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Literature Review

Corrective Cotangent

Introduction - Sines in the Secondary

Sine, cosine, and tangent exhibit their transcendental nature in several ways, particularly within the debate as to when to introduce them in the secondary math curriculum. Trigonometry, where these functions include components of visual and nonvisual application, can be interwoven as soon as Algebra 1 and certainly by the end of Geometry before Algebra 2. Adding to the difficulty of introduction, some students may learn more trigonometry than necessary if they were not trending towards science and engineering routes. Many secondary Content Standards only list standards concerning similar triangles and the Pythagorean Theorem that would conclude by Algebra 2 or Integrated Math 3. While certainly not a case for every high school student to augment their trigonometry with the McLauren series, or even just one additional trig proportion (not even all three - csc, sec, and cot), those that journey on into Precalculus face a particular challenge.

With the extensive demand and application of trigonometry more prevalent in emerging technologies (e.g. AI vector-valued database), many secondary teachers fall short of serving trigonometry beyond the antiquated insensitive mnemonics. Secondary Math Textbooks often come with subtexts such as “late transcendentals," alluding to the amount of trigonometric example exercise problems that are woven into the lessons, or lack thereof. Students are left to then discover foundational concepts of higher math on their own far earlier in their journeys than should be expected. Motivated by the ramifications of students’ lack of trigonometric understanding, an international collection of researchers explore the ramifications of these gaps in their own individual contexts, with the hopes of finding a remedy for their respective student populations.

Trigonometry onto the Tertiary

Hamzić et al. (2025) composed a test of trigonometry concepts with faculty at the University of Sarajevo to analyze the performance of 58 incoming college freshmen from Bosnia and Herzegovina. These students took both a typical cumulative precalculus final and the faculty test which focused on specific trigonometric applications. Hoping the faculty members designed a test of concepts relevant to their level of math, the authors were able to observe a Pearson coefficient of $r = .589$ with $p < 0.01$, suggesting a statistically significant and strong correlation between the performance in the two tests. Namely, with these results, the faculty were successfully able to synchronize with a previous class' final using both the secondary standards and requirements from the University. From the introduction, "the goal of this paper is to develop a test for checking student achievements in triangle trigonometry and the basics of circle trigonometry-those parts of trigonometry that all students should know at the end of their secondary education. (Hamzić, 2025, p. 4)"

Commentary is made on the construction of the test by faculty with discussion of the Ministry of Education of Sarajevo Canton 2022 learning outcomes for Trigonometry. While the comparison test is a final in precalculus, the authors discuss the student's thought process learning trigonometry, and how the Bosnian standards explicitly state difficulty in understanding and usually what subject (Algebra or Geometry) that the trigonometric standard is first encountered. The alignment of standard expectations between secondary and tertiary mathematics is challenging for all countries, even in the case as Walsh (2017) in an Irish University program especially for future secondary math teachers.

Walsh et al. (2017) studied fifty preservice teachers from an Irish University as they progressed through a standard content knowledge breadth for math education in Ireland, which included linear algebra, and real number analysis. Their motivations came from the country's Mathematical Sciences Education Board conclusions that "Irish students are leaving second-level education with significant deficiencies in their understanding of trigonometric concepts as well as an inability to visually represent mathematical content. (Walsh, 2017, p. 4)"

At the end of each year in the three-year undergraduate program, the cohort took a test on concepts of Trigonometry to gauge their understanding of their foundational concepts and how it changed as their knowledge of other mathematical extensions increased. The experiment sought to examine how core trigonometry concepts were grasped by the teachers as they went through tertiary or third-level math content. The results were surprising, as the two final problems were always guaranteed to be abstract trigonometry problems without necessarily a geometric context. The authors comment on how mathematical knowledge changes as an undergraduate math knowledge transitions into presservice teaching, and that advanced math may not fill in gaps in Trigonometric understanding unless there is a targeted approach that anticipates the shortcoming. "This study has shown that pre-service teachers in their final years of study also have difficulties in these areas…therefore, their misunderstandings, procedural approaches, and underdeveloped visualization skills may get passed [onto] their students. (Walsh, 2017, p.38)"

The Ghost in the Error

Despite the diverse entry points into trigonometry through which students first approach these concepts, the same general collection of errors and student rationale for associated errors remains very similar across observed populations, both in secondary and beginning tertiary. Research teams led by Obeng (2024), Fahrudin (2019), and Nanmumpuni (2024) respectively, describe different classroom scenarios where the error analysis undertaken in each suggest a unified need - increased pedagogical content knowledge in trigonometry that leans on both effective student collaboration and the willingness for teachers to see the impetus for new representations of trigonometry as not merely reinventing the wheel.

