Integration Methods and Error Analysis Writing Task

In this assignment, you will apply different integration techniques to evaluate both exact and approximate integrals. You will analyze when certain techniques are appropriate, determine error bounds for approximations, and interpret the meaning of these integrals in applied contexts.

The Assignment

Part 1: Integration by Substitution

Consider the integral $\int x\sqrt{3x^2+4}\,dx$.

1. Use substitution to find the antiderivative. Show all steps of your solution, including:
    - Your choice of substitution variable
    - The resulting expression in terms of the new variable
    - The steps to integrate this expression
    - Your final answer in terms of the original variable
2. Verify your answer by differentiating your result and showing that you obtain the original integrand.
3. Now use your antiderivative to evaluate the definite integral $\int_1^2 x\sqrt{3x^2+4}\,dx$.

Part 2: Integration Involving Special Functions

Consider the following integrals:

A. $\int \frac{e^{2x}}{1+e^{4x}}, dx$

B. $\int \frac{dx}{4+9x^2}$

C. $\int \frac{dx}{\sqrt{25-4x^2}}$

For each integral:

1. Identify the most appropriate integration technique to use (substitution, direct formula for exponential/logarithmic/inverse trigonometric functions).
2. Apply the technique to find the antiderivative, showing all your steps.
3. Explain why your chosen technique is appropriate for this particular integral.

Part 3: Approximation and Error Analysis

Consider the function $f(x) = \frac{x}{\sqrt{x^4 + 1}}$ on the interval [0,1].

1. Determine whether the integral $\int_0^1 f(x)\,dx$ can be evaluated using the techniques we've studied. If it can, find the exact value. If not, explain why not.
2. Use a left-endpoint approximation with *n*=4 subintervals to approximate $\int_0^1 f(x)\,dx$. Show all computations.
3. Use a right-endpoint approximation with *n*=4 subintervals to approximate $\int_0^1 f(x)\,dx$. Show all computations.
4. Based on your approximations, provide an interval that contains the exact value of the integral. Choose the midpoint of this interval as your final approximation and determine the maximum possible error.
5. Analyze how the behavior of *f(x)* on [0,1] affects the relative accuracy of the left-endpoint versus right-endpoint approximations. Which would you expect to be more accurate, and why?

Part 4: Application and Interpretation

1. Consider a particle moving along a straight line with velocity function $v(t) = te^{-t/2}$ meters per second for $t \geq 0$.
   - Set up and evaluate the definite integral that gives the distance traveled by the particle from *t*=0 to *t*=4.
   - Interpret the meaning of any integration techniques you use in terms of this physical context.
   - Explain what the value of the integral represents physically.
2. The rate at which a certain population of bacteria grows is modeled by $r(t) = \frac{100}{1 + t^2}$ bacteria per hour, where *t* is measured in hours since the experiment began.

    - Set up and evaluate the definite integral that gives the total increase in the bacterial population from $t=0$ to $t=3$.
    - Explain your choice of integration technique and interpret the result in the context of the bacteria growth.

Your submission should include:

- Clearly labeled solutions for each part
- Complete mathematical work showing all steps
- Proper mathematical notation
- Thorough explanations that connect your mathematical work to the problems' contexts
- Clear analysis of approximation errors and their significance

This assignment is worth 20 points. Your work will be assessed on mathematical accuracy, clarity of explanations, proper use of integration techniques, and your ability to interpret results in context.

Rubric:

Criteria	Proficient	Developing	Not Evident	Points
Integration by Substitution	Correctly applies substitution technique with appropriate variable choice. All steps are clearly shown with proper notation. Verification by differentiation is complete and accurate. Definite integral is correctly evaluated.	Substitution is attempted with minor errors or unclear steps. Verification has minor errors or is incomplete. Definite integral evaluation contains minor computational errors.	Significant errors in substitution or inappropriate variable choice. Verification missing or contains major errors. Definite integral evaluation incorrect or missing.	___/5
Special Function Integration	Correctly identifies and applies appropriate techniques for all three integrals. Steps are clearly shown and mathematically accurate. Explanations of technique choices are insightful and thorough.	Most techniques correctly identified with minor errors in application. Some steps may be unclear or contain minor errors. Explanations of technique choices are basic but accurate.	Multiple errors in technique identification or application. Missing steps or major mathematical errors. Explanations of technique choices are missing or incorrect.	___/5
Approximation and Error Analysis	Accurately determines whether the integral can be evaluated exactly. Riemann sum approximations are correctly calculated with complete work shown. Error bounds are precisely determined with clear explanation. Analysis of approximation accuracy is insightful and mathematically sound.	Basic determination of integral evaluability. Riemann sum calculations contain minor errors. Error bounds are determined with minor inaccuracies. Analysis of approximation accuracy shows basic understanding.	Incorrect determination of integral evaluability. Significant errors in Riemann sum calculations. Error bounds missing or substantially incorrect. Little or no valid analysis of approximation accuracy.	___/5

Application and Interpretation	Physical applications are correctly modeled with appropriate integrals. Integration techniques are accurately applied with clear work shown. Interpretations of results demonstrate deep understanding of the mathematical and physical connections.	Physical applications are mostly correctly modeled. Integration contains minor errors. Interpretations show basic understanding of connections between mathematics and physical contexts.	Physical applications are incorrectly modeled. Integration contains major errors or missing work. Interpretations show significant misconceptions or are missing.	___/5
Total				___/20