Calculus 1 - Trig Identities Worksheet

Answer Key with Complete Solutions

Basic Angle Reference

Use $tan\theta = \frac{sin\theta}{cos\theta}$ and $cos\theta = sin\left(90^\circ - \theta\right)$.

Angle	0°	30°	45°	60°	90°
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
Sine	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
Tangent	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined

At 90°, tangent is undefined because $cos90^\circ = 0$, so $tan90^\circ = 1/0$.

Bell Work - Thursday Review

1. Write $sin(\alpha - \beta)$.

Answer:

$$sin(\alpha - \beta) = sin\alpha cos\beta - cos\alpha sin\beta$$

2. Write $cos(\alpha + \beta)$.

Answer:

$$cos(\alpha + \beta) = cos\alpha cos\beta - sin\alpha sin\beta$$

3. Find $cos75^\circ$ exactly.

Use $75^\circ = 45^\circ + 30^\circ$ and the cosine sum identity.

$$cos75^\circ = cos\left(45^\circ + 30^\circ\right)$$
$$= cos45^\circ cos30^\circ - sin45^\circ sin30^\circ$$
$$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$$
$$= \frac{\sqrt{6}-\sqrt{2}}{4}$$

4. Complete the cosine and tangent rows.

Answer: Cosine: $1, \frac{\sqrt{3}}{2}, \frac{\sqrt{2}}{2}, \frac{1}{2}, 0$*; Tangent:* $0, \frac{\sqrt{3}}{3}, 1, \sqrt{3}$*, undefined.*

Answer Summary: $sin\alpha cos\beta - cos\alpha sin\beta$; $cos\alpha cos\beta - sin\alpha sin\beta$; $\frac{\sqrt{6}-\sqrt{2}}{4}$; completed table above.

Part I - Sum & Difference Review

1. Find $sin105^{\circ}$ exactly.

Use $105^{\circ} = 60^{\circ} + 45^{\circ}$.

$$sin105^{\circ} = sin\left(60^{\circ} + 45^{\circ}\right)$$
$$= sin60^{\circ}cos45^{\circ} + cos60^{\circ}sin45^{\circ}$$
$$= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$$
$$= \frac{\sqrt{6}+\sqrt{2}}{4}$$

2. Find $cos15^{\circ}$ exactly.

Use $15^{\circ} = 45^{\circ} - 30^{\circ}$.

$$cos15^{\circ} = cos\left(45^{\circ} - 30^{\circ}\right)$$
$$= cos45^{\circ}cos30^{\circ} + sin45^{\circ}sin30^{\circ}$$
$$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$$
$$= \frac{\sqrt{6}+\sqrt{2}}{4}$$

3. Find $sin165^{\circ}$ exactly.

Use $165^{\circ} = 120^{\circ} + 45^{\circ}$.

$$sin165^{\circ} = sin\left(120^{\circ} + 45^{\circ}\right)$$
$$= sin120^{\circ}cos45^{\circ} + cos120^{\circ}sin45^{\circ}$$
$$= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(-\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$$
$$= \frac{\sqrt{6}-\sqrt{2}}{4}$$

4. Given $sinA = \frac{8}{17}$ and $cosB = \frac{12}{13}$, where A and B are acute, find $sin(A - B)$.

Because the angles are acute, the missing trig values are positive.

$$cosA = \frac{15}{17}, \qquad sinB = \frac{5}{13}$$
$$sin(A - B) = sinAcosB - cosAsinB$$
$$= \left(\frac{8}{17}\right)\left(\frac{12}{13}\right) - \left(\frac{15}{17}\right)\left(\frac{5}{13}\right)$$
$$= \frac{96-75}{221}$$
$$= \frac{21}{221}$$

Answer Summary: $\frac{\sqrt{6}+\sqrt{2}}{4}$; $\frac{\sqrt{6}+\sqrt{2}}{4}$; $\frac{\sqrt{6}-\sqrt{2}}{4}$; $\frac{21}{221}$.

