsilon_B+(\omega\mu_B\varepsilon_B\beta/2)^2}),\qquad (2.6)$$

$$k_{LCP}=\omega(\omega\mu_B\varepsilon_B\beta/2+\sqrt{\mu_B\varepsilon_B+(\omega\mu_B\varepsilon_B\beta/2)^2})$$

(2.7)

The formalisms between Born and Engheta are equivalent if $\omega\varepsilon\beta=-2\gamma$, but this correspondence is not adequate because transforms the well defined relations (2.1) in a problematic relations (2.2) where the relation $\mathbf{D}(\mathbf{E})$ is non-local and $\mathbf{B}(\mathbf{H})$ is local. Besides, it is not useful for study of two media or chiral slab.

2.3 Born-Fedorov formalism

Here the relation between $\mathbf{B}$ and $\mathbf{H}$ is non-local:

$$\mathbf{D} = \epsilon_{BF}\left[\mathbf{E}+T(\nabla\times\mathbf{E})\right],\quad \mathbf{B}=\mu_{BF}\left[\mathbf{H}+T(\nabla\times\mathbf{H})\right],$$

(2.8)

the wave propagation is:

$$(1-\omega^2\mu_{BF}\varepsilon_{BF}T^2)\nabla\times\nabla\times\mathbf{E}\text{-}2T\mu_{BF}\varepsilon_{BF}\omega^2\nabla\times\mathbf{E}\text{-}\omega^2(\mu_{BF}\varepsilon_{BF})\mathbf{E}=0, \qquad (2.9)$$

and the propagation constant of the RCP and LCP waves are [12]:

$$k_{RCP}=\omega(-\omega\mu_{BF}\varepsilon_{BF}T+\sqrt{\mu_{BF}\varepsilon_{BF}})(1-\omega^2\mu_{BF}\varepsilon_{BF}T^2)^{-1},$$

(2.10)

$$k_{LCP}=\omega(\omega\mu_{BF}\varepsilon_{BF}T+\sqrt{\mu_{BF}\varepsilon_{BF}})(1-\omega^2\mu_{BF}\varepsilon_{BF}T^2)^{-1}.$$

(2.11)

Relations (2.8) are not only symmetric under time reversibility and duality transformations but also satisfy the test set up by Silverman to distinguish between (2.4) and (2.5) with T as the measure of chirality expressed as unit of length [8]. The validity of (2.8) has been affirmed by studies carried on optically active molecules as well as from the examination of light propagation in optically active crystals. The non-local character of (2.8) needs to be noticed, because the magnetization M depends not only on H, but also on the circulation of H, (rot H).

Here, the chiral parameter is represented by γ , β and T , respectively. The relationship between formalisms 2.1 and 2.3 is well known and has been used in different applications [18-27]. The correspondence between (2.1) and (2.3) holds if $\omega\varepsilon\beta=\gamma$, $\varepsilon_{BF}=\varepsilon_{EJ}$ and $\mu_{EJ}=\mu_{BF}(1-\omega^2\mu_{BF}\varepsilon_{BF}T^2)^{-1}$. Using the vector potential $\mathbf{A}$, equivalence between the Born formalism and Engheta formalisms is not suitable because it produces physical problems in terms of symmetry [15] . For this reason Uckun graphics can be illustrative, but must be corrected using the formalism Fedorov-Born [28]. All chiral wave equations discussed here, have a term of first order in the field that is proportional to the rotor field ($\nabla\times\mathbf{E}$) or $\nabla\times\mathbf{H}$. The question that emerges here is *can you obtain a wave equation which does not appear explicitly this term?* In this paper we show that the answer is positive, that is there are wave equations for the set $\{\mathbf{A},V\}$ fields, where no term appears proportional to the rotor $\nabla\times\mathbf{A}$. For the magnetic potential field, chirality appears as a hidden variable.

3. Like Lorenz gauge in chiral media

The introduction of auxiliary potentials is a common procedure in dealing with electromagnetic problems. In this way one can recast Maxwell's equations with chiral constitutive relations into simple forms that can be numerically treated by a variety of techniques. Due to a lack of uniqueness in the definition of potentials, we can conveniently impose conditions on potentials without affecting the results of electromagnetic fields. Such choices are gauges, and the most common ones are Lorenz and Coulomb gauges. It is shown that the well-known procedure for proving the equivalence of the expressions for the electric field calculated using the Lorenz and Coulomb gauges is incorrect. The difference between the two gauges is due to the difference in the speed of propagation of a disturbance of the scalar potential. As an auxiliary result, it is proven that the solution for the electric field cannot be obtained directly from the Maxwell equations, i.e. without introducing the scalar and vector potentials.