Obeng et. al (2024) conducted a descriptive qualitative study composed of ninety high school seniors and three math teachers who taught these students. Using the Trigonometry Achievement Test (TAT) and interviews where the students had an opportunity to discuss errors they made on the test, observed student errors were classified into Newman's (1977) Error Hierarchical Model. The referenced Model defines the progression of the errors as Reading, Comprehension, Transformation, Processing, and Encoding. In the six sample problems from the TAT which were stratified, 7.4%, 16.2%, 36.9%, 35.9%, and 3.5% of the errors fell into the respective categories. From the interviews conducted of the students after taking the test "the causes of the students’ errors were a result of the negative perception they have about trigonometry as being difficult, also the strategies and methods employed by teachers while teaching the concepts of trigonometry. (Obeng, 2024, p. 9)” Participant 3’s comment draws particular concern, as they say “most students do not take learning trigonometry seriously as they think it’s irrelevant and also difficult. (Obeng, 2024, p. 9)”

Reviewing the types of errors observed in the TAT results, these authors recommend “using manipulatives and also starting with specific examples or problems during the teaching and learning process…(Obeng, 2024, p. 14)” Independently, Fahrudin et al. classified their own five types of trigonometric errors - interpretive, conceptual, strategic, evaluative, and conclusionary. As classified and derived through both the qualitative data of 203 ninth grade students’ interviews and classroom test results, a discussion of the errors in thought process showed distinct conceptual shortcomings. Depending on the way the problem was phrased, oftentimes it was the interpretive phase where the errors started forming if the student did not cease efforts on the problem entirely. Interestingly, more than 40% of the students for both the second and third problem made a conclusionary error, meaning the students did all of the required calculations in order to solve the problem but may not have realized it.

Nanmumpuni et al. (2021) followed five sophomores through their experiences learning trigonometry, and how their difficulties manifested over time in the classroom. A series of essays in interviews were conducted immediately after the students took a small trigonometry test consisting of five questions. As all five students ended up not scoring well on the test, implications on how the teacher taught were also made. Reviewing the results, the authors comment “In the trigonometry textbook, there are already problem solving, but the [teacher] does not discuss specifically and asked the students to study on their own, discuss it with classmates, and ask the teacher for any queries. (Nanmumpuni, 5)"

Baidoo et al. (2023) documented 381 high school seniors across five different math classes. The secondary math teachers taught three dimensional trigonometry during the units, which usually appears around the precalculus level. Baidoo (2019) had previously designed the MPCA error protocol to categorize errors in mathematical problems. Under this four category model, the errors students make are sorted as one of mathematical language, procedural, conceptual, and application. From the introduction "the fundamentally research gap in this area is the lack of a comprehensive understanding of the errors made by learners when performing 3D trigonometry problems.(Baidoo, 2023, p. 7)" Commenting on the skillsets of the successful teachers "learners who use concrete materials develop more comprehensive mental representations…they often show more motivation and on-tack behavior, understand mathematical ideas, and better apply these ideas to life situations.(Baidoo, 2023, p. 12)"

The results demonstrate a uniform distribution of error within the four categories, with 66% of all total question answered correctly without any error. The authors further report that they use the approach of building on prior knowledge. The teachers were also comfortable in scenarios when students needed extra time to remediate introductory concepts, including the two dimensional applications. The authors also comment on "flipped" classrooms, where students are given more agency through groupwork and multimodal design. The authors do acknowledge a limitation in the experiment by only conducting one emotional questionnaire at the beginning and then pairing with with one cumulative final grade, rather than more periodic and concurrent assessment of both emotional well-being as well as assessed performance.

Babaali et al. (2015) analyzed the final exam grades from 203 students enrolled in the Spring 2010 precalculus sections of York College, within the City University of New York system of public colleges. Each student was enrolled and graded using an online content suite - the Paul Sisson's Pre-calculus module of the Hawkes Learning Systems. Commenting on the Adoption of the software package, the authors mention "the consistent high rate of failure in the course, the attrition rate that such a pass rate adds to with respect to majors that require the course and the importance placed upon it as a gatekeeper to further coursework in many majors (Babaali, 2015, p.2)" as motivation for faculty to adopt the online system. In contrast to other online platforms such as MyMathlab and WebAssign, Hawkes uses a mastery-oriented approach with certificates for each precalculus section.