Practice A - Independent Check

1. Find $cos105^{\circ}$ exactly.

$$cos105^{\circ} = cos\left(60^{\circ} + 45^{\circ}\right)$$
$$= cos60^{\circ}cos45^{\circ} - sin60^{\circ}sin45^{\circ}$$
$$= \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$$
$$= \frac{\sqrt{2}-\sqrt{6}}{4}$$

2. Find $sin135^{\circ}$ exactly.

$$sin135^{\circ} = sin\left(90^{\circ} + 45^{\circ}\right)$$
$$= sin90^{\circ}cos45^{\circ} + cos90^{\circ}sin45^{\circ}$$
$$= 1\left(\frac{\sqrt{2}}{2}\right) + 0\left(\frac{\sqrt{2}}{2}\right)$$
$$= \frac{\sqrt{2}}{2}$$

3. Given $cosA = \frac{5}{13}$ and $sinB = \frac{3}{5}$, with A and B acute, find $sin(A + B)$.

$$sinA = \frac{12}{13}, \qquad cosB = \frac{4}{5}$$
$$sin(A + B) = sinAcosB + cosAsinB$$
$$= \left(\frac{12}{13}\right)\left(\frac{4}{5}\right) + \left(\frac{5}{13}\right)\left(\frac{3}{5}\right)$$
$$= \frac{48+15}{65}$$
$$= \frac{63}{65}$$

4. Given $sinA = \frac{5}{13}$ and $cosB = \frac{3}{5}$, with A and B acute, find $cos(A - B)$.

$$cosA = \frac{12}{13}, \qquad sinB = \frac{4}{5}$$
$$cos(A - B) = cosAcosB + sinAsinB$$
$$= \left(\frac{12}{13}\right)\left(\frac{3}{5}\right) + \left(\frac{5}{13}\right)\left(\frac{4}{5}\right)$$
$$= \frac{36+20}{65}$$
$$= \frac{56}{65}$$

Answer Summary: $\frac{\sqrt{2}-\sqrt{6}}{4}$; $\frac{\sqrt{2}}{2}$; $\frac{63}{65}$; $\frac{56}{65}$.

Part II - Derive the Tangent Sum & Difference Identities

Tangent Sum Identity

Start with $tan\theta = \frac{sin\theta}{cos\theta}$ and substitute the sine and cosine sum identities.

$$tan(\alpha + \beta) = \frac{sin(\alpha+\beta)}{cos(\alpha+\beta)}$$

$$= \frac{sin\alpha cos\beta + cos\alpha sin\beta}{cos\alpha cos\beta - sin\alpha sin\beta}$$

Divide every term in the numerator and denominator by $cos\alpha cos\beta$.

$$tan(\alpha + \beta) = \frac{\frac{sin\alpha cos\beta}{cos\alpha cos\beta} + \frac{cos\alpha sin\beta}{cos\alpha cos\beta}}{\frac{cos\alpha cos\beta}{cos\alpha cos\beta} - \frac{sin\alpha sin\beta}{cos\alpha cos\beta}}$$

$$= \frac{tan\alpha + tan\beta}{1 - tan\alpha tan\beta}$$

Tangent Difference Identity

$$tan(\alpha - \beta) = \frac{sin(\alpha - \beta)}{cos(\alpha - \beta)}$$

$$= \frac{sin\alpha cos\beta - cos\alpha sin\beta}{cos\alpha cos\beta + sin\alpha sin\beta}$$

$$= \frac{tan\alpha - tan\beta}{1 + tan\alpha tan\beta}$$

What sign pattern do you notice?

The numerator keeps the original sign, while the denominator uses the opposite sign: sum over difference, and difference over sum.

Answer Summary: $tan(\alpha + \beta) = \frac{tan\alpha + tan\beta}{1 - tan\alpha tan\beta}$; $tan(\alpha - \beta) = \frac{tan\alpha - tan\beta}{1 + tan\alpha tan\beta}$.