It has been established in classical electrodynamics that the EM potential cannot be treated as a physical quantity, but as a mathematical tool for calculating EM fields [2] (Ch. 6.5, 3). Therefore, the solutions of the wave equation can be chosen so that the speed of propagation v of the scalar potential can vary from zero to infinity. By choosing such solutions, the gauge is also determined [4]. In this paper we study the velocity gauge since it specifies the divergence of the vector potential in terms of the velocity of propagation of the scalar potential. This velocity is the speed of light for the Lorenz gauge and infinite for the Coulomb gauge; with the velocity gauge, the selection of a gauge can be explained in physical terms—the specification of the velocity of propagation of the scalar potential—which engineering students might find more intuitive than a suspiciously arbitrary choice made for mathematical convenience. A class of gauges (velocity gauge) is described as in which the scalar potential propagates at an arbitrary speed v relative to the speed of light. The Lorenz and Coulomb gauges are special cases of the v-gauge. For the formalisms presented in Sec. 2 we have a velocity gauge because v is different to c, the light velocity, because: $\mathrm{v}=\mathrm{v}(\gamma)$ or $\mathrm{v}=\mathrm{v}(\beta)$, $\mathrm{v}=\mathrm{v}(T)$, respectively. The proposed approach of the chiral formalism is appropriate to describe electromagnetic waves in terms of vector and scalar potential where v is not equal to c. The Lorenz gauge condition is covariant with respect to the Lorentz transformations are studied in [18, 19].

Let the Maxwell equations $\nabla\times\mathbf{E}=-\frac{\partial}{\partial t}\mathbf{B}$, $\nabla\cdot\mathbf{B}=0$, $\nabla\times\mathbf{H}=\frac{\partial}{\partial t}\mathbf{D}$, $\nabla\cdot\mathbf{D}=\rho$ with the non locality definitions:

$$\mathbf{B}=\mu_{T}\left[\mathbf{H}+T(\nabla\times\mathbf{H})\right],$$

(3.1)

$$\mathbf{D}=\epsilon_{T}\left[\mathbf{E}+T(\nabla\times\mathbf{E})\right],$$

(3.2)

$$\mathbf{B}=\nabla\times\left(\mathbf{A}+T\nabla\times\mathbf{A}\right)=\nabla\times\mathbf{F}, \tag{3.3}$$

$$\left(\mathbf{A}+T\nabla\times\mathbf{A}\right)=\mathbf{F},$$

(3.4)

$$\nabla \cdot \mathbf{F} = \nabla \cdot \mathbf{A} \, ,$$

(3.5)

$$\mathbf{E} = -\frac{\partial}{\partial t}\mathbf{F} - \nabla V \, .$$

(3.6)

The non local wave equation for $\mathbf{B}$ is:

$$\nabla^2 \mathbf{B}(T) + \omega^2 \mu_0 \varepsilon_0 \mathbf{B}(T) = 0 \, , \qquad (1.4b)$$

and if we compare the local wave equation (1.4a) with the non-local wave equation (1.4b) we can see that the chiral effect T is like a hidden parameter.

Now, we can make $\mu_T = \mu_0$, $\varepsilon_T = \varepsilon_0$ so we have that the general wave equation for $\mathbf{F}$ and V is:

$$(1 - \omega^2 \mu_0 \varepsilon_0 T^2)\nabla^2 \mathbf{F} + 2\omega^2 \mu_0 \varepsilon_0 \nabla \times \mathbf{F} + \omega^2 \mu_0 \varepsilon_0 \mathbf{F} \, ,$$

(3.7)
and:

$$\nabla(\nabla \cdot \mathbf{F})(1 - \omega^2 \mu_0 \varepsilon_0 T^2) + i\omega \mu_0 \varepsilon_0 \nabla V = 0 \, ,$$

