Math PBAT

Game of X Tokens

Assessment: Good/Outstanding

Game of X tokens

This problem is going to be an extension of the Game of 27 that we explored in class. You have a cup with *x* gold tokens. There are two people drawing gold tokens out of this cup. You may take between 1 and *t* gold tokens out of the cup at any time. The person who takes the last gold token out of the cup wins and keeps all gold tokens.

Would you like to be the first player or second player? Why?

In this PBAT, you are going to need to prove your findings. In your write up you'll discuss your finding and use any proof methods discussed in class to prove your findings.

<u>Game of X tokens</u>

Problem statement

In this PBAT, the problem I will be solving is would you like to be the first player or second player when playing the Game of X Tokens? Why? The Game of X tokens is a two player game in which both players take turns removing tokens and the player to remove the last token wins. In playing the game there's one simple rule that both players need to follow and that is that they have a fixed amount of tokens to take.

Procedure

For this question, I assumed that being player 1 is the best option because you get to go first but I realized that going first isn't always the best. Then I thought that this problem was a trick question because deciding what player you want depends on how many tokens there are and how many tokens you can take. For example, if there was a maximum of 2 tokens and the most you can take is 2 tokens then it would be obvious that being player 1 is best for this scenario. However, this obvious scenario is not as obvious when the number of tokens are changed because there would be more mathematical thinking into how you can take the last token and win.

Solutions and Findings

The solution to this problem depends on how many tokens there are and how many tokens you can take. Strategies I considered using were conjectures like "the amount of tokens you can take +1 reveals the amount of tokens you are supposed to leave the other player at for a win." This is a good strategy to use because if t equals the amount of tokens you can take then you can set up an expression like *t*+1 which equals the amount of tokens you need to leave your opponent with to guarantee them to automatically lose. For example, if two players are playing the game and they start with 23 tokens and the maximum amount of tokens they can take is 4, then each player's goal is to leave the other player with 5. This strategy works because the most you can take is 4, so whoever needs to pick up from 5 tokens loses. This is a strategy worth using because it allows you to start thinking about the moves needed to be made in order to get your opponent to the numbers of tokens that will guarantee their loss. In your mind you have an objective which is to win by leaving the player at a specific number. The table below shows an example of when I played this game with another player and I left them with 5 tokens and was winner.

Player 1 Token Remove	Player 2 Token Remove	Amount of Tokens Left
		23
3		20
	4	16
1		15
	3	12
2		10
	1	9
4		5
	2	3
3		0

Also, in order to do this it can't come down to whoever is left with that number but I have to keep choosing multiples of a number as the player. I was able to further explore and create different conjectures in playing the game with 23 tokens and up to 4 tokens to take during a given turn. Since I knew that leaving your opponent with 5 tokens will guarantee you a win, I wonder what the next number of tokens will continue to guarantee you a win? I noticed that if it is your turn to play and there are 6, 7, 8, or 9 tokens left you can leave your opponent at 5 tokens and guarantee a win. However, if there are 10 tokens left you are not able to leave your opponent at 5. In fact, if there are 10 tokens left your opponent will be able to leave you at 5 tokens after you make a move. The reason this is true is because at most you can take 4 tokens leaving 6 tokens on the table and your opponent will be able to take 1 token. Leaving your opponent at 10 tokens now became the new goal of the game. In a similar process I discovered the same was true if you left your opponent to select from 15 or 20 tokens, which are multiples of 5. The table above shows this. As you can see in the highlighted numbers in red are the amount of token that I left on the table which are multiples of 5.

I already knew that you can guarantee a win if you leave your opponent at $t+1$ tokens, where t is maximum number of tokens allowed to take. What I now discovered is that leaving your opponent at multiples of $t+1$ will always guarantee you a win. The player that is always able to leave their opponent to choose from $t + 1$ is decided by the amount of tokens you start with. For example, for the first move, if there was 23 tokens and the maximum amount of tokens I can take is 4, then I would have to pick up 3 tokens and leave the opponent at 20 tokens because if I take my expression $t + 1$ and plug it in with 4 + 1 which equals 5, this tells me I have to leave my opponent at 20 since it is a multiple of 5. Another example is if the game starts with 25 tokens and each player can pick 1 to 6 tokens then player 1 would be guaranteed a win because if 4 tokens is taken from the coins it would leave the other player at 21 tokens which is a multiple of 7.