Part III - Tangent Practice

5. Find $tan75^\circ$ exactly.

$$tan75^\circ = tan\left(45^\circ + 30^\circ\right)$$

$$= \frac{1 + \frac{\sqrt{3}}{3}}{1 - \frac{\sqrt{3}}{3}}$$

$$= \frac{\sqrt{3} + 1}{\sqrt{3} - 1}$$

$$= \frac{\left(\sqrt{3} + 1\right)^2}{3 - 1}$$

$$= 2 + \sqrt{3}$$

6. Find $tan15^\circ$ exactly.

$$tan15^\circ = tan\left(45^\circ - 30^\circ\right)$$

$$= \frac{1 - \frac{\sqrt{3}}{3}}{1 + \frac{\sqrt{3}}{3}}$$

$$= \frac{\sqrt{3} - 1}{\sqrt{3} + 1}$$

$$= \frac{\left(\sqrt{3} - 1\right)^2}{3 - 1}$$

$$= 2 - \sqrt{3}$$

Answer Summary: $2 + \sqrt{3}$; $2 - \sqrt{3}$.

Practice B - Independent Tangent Check

1. Find $tan105^{\circ}$ exactly.

$$tan105^{\circ} = tan\left(60^{\circ} + 45^{\circ}\right)$$
$$= \frac{\sqrt{3}+1}{1-\sqrt{3}}$$
$$= \frac{\left(\sqrt{3}+1\right)\left(1+\sqrt{3}\right)}{\left(1-\sqrt{3}\right)\left(1+\sqrt{3}\right)}$$
$$= \frac{4+2\sqrt{3}}{-2}$$
$$=- 2 - \sqrt{3}$$

2. Find $tan30^{\circ}$ using the reference table.

$$tan30^{\circ} = \frac{sin30^{\circ}}{cos30^{\circ}}$$
$$= \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}$$
$$= \frac{1}{\sqrt{3}}$$
$$= \frac{\sqrt{3}}{3}$$

3. If $tanA = \frac{1}{3}$ and $tanB = \frac{1}{2}$, find $tan(A + B)$.

$$tan(A + B) = \frac{tanA+tanB}{1-tanAtanB}$$
$$= \frac{\frac{1}{3}+\frac{1}{2}}{1-\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)}$$
$$= \frac{\frac{5}{6}}{\frac{5}{6}}$$
$$= 1$$

Answer Summary: $- 2 - \sqrt{3}; \frac{\sqrt{3}}{3}; 1.$

Part IV - What If the Two Angles Are the Same?

Sine Double-Angle Identity

$$sin(x + x) = sinxcosx + cosxsinx$$
$$sin(2x) = 2sinxcosx$$

Cosine Double-Angle Identities

$$cos(x + x) = cosxcosx - sinxsinx$$
$$cos(2x) = cos^{2}x - sin^{2}x$$

Using $sin^{2}x + cos^{2}x = 1$ gives two equivalent forms.

$$cos(2x) = 2cos^{2}x - 1$$
$$cos(2x) = 1 - 2sin^{2}x$$

Tangent Double-Angle Identity

$$tan(x + x) = \frac{tanx+tanx}{1-tanxtanx}$$

$$tan(2x) = \frac{2tanx}{1-tan^2x}$$

Answer Summary: $sin2x = 2sinxcosx$; $cos2x = cos^2x - sin^2x = 2cos^2x - 1 = 1 - 2sin^2x$; $tan2x = \frac{2tanx}{1-tan^2x}$.