(3.8)

where we are removed the coupling term by introducing the so-called like Lorenz gauge:

$$\nabla \cdot \mathbf{F}(1 - \omega^2 \mu_0 \varepsilon_0 T^2) + i\omega \mu_0 \varepsilon_0 V = 0 \, . \qquad (3.9)$$

This like Lorenz gauge condition is covariant with respect to the Lorentz transformations; here we obtain the well known result with $\omega^2 / c^2 = k^2$. As $(\mathbf{A} + T\nabla \times \mathbf{A}) = \mathbf{F}$ is a linear combination of $\mathbf{A}$, finally we have:

$$(1 - \omega^2 \mu_0 \varepsilon_0 T^2)\nabla^2 \mathbf{A} + 2\omega^2 \mu_0 \varepsilon_0 \nabla \times \mathbf{A} + \omega^2 \mu_0 \varepsilon_0 \mathbf{A} \, ,$$

(3.10)

$$\nabla \cdot \mathbf{A}(1 - \omega^2 \mu_0 \varepsilon_0 T^2) + i\omega \mu_0 \varepsilon_0 V = 0 \, ,$$

(3.11)

this equation can be expressed as:

$$\nabla\cdot\mathbf{A}+(i\omega\mu_0\varepsilon_0/(1-\omega^2\mu_0\varepsilon_0T^2)V=0,$$

(3.11')

$$\nabla\cdot\mathbf{A}+(i\omega/(v^2)V=0,$$

(3.12)

where v is the velocity gauge and the Lorenz gauge is a special case of the v-gauge.

Equations (3.7) and (3.9) or (3.10) and (3.11') are decoupled for numerical calculation. A special case results if we consider spatial parallel field configuration (Beltrami regimen). If we choose $\mathbf{F}=\frac{1}{2}\mathbf{A}$ we have:

$$\nabla\times\mathbf{D}=-\frac{1}{2T}\mathbf{D},\quad \nabla\times\mathbf{H}=-\frac{1}{2T}\mathbf{H},$$

(3.13)

$$\nabla\times\mathbf{E}=-\frac{1}{2T}\mathbf{E},\quad \nabla\times\mathbf{B}=-\frac{1}{2T}\mathbf{B},$$

(3.14)

$$\Rightarrow\{\mathbf{E}\parallel\mathbf{B}\parallel\mathbf{H}\parallel\mathbf{D}\parallel\mathbf{F}\},\quad \nabla\times\mathbf{F}=-\frac{1}{2T}\mathbf{F},\qquad \nabla\times\mathbf{A}=-\frac{1}{2T}\mathbf{A}.$$

(3.15)

This system of equations (3,1-3.15) is useful to apply the electric field integral equation (EFIE) method for modern simulations of scattering, radiation, and circuit problems [20-24]; when the sources ρ and J are included also are invariants under gauge transformations.

In general, there is a peculiar kind of *quantum nonlocality* in nature. For the zero mass case, the four spinor ψ_μ satisfies the equation $\gamma^\nu\partial_\nu\psi_\mu=0$ which is the zero mass Dirac equation in the standard chiral representation. The 4-potential play the role of the 4-spinor, and subsequently we can derive the resulting electric and magnetic field components in the frequency domain $\gamma^\nu\partial_\nu F_\mu=0$ with $\partial_\mu F^\mu=0$ the velocity gauge condition (see equations 4.12) with $(\mathbf{J},\rho)=0$) [19]. The four-potential can be assumed as a spinor solution, provided that the latter satisfies the Lorenz gauge. A crucial choice is needed: the form of the electromagnetic potential is to be assumed as a combination of positive- and negative-energy (frequency) solutions of the spinor. The present work may help to clarify the controversial relation between Maxwell and Dirac equations, while presenting an original way to derive the electromagnetic fields, leading, perhaps, to novel concepts in EM simulations. The explicit derivation of the electromagnetic-fields solution of Maxwell equations starting from the Dirac equation can be obtained, that describes the so called spinor wave function of quantum particles. In particular we show that if the four-component vector (spinor) solution of the Dirac equation for zero mass is identified with the four-potential of the EM field, then, under the like Lorenz gauge, fields derived from that potential satisfy Maxwell equations [25-30]. On the other hand, the possibility of representing the observables of the Maxwell theory by different (gauge-equivalent) potentials means that within the Maxwell theory the electrodynamic potentials **A** and V possess *no physical reality*. The same holds for the vector potential **A** in context of the Aharonov-Bohm effect where only that part of information contained in **A** is *also contained in the observable* **B** = curl **F** so it *is of physical relevance*. However, in our case $\{\mathbf{E}, \mathbf{B}, \mathbf{H}, \mathbf{D}, \mathbf{F}\}$, **F** and V possess as **B** = curl **F** a *physical reality,* here the Aharonov-Bohm effect for dependent time **B(t)** is obtained if **B** = curl **F=0.**