If there are 1-4 tokens and there are x tokens I noticed that player 2 can win under certain circumstances. Player 2 is a guaranteed win for example if x is 20, which is a multiple of 5, then player 1 would instantly lose if the right moves are made. The case for player 1 would be an instant loss because there are no remainders for player 1 to leave player 2 at a multiple of 5. Therefore, deciding which player you want to be depends on how many tokens there are and how many tokens you can take.

Connections

After having solved your problem, what concepts can you make between your specific problem and other mathematical concepts? The game of 27 is connected to modular arithmetic because modular arithmetic has to do with whole numbers, division, and remainders. Modular arithmetic specifically focuses on the remainder when two whole numbers are divided. For example, 7 mod 3 = 1 because given 7 objects 2 groups of three can be made and 1 object will remain without a group. In the game of 27 each player can pick up 1 to 4 tokens and player 1 is able to find out how many tokens to take first by calculating 27 mod 5 = 2. This makes sense because given 27 tokens you can make 5 groups of 5 tokens and 2 tokens will remain without a group. Again, we used mod 5 since in order to win the game you need to leave your player to select from 4+1 tokens. So, if there is a max of 27 tokens and I can pick up 1-4 tokens then on my first move I would take 2 tokens because 27 mod 5 = 2.

What relationships can you make between mathematical concepts, procedures, and strategies? Modular arithmetic in the game of 27 works the same as the game of x tokens because in order for the player to be guaranteed a win it all depends on whether the first move will have any remainders. For example, here are the cases for each player:

Case 1: x mod (*t*+1) = 0 player 2 wins

Case 2: x mod (*t*+1) ≠ 0 player 1 wins

For case 1, player 1 instantly loses because there are no remainders when the amount of tokens starts off as a multiple of *t*+1. For example, if there is a maximum of 30 tokens and 1 to 4 tokens can be picked up then player 1 instantly loses because 30 mod 5 = 0. For case 2, player 1 wins when there are remainders for him to leave player 2 at a multiple of *t*+1.

What would you want to investigate further? Now I want to investigate modular arithmetic when the game rules changes to the last one who takes the tokens loses. In playing variation with this new rule my conjecture is that if I leave my opponent at (t+1)+1 I will be guaranteed a win. For the player to be guaranteed a win it still depends on the amount of tokens you start with and the amount of tokens you can take. For example, here is a game in which I used this strategy. There were 32 tokens and I could pick up 1-5 tokens.

PLAYER 1: 1, 1, 5, 2, 3, 3, Winner

PLAYER 2: 5, 1, 4, 3, 3, 1 Loser

Other strategies I used while playing this game was putting the tokens into groups of 6 and leaving the opponent with one because (5+1)+1=7.

Metacognition

What did you learn about yourself as a student through the pbat process?

Throughout the pbat process I've learned that in order to master a problem you need to provide 100% proof and examples to every conjecture you make. Also, I've learned that in order to solve the problem you have to explain why your solution is always true which requires a lot of mathematical thinking.

What were you successful at?

I was most successful at answering the question “Would you like to be the first player or second player?” because in my solutions and findings I proved that the solution to this problem depends on how many tokens there are and how many tokens you can take. For example, one conjecture I provided that proves this answer is “leaving your opponent at multiples of t+1 will always guarantee you a win.” After stating the conjecture I explained how this conjecture is related to the question.

What was most challenging for you?

The most challenging part for me was the solutions and findings because I had to break down all the strategies for game and make new discorives. For example, I had to break down conjectures like “leaving your opponent at multiples of t+1 will always guarantee you a win” and explain how it is always true. In addition, when I thought I was done with the problem there was always another strategy I haven't discovered yet.

What would you do differently next time?

Next time I would talk about the different variations of the game. I would talk about how the game of x tokens would work when the game rules are changed. For example, I would solve different variations like “would t+1 change if the player can skip his/her turn?” In doing so this will help me see how the two different games are different and connected. Also, I wonder if the concept of modular arithmetic will continue to work if the rule changes.