1.

One root of the equation $x^2 + px + q = 0$ is $2 - 3\mathrm{i}$. Find the values of *p* and *q*.

$$\begin{aligned} 2 - 3\mathrm{i} + 2 + 3\mathrm{i} &= -p \\ p &= -4 \\ (2 - 3\mathrm{i})(2 + 3\mathrm{i}) &= q \\ q &= 13 \end{aligned}$$

(3)

2.

Given $z^2 = 15 + 8\mathrm{i}$, $z = a + b\mathrm{i}$. Find the possible combinations of *a* and *b*, where *a* and *b* are integers.

$$\begin{aligned} (a + b\mathrm{i})^2 &= 15 + 8\mathrm{i} \\ a^2 + 2ab\mathrm{i} - b^2 &= 15 + 8\mathrm{i} \\ 2ab &= 8 \\ a &= \frac{4}{b} \\ \left(\frac{4}{b}\right)^2 - b^2 &= 15 \\ \frac{16}{b^2} - b^2 &= 15 \\ &= b^4 + 15b^2 - 16 \\ &= \left(b^2 + 16\right)\left(b^2 - 1\right) \\ b &= \pm 1 \\ a &= \pm 4 \\ z_1 &= 4 + \mathrm{i} \\ z_2 &= -4 - \mathrm{i} \end{aligned}$$

(6)

3.

a. Find all possible values of the real numbers *a* and *b* which

$$2 + a\mathrm{i} = \frac{6 - 2\mathrm{i}}{b + \mathrm{i}}$$

$$\begin{aligned} 2 + a\mathrm{i} &= \frac{6 - 2\mathrm{i}}{b + \mathrm{i}} \times \frac{b - \mathrm{i}}{b - \mathrm{i}} \\ 2 + a\mathrm{i} &= \frac{6b - 6\mathrm{i} - 2b\mathrm{i} - 2}{b^2 + 1} \\ 2 &= \frac{6b - 2}{b^2 + 1} \\ 2b^2 - 6b + 4 &= 0 \\ b^2 - 3b + 2 &= 0 \\ (b - 1)(b - 2) &= 0 \\ b &= 1, 2 \\ a &= \frac{-6 - 2b}{b^2 + 1} \\ &= \frac{-8}{2} \text{ or } \frac{-10}{5} \\ a &= -4, \ -2 \end{aligned}$$

(4)

b. Given that $w = -\frac{1}{2} + \frac{1}{2}\mathrm{i}$, find the modulus and the argument of $\frac{1}{1+w}$, giving the argument in radians between $-\pi$ and π

$$\begin{aligned}\frac{1}{1+w} &= \frac{1}{1-\frac{1}{2}+\frac{1}{2}\mathrm{i}}\\ &= \frac{1}{\frac{1}{2}+\frac{1}{2}\mathrm{i}} \times \frac{\frac{1}{2}-\frac{1}{2}\mathrm{i}}{\frac{1}{2}-\frac{1}{2}\mathrm{i}}\\ &= \frac{\frac{1}{2}-\frac{1}{2}\mathrm{i}}{\frac{1}{4}+\frac{1}{4}}\\ &= 1-\mathrm{i}\\ \left|\frac{1}{1+w}\right| &= \sqrt{2}\\ \arg\left(\frac{1}{1+w}\right) &= -\frac{\pi}{4}\end{aligned}$$

(4)

4.

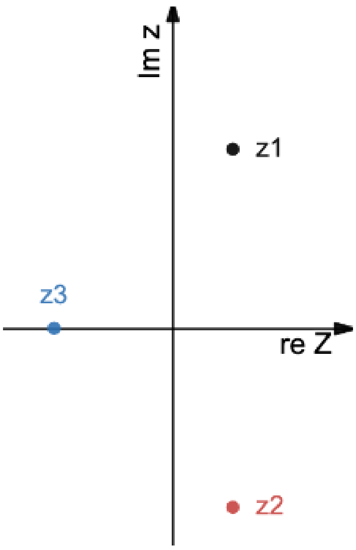
Given that $1+3\mathrm{i}$ is a root of the equation $z^3 + 6z + 20 = 0$

a. Find the other two roots of the equation,

$$\begin{aligned}z_1 &= 1+3\mathrm{i}\\ z_2 &= 1-3\mathrm{i}\\ z_1+z_2 &= 2\\ z_1z_2 &= 10\\ z^3+6z+30 &= \left(z^2-2z+10\right)(z+2)\\ z_3 &= -2\end{aligned}$$