Comparing results from the previous year that did not utilize the online platform, it was revealed that average test scores and standard deviation remained the same, even though the online sections had more documented homework that the undergraduate students would submit. The Article is relevant, as it shows the persistent fate of attrition that appears when trigonometry gaps persist into remedial college courses. With online grading systems easily adjustable, college faculty are more precisely able to request their own contextual mindsets of their students' trigonometric background. In this case, technology can further facilitate the recording of error and offer suggestions, with one particular strategy in agreement with pioneering studies in the past.

The Law of Cosines and Community

Fullilove et al. (1990) studied the performance of incoming UC Berkeley African American Students enrolling in the introductory Math 1A Calculus class during the mid 1970s. Each of the students were invited to build study groups amongst their peers, which were then supported with activities and ideas from East Asian groups that were more comfortable working together in the University setting. Comparing student performance with control data from the early 70s without the support groups (approximately 500 students from 1973-78 and 400 from ‘78-’80), African American students through the next three years were more successful in their math and engineering classes with increased graduation rates and less changing into other majors.

In these Math Workshop Groups, the African American students were encouraged to spend time with each other discussing things not related to math, and to build a routine around learning a concept with as little concealed anxiety as possible and remaining mindful of the community of their Workshop Group. Fulilove’s implementations were groundbreaking, as they offered additional insights into the benefits of social emotional learning by creating a community beyond the specific classroom.

Villavicencio et al. (2016) examined entry interviews of college precalculus students from a Filipino University and cross referenced them with their final grades at the end of the course. Discussing the results, “the experience of positive emotions fosters cognitive-behavioral processes (self-regulation) and psychological resources (self-efficacy) that are important components of optimal performance.” One thousand three hundred forty-five Filipino university students were randomly selected out of thirty-one different class sections across the country. Class finals were correlated with the Academic Emotions Questionnaire-Math (AEQ-M, Frenzel et al.).

Reviewing the results, additional discussions were made on further affecting positive outlook through pride in the general class performance and fellow classmates. Making note of cultural aspects that exist independent of class content, the Authors encourage activities that further engage students to also reduce their general anxiety. From the concluding discussions, “even after accounting for the influence of anxiety, the results show the student’s experiences of enjoyment and pride explain an additional amount of variance. (Villavicencio, 2016, p.3)”

Together, the researchers make a case that multimodal approaches can serve as potential remedies for the disconnect in trigonometric application. “If trigonometric material is only taught using the lecture method, students will feel that trigonometric material is very abstract…If trigonometry is taught correctly, the concept of trigonometric periods or angles in various quadrants will be easier for students to understand. (Fahrudin, 2019, p. 6)” Fahrudin hints on what higher math considers useful that a math student should know about trigonometry. It is also one case of technology that does not need to be cutting-edge (Geogebra), and highlights technologies that exist that can serve as evergreen manipulatives independent of AI. Lest the literal wooden building block would also see its last day. The latter portion of Nanmumpuni’s remarks offer other solutions that do not even require a computer. Namely, student conversations on the nuances of trigonometry can be modeled and encouraged, such that the analysis of their own wrongs become their own enrichment and rite - a process which can be further facilitated and recordkept by technology.

Conclusion - Algebra 2 Wrongs Make a Rite

Cavey and Berenson (2005) examined an educator's experience developing a Trigonometry lesson through a Japanese Development Model of "Lesson Plan Study", where the case study works with other educators through praxis and repetition to also instill confidence that the teacher has sufficient content knowledge to present the lesson. From the abstract, "Japanese Lesson Study method of professional development, is a sequence of activities designed to engage prospective teachers in broadening and deepening their understanding of school mathematics and teaching strategies. LPS occurs over 5 weeks on the same lesson topic and includes four opportunities to revisit one's own ideas and the ideas of others." The entire experience had both qualitative and survey style metrics, as part of the evaluation also involved dry trigonometry battery tests to gauge the educator's baseline, as well as more subjective reflection activities. The end results of the study and Molly's experience were positive, with her more comfortable with the gaps in her trigonometry background, and knowing methods the mentors in the program gave her to ramp up knowledge quickly.

With teachers as models of constant praxis, this final example from research of Molly's experience encompasses the experience that every explorer through math should have the pleasure of being a part of, including those in secondary who initially are not interested in the sciences. Even if some students see the triangle as their terminal experience with trigonometry, much effort is still necessary to ensure that a geometric context is extended beyond the paper. To gather around the sand like Euclid and Pythagoras and form a circle is the essential enriching spirit of mathematics.The word itself comes from the Greek *mathema* meaning “to teach”. Hence, there are more misconceptions in modern math than what is a suitable trigonometric understanding. What was the line in the sand is now a gauntlet, as all teachers from primary and onward must assure that we never simultaneously forget the trigonometry the built our AI and become hollowly artificial ourselves.

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