4. Graphene like a chiral media

From the point of view of its electronic properties, graphene is a two-dimensional zero-gap semiconductor with the cone energy spectrum, and its low-energy quasi-particles are formally described by the Dirac-like Hamiltonian [28, 31, 32]:

$$H_0 = -i\hbar v_F \sigma \cdot \nabla \quad , \tag{4.1}$$

where $v_F \approx 10^6 ms^{-1}$ is the Fermi velocity and $\sigma = (\sigma_x, \sigma_y)$ σ are the Pauli matrices. The fact that charge carriers are described by the Dirac-like equation, rather than the usual Schrödinger equation, can be seen as a consequence of graphene's crystal structure, which consists of two equivalent carbon sub-lattices [28]. Quantum mechanical hopping between the sub-lattices leads to the formation of two cosine-like energy bands, and their intersection near the edges of the Brillouin zone yields the conical energy spectrum. As a result, quasi-particles in graphene exhibit the linear dispersion relation $E_G = E = \hbar k v_F$ as if they were massless relativistic particles with momentum k (for example, photons) but the role of the speed of light is played here by the Fermi velocity $v_F \approx c/300$. Owing to the linear spectrum, it is expected that graphene's quasi-particles will behave differently from those in conventional metals and semiconductors where the energy spectrum can be approximated by a parabolic (free-electron-like) dispersion relation. The low-energy excitations of this system are then described by the massless two-dimensional Weyl-Dirac equation and its energy dispersion relation $\omega = v_F k$, k is that of relativistic massless fermions with particle-hole symmetry.

In graphene these massless fermions propagate with a velocity v_F . The maths is simple but the principles are deep. We will review the formulation of graphene's massless Dirac Hamiltonian, under the chiral electromagnetism approach, like a metamaterial media, hopefully demystifying the material's unusual chiral, relativistic, effective theory. The novel result here is that in our theory we do not make $c \to v_F$ [31, 32], but we obtain v_F as $v_F = v = c\sqrt{1-(k_0T)^2}$ This result is derived of equation (3.11') with T as the chiral parameter and $k_0 = \omega / c$. Making $v_F / c = v / c = \alpha = \sqrt{1-(k_0T)^2}$, where α is the fine-structure constant $\alpha = 1/137$, we find that $k_0T = 0.9973 \approx 1$, here the electric wave *E* is quasi-parallel to the magnetic wave *H.*

5. Conclusions

In this paper we have proposed a non-local constitutive relation for $\mathbf{D}$, $\mathbf{B}$ in terms of $\mathbf{E}$, $\mathbf{H}$ having as a chiral hidden parameter T for the vector potential $\mathbf{A}$ which satisfies a velocity gauge. This condition is not satisfied by the other chiral formalisms so the proposed approach is useful to numerical calculations and applications of scalar and vector potentials in chiral devices, and metamaterials. Here, we have non local constitutive relations that we need to study the non local AB effect. On the other hand the Born Fedorov formulation for chiral media is the only one that can be work at zero frequency.
When we compare with other formulations like the Uckun'work [15], where he mix a local constitutive relation of Engetha formulation [13] with a nonlocal one of Born [10], we can see that this approach is not auto consistent, so it is not possible to study the nonlocal AB effect. For this reason Uckun paper must be corrected using the complete formalism of Fedorov Born [17] by considering a factor $1-(k_0T)^2$ where T is the nonlocal chiral parameter.

Also with this approach it is possible reconsider the relation between Maxwell and Dirac equations because the connection between four-spinor and four-potential, instead of the electromagnetic field leading, perhaps, to novel concepts in EM simulations, i.e. modeling the combined electromagnetic (EM)/quantum transport problems in graphene circuits that requires the development of novel concepts and novel unified tools. For graphene systems we find analytically the Fermi velocity in terms of the velocity gauge and the fine-structure constant $\alpha = 1/137$.

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