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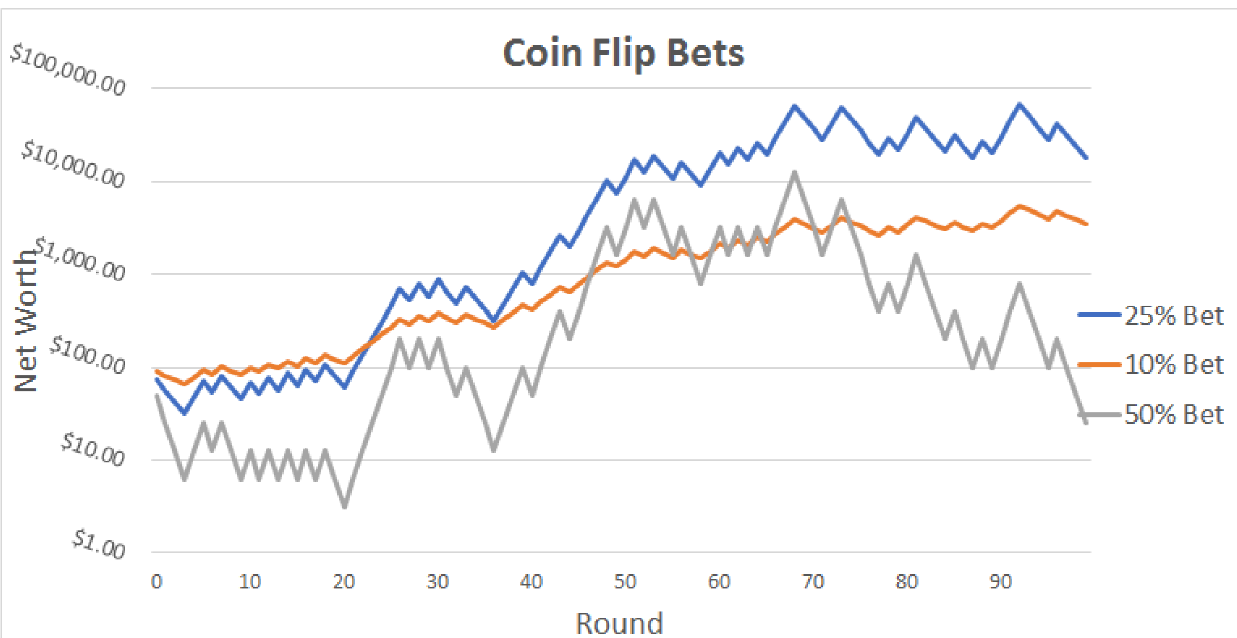
Investigation of the Kelly Criterion

Since the dawn of gambling, people have attempted to devise strategies in order to game the system. The most famous method is the Martingale method, in which a player constantly doubles their bets on each loss until they win. In the event of an odds where people believe that they have an overall chance of winning it is recommended to use the Kelly Criterion. The biggest issue with the Martingale is that it doesn't take into account the finite capital that one has. It reduces the chance of losses, while also maximizing the ability of profits. We will be investigating the derivation of the Kelly Criterion and compare it to different gambling techniques.

Derivation of the Kelly

The idea of the Kelly Criterion is to devise a method in order to optimize your profits off of a bet in which the estimated value of the bet is positive. We will view the example of a simple coin flip. We will make the assumption of that this is a fair coin, however if you guess the landing face correctly you stand to win twice as much as you bet, in other words 2:1 odds. We can calculate the estimated value of the bet to be EV(X)=(0.5*2) - (0.5-1) = 0.5. After an infinite series of bets, we expect to win about 50% more than what we put in. But from this information we simply do not know how much money to bet each time. Below we have three situations, one in which a player bet half of his worth each time, another bets a quarter and the last bets one tenth.



As we see from the data above it appears that the player who bet a quarter of their net worth every time outperformed the other two players. The player that only put down only a tenth at a time was still able to have net gain in profit, however it pales in comparison to the quarter at a time player. It appears that putting down half at a time results in a net loss in wealth. It appears that the bold choice creates more volatility in the wealth, as we that although increases are magnified so are the drops.