(4)

b. Show, on a single Argand diagram, the three points representing the roots of the equation,



(1)

c. Prove that these three points are the vertices of a right-angled triangle.

$$\begin{aligned} \text{gradient of } z_1z_3 &= \frac{3-0}{1+2} = 1 \\ \text{gradient of } z_2z_3 &= \frac{-3-0}{1+2} = -1 \\ (\text{gradient of } z_1z_3)(\text{gradient of } z_2z_3) &= -1 \\ \therefore \angle z_1z_3z_2 &= 90^\circ \end{aligned}$$

(2)

5.

Show that

$$\sum_{r=1}^{n+2}(2r+3) = (n+2)(n+6)$$

$$\begin{aligned} \sum_{r=1}^{n}(2r+3) &= 2\sum_{r=1}^{n} r + \sum_{r=1}^{n} 3 \\ &= 2\left(\frac{1}{2}n(n+1)\right) + 3n \\ &= n(n+4) \\ \sum_{r=1}^{n+2}(2r+3) &= (n+2)(n+6) \end{aligned}$$

(3)

6.

Show that

$$\sum_{p=3}^{n}(4p+5) = (2n+11)(n-2)$$

$$\begin{aligned} \sum_{p=3}^{n}(4p+5) &= \sum_{p=1}^{n}(4p+5) - \sum_{p=1}^{2}(4p+5) \\ &= 4\left(\frac{1}{2}n(n+1)\right) + 5n - 9 - 13 \\ &= 2n^2 + 7n - 22 \\ &= (2n+11)(n-2) \end{aligned}$$

(3)

7.

a. Show that $\sum_{r=1}^{k}(4r-5) = 2k^2 - 3k$

$$\begin{aligned}\sum_{r=1}^{k}(4r-5) &= 4\sum_{r=1}^{k} r - \sum_{r=1}^{k} 5 \\ &= 4\left(\frac{1}{2}k(k+1)\right) - 5k \\ &= 2k^2 + 2k - 5k \\ &= 2k^2 - 3k\end{aligned}$$

(2)

b. Find the smallest value of k for which $\sum_{r=1}^{k}(4r-5) > 4850$

$$\begin{aligned}2k^2 - 3k &> 4850 \\ 2k^2 - 3k - 4850 &> 0 \\ (k-50)(2k+97) &> 0 \\ k &> 50\end{aligned}$$

(2)

8.

a. Show that $\sum_{r=1}^{n} r(r+2) = \frac{n}{6}(n+1)(2n+7)$

$$\begin{aligned}\sum_{r=1}^{n} r(r+2) &= \sum_{r=1}^{n} r^2 + 2\sum_{r=1}^{n} r \\ &= \frac{1}{6}n(n+1)(2n+1) + \frac{2}{2}n(n+1) \\ &= \frac{n}{6}(n+1)[2n+1+6] \\ &= \frac{n}{6}(n+1)(2n+7)\end{aligned}$$

(3)

b. Use the result, or otherwise, find in terms of n, the sum of

$3\log 2 + 4\log 2^2 + 5\log 2^3 + \ldots + (n+2)\log 2^n$

$$\begin{aligned}3\log 2 + 4\log 2^2 + 5\log 2^3 + \ldots + (n+2)\log 2^n &= \log 2[3\cdot 1 + 4\cdot 2 + 5\cdot 3 + \ldots + (n+2)n] \\ &= \log 2\sum_{1}^{n} r(r+2) \\ &= \frac{n\log 2}{6}(n+1)(2n+7)\end{aligned}$$

(2)

9.

If the roots of the equation $4x^3+7x^2-5x-1=0$ are $\alpha,\ \beta$ and γ, find the equation whose roots are $2\alpha+1,\ 2\beta+1$ and $2\gamma+1$

$$w=2x+1$$
$$x=\frac{w-1}{2}$$
$$4\left(\frac{w-1}{2}\right)^3+7\left(\frac{w-1}{2}\right)^2-5\left(\frac{w-1}{2}\right)-1=0$$
$$\frac{4}{8}\left(w^3-3w^2+3w-1\right)+\frac{7}{4}\left(w^2-2w+1\right)-\frac{5}{2}(w-1)-1=0$$
$$2\left(w^3-3w^2+3w-1\right)+7\left(w^2-2w+1\right)-10(w-1)-4=0$$
$$2w^3+w-18w+11=0$$

(5)

10.