Part V - Double-Angle Practice

7. If $sinx = \frac{3}{5}$ and x is acute, find $sin(2x)$.

Because x is acute, $cosx = \frac{4}{5}$.

$$\begin{aligned} sin(2x) &= 2sinxcosx \\ &= 2\left(\frac{3}{5}\right)\left(\frac{4}{5}\right) \\ &= \frac{24}{25} \end{aligned}$$

8. Under the same condition, find $cos(2x)$.

$$\begin{aligned} cos(2x) &= cos^2x - sin^2x \\ &= \left(\frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2 \\ &= \frac{16-9}{25} \\ &= \frac{7}{25} \end{aligned}$$

9. If $tanx = \frac{1}{2}$, find $tan(2x)$.

$$\begin{aligned} tan(2x) &= \frac{2tanx}{1-tan^2x} \\ &= \frac{2\left(\frac{1}{2}\right)}{1-\left(\frac{1}{2}\right)^2} \\ &= \frac{1}{\frac{3}{4}} \\ &= \frac{4}{3} \end{aligned}$$

10. If $sinx = \frac{4}{5}$, find $cos(2x)$ without finding $cosx$.

Choose the form containing only sine.

$$\begin{aligned} cos(2x) &= 1 - 2sin^2x \\ &= 1 - 2\left(\frac{4}{5}\right)^2 \\ &= 1 - \frac{32}{25} \\ &= -\frac{7}{25} \end{aligned}$$

11. If $cosx =- \frac{3}{5}$ and x is in Quadrant II, find $sin(2x)$.

In Quadrant II, sine is positive. Therefore $sinx = \frac{4}{5}$.

$$sin(2x) = 2sinxcosx$$
$$= 2\left(\frac{4}{5}\right)\left(-\frac{3}{5}\right)$$
$$=- \frac{24}{25}$$

Answer Summary: $\frac{24}{25}; \frac{7}{25}; \frac{4}{3}; -\frac{7}{25}; -\frac{24}{25}$.

Practice C - Independent Double-Angle Check

1. If $cosx = \frac{12}{13}$ and x is acute, find $sin(2x)$.

Because x is acute, $sinx = \frac{5}{13}$.

$$sin(2x) = 2sinxcosx$$
$$= 2\left(\frac{5}{13}\right)\left(\frac{12}{13}\right)$$
$$= \frac{120}{169}$$

2. If $cosx = \frac{4}{5}$, find $cos(2x)$ without finding $sinx$.

Choose the form containing only cosine.

$$cos(2x) = 2cos^2x - 1$$
$$= 2\left(\frac{4}{5}\right)^2 - 1$$
$$= \frac{32}{25} - \frac{25}{25}$$
$$= \frac{7}{25}$$

3. If $tanx = \frac{1}{3}$, find $tan(2x)$.

$$tan(2x) = \frac{2tanx}{1-tan^2x}$$
$$= \frac{\frac{2}{3}}{1-\frac{1}{9}}$$
$$= \frac{\frac{2}{3}}{\frac{8}{9}}$$
$$= \frac{3}{4}$$

4. If $sinx = \frac{5}{13}$ and x is acute, find both $sin(2x)$ and $cos(2x)$.

Because x is acute, $cosx = \frac{12}{13}$.

$$sin(2x) = 2sinxcosx$$

$$= 2\left(\frac{5}{13}\right)\left(\frac{12}{13}\right)$$

$$= \frac{120}{169}$$

$$cos(2x) = cos^2x - sin^2x$$

$$= \left(\frac{12}{13}\right)^2 - \left(\frac{5}{13}\right)^2$$

$$= \frac{144-25}{169}$$

$$= \frac{119}{169}$$

Answer Summary: $\frac{120}{169}$; $\frac{7}{25}$; $\frac{3}{4}$; $\frac{120}{169}$ and $\frac{119}{169}$.

Homework - Cumulative Practice

1. Find $sin135^{\circ}$ exactly.

$$sin135^{\circ} = sin\left(90^{\circ} + 45^{\circ}\right)$$

$$= sin90^{\circ}cos45^{\circ} + cos90^{\circ}sin45^{\circ}$$

$$= \frac{\sqrt{2}}{2}$$

2. Find $cos165^{\circ}$ exactly.

Use $165^{\circ} = 180^{\circ} - 15^{\circ}$.