This leads to the idea that there is an optimal amount to bet at each turn, in this example it appears to be greater than a tenth and lower than half. We can determine that the net amount of wealth can be characterized as wn=w0X1X2X3...Xn (cite) after n iterations. If we look at the math we arrive that the average return is going to be exp(E(log(X))-1), where E(log(X))=log(w0)+...+log(wn), where w is the chance winning in each situation.

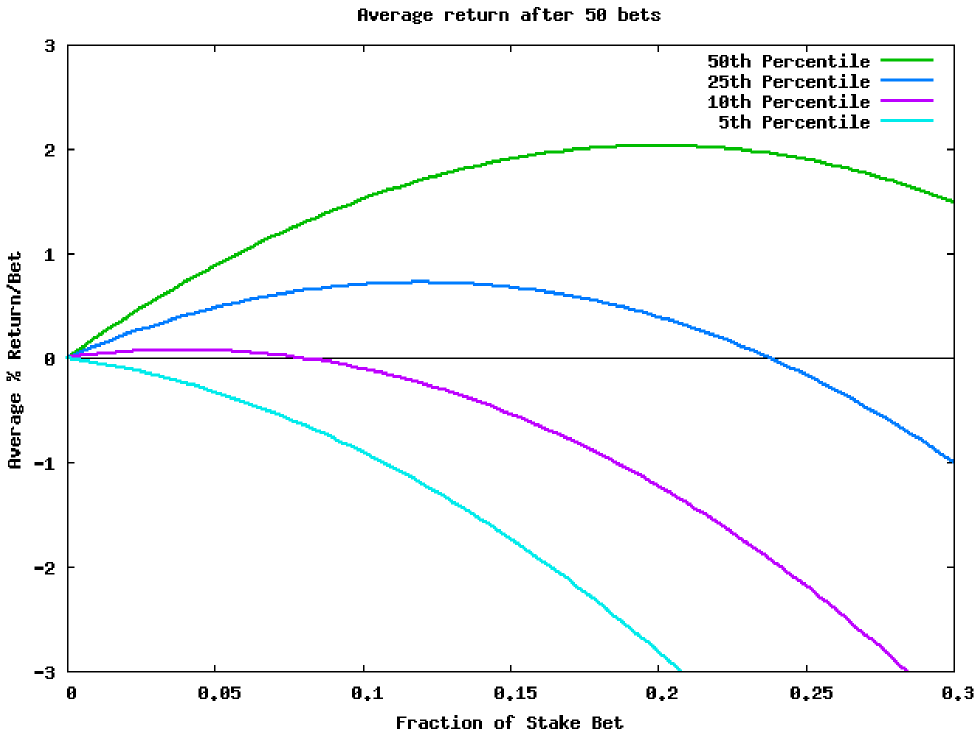
In order to arrive to the Kelly Criterion we will maximize the expected value of the bet. We will take *p* to be the probability of winning the bet, *b* to be the percentage that can be won in the bet, and *x* to be the fraction of our wealth that we wish to bet. Therefore E(log(X))= *p*log(1+*bx*) + (1-*p*)log(1-*x*). Using the derivative as shown below we can show that it yields the Kelly Criterion equation.