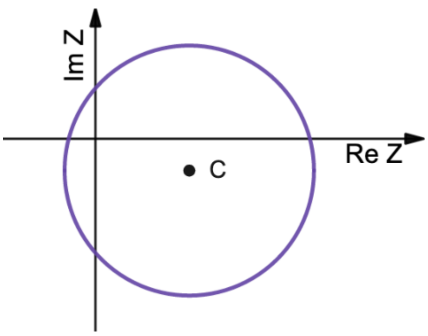
The root of the equation $x^3+px^2+qx+30=0$ are in the ratios 2:3:5. Find the values of p and q.

$$\text{roots are } 2\alpha,\ 3\alpha \text{ and } 5\alpha$$
$$2\alpha+3\alpha+5\alpha=-p$$
$$p=-10\alpha$$
$$(2\alpha)(3\alpha)+(2\alpha)(5\alpha)+(3\alpha)(5\alpha)=q$$
$$q=31\alpha^2$$
$$\mathrm{f}(2\alpha)=0$$
$$(2\alpha)^3-10\alpha(2\alpha)^2+31\alpha^2(2\alpha)+30=0$$
$$30\alpha^3+30=0$$
$$\alpha=-1$$
$$p=10$$
$$q=31$$

(5)

11.

a. Sketch the locus $|z-3+\mathrm{i}|=4$



$$\text{centre}=(3,-1)$$
$$\text{radius}=4$$

(2)

b. Find the maximum value of $|z|$ correct to 3 decimal places

$$y = \frac{-1-0}{3-0}x$$
$$y = -\frac{1}{3}x$$
$$(x-3)^2 + (y+1)^2 = 16$$
$$(x-3)^2 + \left(-\frac{1}{3}x + 1\right)^2 = 16$$
$$x^2 - 6x + 9 + \frac{x^2}{9} - \frac{2}{3}x + 1 = 16$$
$$\frac{10}{9}x^2 - \frac{20}{3}x - 6 = 0$$
$$x = \frac{15 - 6\sqrt{10}}{5}$$
$$y = \frac{-5 + 2\sqrt{10}}{5}$$
$$\text{minimum } |z| = \sqrt{\left(\frac{15-6\sqrt{10}}{5}\right)^2 + \left(\frac{-5+2\sqrt{10}}{5}\right)^2} = 0.838\,(3\text{dp})$$

(5)

12.

Find the cartesian equation for

a. $|z-4| < |z-2\mathrm{i}|$

$$|z-4| < |z-2\mathrm{i}|$$
$$\sqrt{(x-4)^2 + y^2} < \sqrt{x^2 + (y-2)^2}$$
$$x^2 - 8x + 16 + y^2 < x^2 + y^2 - 4y + 4$$
$$4y < 8x - 12$$
$$y < 2x - 3$$

(2)

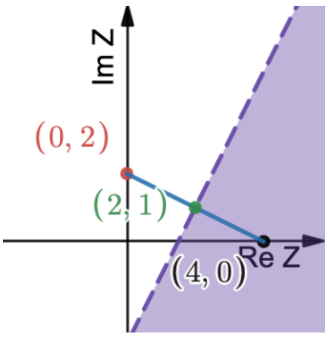
b. $\arg(z - 2 + \mathrm{i}) = \frac{\pi}{3}$

$$\tan\frac{\pi}{3} = \sqrt{3}$$
$$y + 1 = \sqrt{3}(x-2)$$
$$y = \sqrt{3}x - 2\sqrt{3} - 1,\ x \geqslant 2$$

(3)

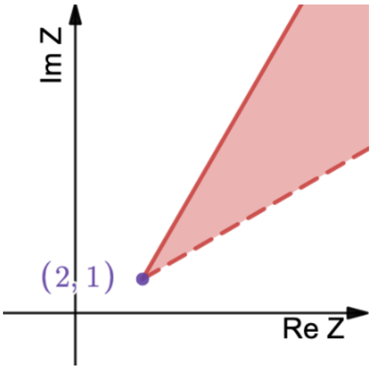
Shaded the region on separated Argand diagrams

a. $|z-4| < |z-2\mathrm{i}|$



(4)

b. $\frac{\pi}{6} < \arg(z - 2 + \mathrm{i}) \leqslant \frac{\pi}{3}$



(4)