$$cos165^{\circ} = cos\left(180^{\circ} - 15^{\circ}\right)$$

$$= cos180^{\circ}cos15^{\circ} + sin180^{\circ}sin15^{\circ}$$

$$= -1\left(\frac{\sqrt{6}+\sqrt{2}}{4}\right) + 0$$

$$= -\frac{\sqrt{6}+\sqrt{2}}{4}$$

3. Find $tan105^{\circ}$ exactly.

$$tan105^{\circ} = tan\left(60^{\circ} + 45^{\circ}\right)$$

$$= \frac{\sqrt{3}+1}{1-\sqrt{3}}$$

$$= -2 - \sqrt{3}$$

4. Given $cosA = \frac{12}{13}$ and $sinB = \frac{8}{17}$, where A and B are acute, find $cos(A + B)$.

$$sinA = \frac{5}{13}, \qquad cosB = \frac{15}{17}$$

$$cos(A + B) = cosAcosB - sinAsinB$$

$$= \left(\frac{12}{13}\right)\left(\frac{15}{17}\right) - \left(\frac{5}{13}\right)\left(\frac{8}{17}\right)$$

$$= \frac{180-40}{221}$$

$$= \frac{140}{221}$$

5. If $tanx = \frac{2}{3}$, find $tan(2x)$.

$$tan(2x) = \frac{2tanx}{1-tan^2x}$$
$$= \frac{\frac{4}{3}}{1-\frac{4}{9}}$$
$$= \frac{\frac{4}{3}}{\frac{5}{9}}$$
$$= \frac{12}{5}$$

6. If $sinx = \frac{8}{17}$ and x is acute, find $sin(2x)$.

Because x is acute, $cosx = \frac{15}{17}$.

$$sin(2x) = 2sinxcosx$$
$$= 2\left(\frac{8}{17}\right)\left(\frac{15}{17}\right)$$
$$= \frac{240}{289}$$

7. Under the same condition, find $cos(2x)$.

$$cos(2x) = cos^2x - sin^2x$$
$$= \left(\frac{15}{17}\right)^2 - \left(\frac{8}{17}\right)^2$$
$$= \frac{225-64}{289}$$
$$= \frac{161}{289}$$

8. If $cosx = \frac{7}{25}$, find $cos(2x)$ without finding $sinx$.

Choose the form containing only cosine.

$$cos(2x) = 2cos^2x - 1$$
$$= 2\left(\frac{7}{25}\right)^2 - 1$$
$$= \frac{98}{625} - \frac{625}{625}$$
$$= -\frac{527}{625}$$

9. Use the sine sum identity to show why $sin(2x) = 2sinxcosx$.

Let both angles in the sine sum identity equal x.

$$sin(\alpha + \beta) = sin\alpha cos\beta + cos\alpha sin\beta$$
$$sin(x + x) = sinxcosx + cosxsinx$$
$$sin(2x) = sinxcosx + sinxcosx$$
$$= 2sinxcosx$$

10. Explain why the three forms of $cos(2x)$ are equivalent.

Begin with $cos(2x) = cos^2x - sin^2x$ *and use the Pythagorean identity* $sin^2x + cos^2x = 1$.

Replacing sin^2x *with* $1 - cos^2x$ *gives:*

$$cos(2x) = cos^2x - \left(1 - cos^2x\right)$$
$$= 2cos^2x - 1.$$

Replacing cos^2x *with* $1 - sin^2x$ *gives:*

$$cos(2x) = \left(1 - sin^2x\right) - sin^2x$$
$$= 1 - 2sin^2x.$$

Therefore,

$$cos(2x) = cos^2x - sin^2x = 2cos^2x - 1 = 1 - 2sin^2x.$$

Answer Summary: $\frac{\sqrt{2}}{2}$; $-\frac{\sqrt{6}+\sqrt{2}}{4}$; $-2-\sqrt{3}$; $\frac{140}{221}$; $\frac{12}{5}$; $\frac{240}{289}$; $\frac{161}{289}$; $-\frac{527}{625}$; $2sinxcosx$; the three cosine forms are equivalent by $sin^2x + cos^2x = 1$.