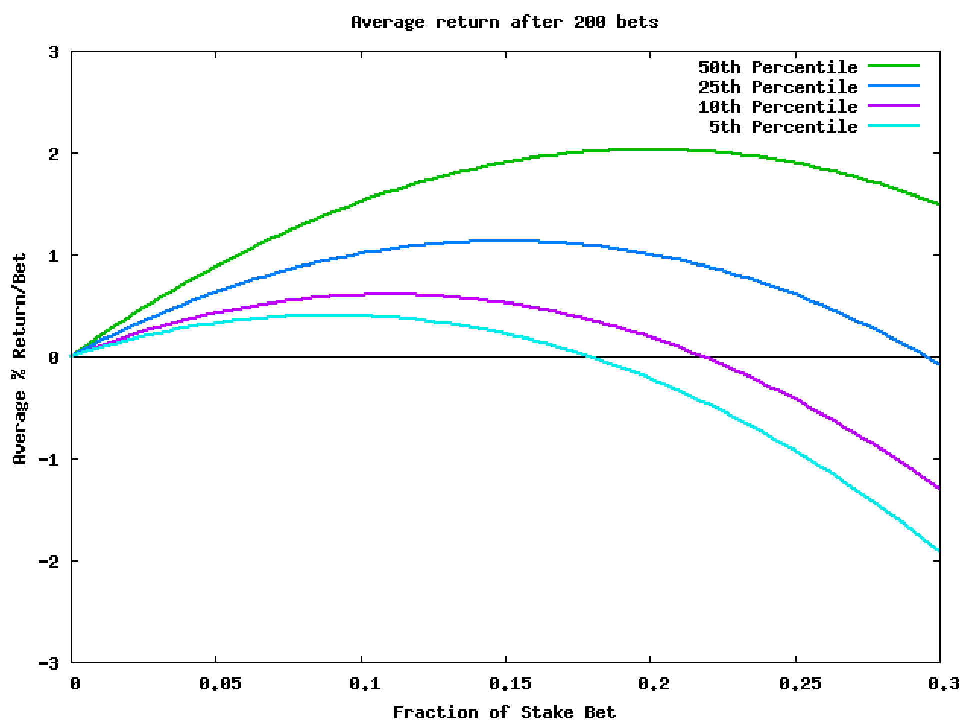
$$\begin{aligned}\frac{d}{dx}(E(\log(X))) &= \frac{d}{dx}(p\log(1+bx)+(1-p)\log(1-x))\\ &= pb\frac{1}{1+bx}+(1-p)(-1)\frac{1}{1-x}\\ &= \frac{pb}{1+bx}-\frac{1-p}{1-x}\end{aligned}$$

$$\begin{aligned}0 &= pb(1-x)-(1-p)(1+bx)\\ 0 &= pb-pbx-1-bx+p+pbx\\ bx &= pb+p-1\\ x &= \frac{pb-(1-p)}{b}\end{aligned}$$

Note that this applies to the simplest form of a bet, in which there are only two outcomes, winning and losing. In events in which there are multiple outcomes, for example card games or the stock market, this derivation needs to be redone. Often there will be multiple roots, which will lead to multiple solutions, most of them nonsensical. However in higher order problems there may be multiple real solutions in which it would need to be tested in order to arrive at the optimal solution.

Another way to view this is to plot the expected values versus the fraction betted. Below is a graph displaying the various results in changing proportion betted in a game in which there is a 60% chance of doubling odds. If we look at the averages the peak is located at near criterion. As we can see there is a large variation in the results for the average return when there is a limited amounts of bet. We see that if our predictions are slightly off with a smaller the set of bets the more likely we are to have a loss. Below we see that with 200 bets the likelihood for the loss drops significantly, as the spread becomes tighter.





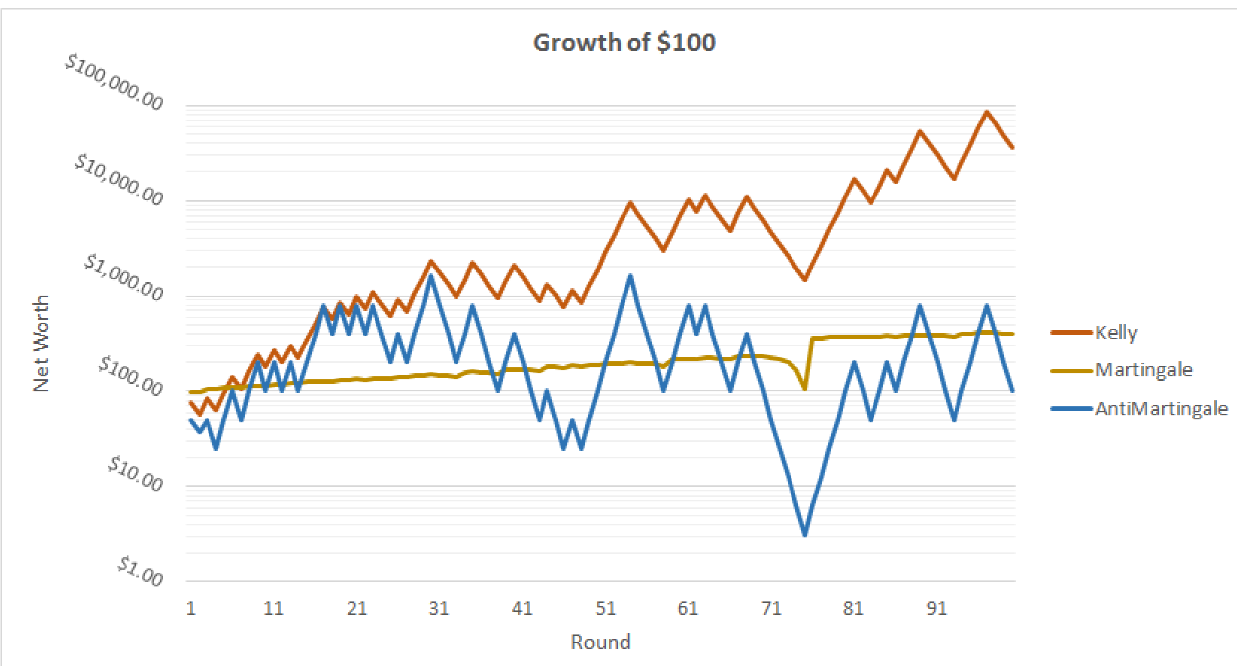
<u>Comparison to other Gambling Techniques</u>

