Superfluidity and Superconductivity - Exercise 1

1. **Coherent state path integrals**

   a. Prove the identity $\int d(\bar{\phi}, \phi) e^{-\bar{\phi} A \phi} = (det\, A)^{\zeta}$

   b. Consider two real bosonic fields $\phi_{1,2}$. Governed by the action

   $$S[\phi_1, \phi_2] = \int_0^{\beta} d\tau\, \frac{1}{2} (\Delta_1 \phi_1^{\ 2} + \Delta_2 \phi_2^{\ 2})$$

   Compute the thermal fluctuations of the fields $< \phi_j^{\ 2} >$ as a function of temperature. Why are these "fluctuations"? What happens at T = 0? Now add the term

   $$\delta S[\phi_1, \phi_2] = \int_0^{\beta} d\tau\, \phi_1 \partial_\tau \phi_2$$

   and repeat the calculation of the fluctuations. What happens at zero temperature? Identify the part of the fluctuations related to quantum "zero point motion" and relate them to the term $\omega_n = 0$ in the action in Matsubara frequency.

   c. Compute the Fermionic Matsubara sums

   $\Pi(i\Omega_m, q) = \frac{1}{\beta V} \sum_n \sum_k \frac{1}{-i\omega_n + \epsilon_k} \frac{1}{-i(\omega_n + \Omega_m) + \epsilon_{k+q}}$ and

   $P(i\Omega_m, q) = \frac{1}{\beta V} \sum_n \sum_k \frac{1}{-i\omega_n + \epsilon_k} \frac{1}{-i(\omega_n + \Omega_m) - \epsilon_{k+q}}$

   d. Consider the fermionic action

   $S[\bar{\psi}, \psi, \bar{J}, J] = \int_0^{\beta} d\tau \int d^d x \{ \bar{\psi} [\partial_\tau - \frac{\nabla^2}{2m} - \mu] \psi - \bar{\psi} J - \bar{J} \psi \}$. Use the identity

   $< \bar{\psi}(x, \tau) \psi(x, \tau) > = \frac{1}{Z[\bar{\psi}, \psi, 0, 0]} \frac{\delta^2 Z[\bar{\psi}, \psi, \bar{J}, J]}{\delta \bar{J}(x,\tau) \delta J(x,\tau)} |_{J, \bar{J} = 0}$ to compute the expectation value above at zero temperature.

2. **The weakly interacting Bose gas**

   a. Consider free bosons $S[\bar{\psi}, \psi] = \int_0^{\beta} d\tau \int d^d x\, \bar{\psi} (\partial_\tau - \frac{\nabla^2}{2m} - \mu) \psi$. Compute the Bose-Einstein condensation temperature T_{BEC} by computing $\mu(T)$ and find when it becomes zero. Assuming it remains zero (for stability), compute the number of Bosons in the ground state as a function of temperature.

   b. Now add the repulsive interaction: $S'[\bar{\psi}, \psi] = \int_0^{\beta} d\tau \int d^d x\, V_0 |\psi|^4$. Decouple the interaction term using the mean field $\psi_k = \sqrt{\rho_0} \delta_{k0} + \phi_k$ and expad to quadratic order in the small deviations ϕ_k. Diagonalize the action, find the spectrum and compute self-consistently the values of $\rho_0(T)$ & $\mu(T)$ by demanding number conservation and minimizing the action with respect to ρ_0. How does $\mu(T)$ behave close to the condensation temperature?

c. Consider the vortex solution in a **two-dimensional** superfluid: $\psi(r) = \sqrt{\rho_0}e^{i\theta(r)}$, where $\theta(r) = n\phi$, and where $\phi = arctan(y/x)$. Compute the energy of the vortex

$$E_1 = \rho_s \int d^d r(\nabla\theta)^2$$

Note that you may assume the vortex core has a finite size ξ and the sample is finite with typical size L. Now consider two vortices of equal winding of strength unity that are displaced by distance d, i.e. $\theta(r) = \phi(x, y) + \phi(x - d, y)$. Compute the energy of the this configuration

$$E_2(d) = \rho_s \int d^d r(\nabla\theta)^2 - 2E_1$$

where E_1 is computed for n = 1. Do this by introducing the appropriate branch cuts such that θ become a single valued function. Using integration by parts, map the area integral into a boundary integral. Excluding the cuts, estimate the energy using a contour integral over analytic functions.

3. Spontaneous symmetry breaking:

a. Using the Gaussian fluctuations of the XY model, compute the contribution of phase fluctuations to the correlation function

$$< \psi^+(r,\tau)\psi(r',\tau) > \approx \rho_0 < e^{i(\theta(r',\tau)-\theta(r,\tau))} >$$

in one, two and three dimensions. Explain why this implies that long-ranged order is not possible in one and two dimensions.
Note: you may neglect the compactness of the field θ.

b. Consider a cube of ^{4}He at T=0. Imagine the initial phase is set at $< \theta > = 0$ and the variance at $< \theta^2 > = (\pi/10)^2$. Estimate how long will it take the phase to become completely disordered, i.e. $< \theta^2 > = 16\pi^2$. Use the fact that the velocity of sound in a superfluid ^{4}He is 20 cm/sec, the density is 0.125 g/cm^3 and the mass is 4 amu.