The power of the Kelly Criterion becomes apparent when compared to other betting techniques. According to Alon Bochman, “It is the only formula I’ve seen that comes with a mathematical proof explaining why it can deliver higher long-term returns.” Let’s compare the Kelly to the most famous of gambling techniques, the Martingale and the Anti-Martingale. First a quick overview of both techniques, and the justifications on why they are still used and where they fall flat.

The Martingale technique is the idea that by doubling one's bets for each loss, eventually when one arrives to a win, they have won what they have lost in previous rounds, then returning to the initial small bet and continuing. The idea is that with technique a player cannot lose money. Again we'll use the example in which there is a coin flip and winning means doubling the bet. Taking a look in the case in which a player has lost 4 rounds from the start, and then finally wins. Starting with a dollar bet, the player has lost a total of $31 and now with there next bet of $32 stands to either win $64 while investing $63, making a grand total of a dollar profit. Assuming there is no bottom limit to one's wealth profit is guaranteed, however the gain is very minimal.

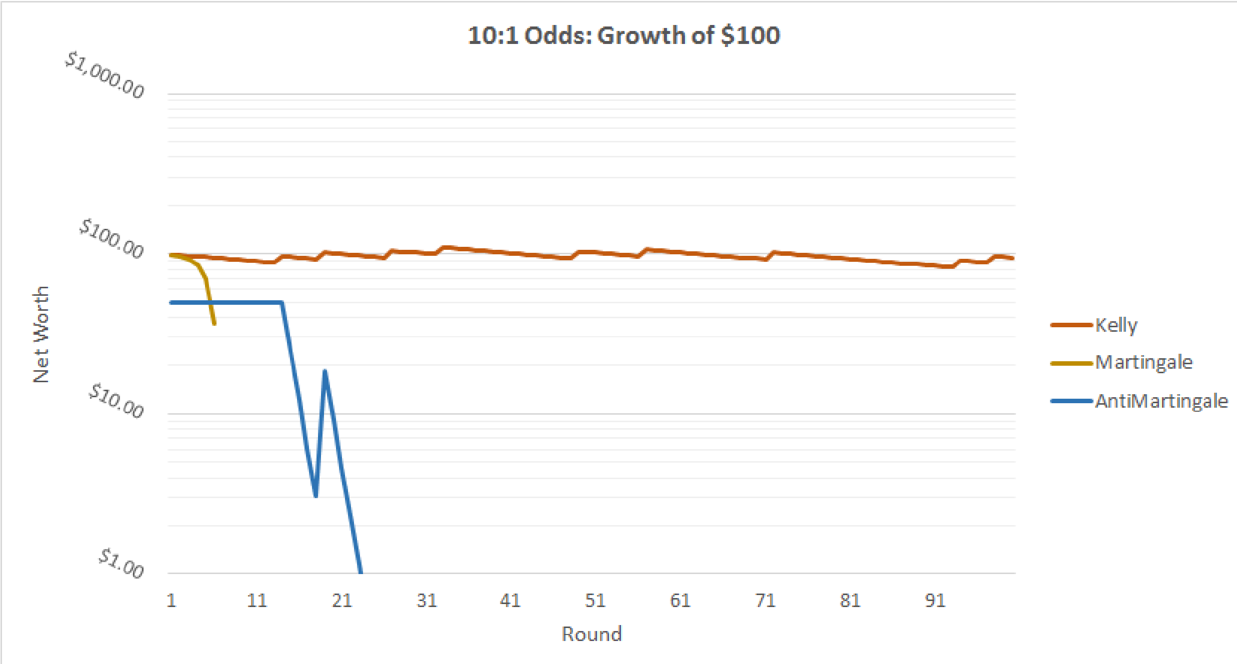
The Anti-Martingale is the idea in which one bets a large portion of their wealth, and for each losing round one bets half as much as before. This plays into the gambler's fallacy, the idea that after a large set of the same occurrence, it is expected that it'll become less likely in the next round. This fails to account that each event is independent. Looking at the scenario detailed in the Martingale explanation, starting with a $50 bet, after four rounds the player has lost $93.75, and with the next bet hopes to win a grand total of $6.25, after investing $96.88. It only prolongs play time, as it is impossible to place a bet that will lead to a player reaching a negative value.

An Excel simulation was ran trying to compare the various outcomes of each of the betting techniques. The first is the situation of the coin flip, where winning the flip earns you twice what was betted in that round. This simulation shows only a single round of betting, in which 100 bets were made. The other stipulation is that once a player has a negative net worth they are no longer allowed to play. This limits the strength of the players, as no bottom limits would just lead to the same situation as infinite wealth.



Due to the positive estimated value of the bet, the probability of loss becomes lower, as we see the Martingale leads to a slow but positive gain. Curious to note is the Anti-Martingale's property of exponentially decreasing one's wealth in the events of so called "losing streaks."

This narrative changes when the stakes become much higher. Let's suppose the situation of horse racing, in which a horse has a 10% chance of winning, with ten to one odds. In other words one stands to win ten times as much per bet. The estimated value of the bet is E.V.=(10*.1)-(1*.9)=0.1. This is still a positive gain bet, however the other two strategies start to fall apart.



The Martingale and the Anti Martingale lead to a quick drop in wealth, with the Martingale losing all money in less than ten rounds of betting. These odds are closer to what one would find in a casino, where both the Martingales are popular. These very minimal odds lead to very minimal growth for the Kelly, as it requires a very small risk, making it uneventful and useless, as the bets were around a dollar each round.

Caveats to the Kelly

A common warning for the Kelly is to bet less than it prescribes. The formula more often than not requires the player to estimate the odds of winning and the odds of losing. People tend to overestimate their chances of winning and underestimate their odds of losing, without this the entire gambling industry would not be able to exist. A common example is that people tend to believe that unlikely odds, like a plane crash, is less likely to happen in comparison to being involved in a car crash, even though you are more than 2000 times likely to die in the latter.(cite) Therefore investors often recommend to bet significantly lower than the value given by the Kelly Criterion. This leads one to be closer to the true Kelly Value, and it reduces one's chance to be in that negative gain area. Even in the case where one correctly estimated the values, being below the Kelly value still yields gain, although be it suboptimal growth.

In events where the estimated value of the bet are high, the Kelly will often prescribe a rather large portion of one's wealth to be put on the line. This should cause most investors to take a step back and reevaluate their odds in order to get a more reasonable and comfortable investment amount.

The False Kelly

One should note that there is an often misquoted version of the Kelly, in which it is written as the expected value over the ratio of win vs loss. The issue with this formula is that it will usually yield a much lower value than expected, as shown by Bochman below.

Bochman placed the scenario in which there is a 60% chance of yielding a 20% profit, and a 40% chance of a 20% loss. The true Kelly would yield that the optimal value to invest is 100%, and the false Kelly would claim to invest 20%. As we determined from the previous derivations, while a lower investment amount is the least likely to yield loss, it is by far not the optimal amount.

	Red: Invest 20%	Blue: Invest 100%	Green: Invest 150%
Minimum	0.59	0.01	0.00
First Quartile	1.55	1.85	0.52
Median	**1.98**	**6.24**	**3.36**
Mean	2.18	40.55	184.02
Third Quartile	2.72	31.60	39.99
Maximum	6.57	2733.14	36245.90