# Assignment 4

## PM1/04 (Group B)

*(Symbols have their usual meanings)*

1. (a) What do you mean by an orientable surface?

   (b) Show that Mobius band is not orientable.

   (c) Find the equation of the tangent plane to the surface patch given by $\sigma(u,v)=(u,v,u^2-v^2)$ at the point (1,1,0).

2. Define smooth surface. Show that $S=\left\{(x,y,z)\in \mathrm{R}^3 : \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1\right\}$, where a,b,c are non-zero constant is a smooth surface.

3. Define first fundamental form of a surface S at any point p in $R^3$. Find out the first fundamental form of the following:

a) Generalized cylinder $\sigma(u,v)=\gamma(u)+va$, a is a point in $R^3$

b) Unit sphere $S^2$

c) $\sigma(u,v)=(f(u)\cos v, f(u)\sin v, g(u))$ (surface of revolution)

d) $\sigma(u, v) = (\sinh u \sinh v, \sinh u \cosh v, \sinh u)$.

4. When two surfaces $S_1$ and $S_2$ are locally isometry? Prove that a smooth map $f:S_1\rightarrow S_2$ is local isometry if and only if the symmetric bilinear form $\langle\, ,\rangle_p$ and $f^*\langle\, ,\rangle_p$ on $T_pS_1$ are equal for all p in $S_1$.

5. Prove that a local diffeomorphism $f:S_1\rightarrow S_2$ is a local isometry if and only if for any surface patch $\sigma_1$ of $S_1$, the patches $\sigma_1$ and $f_o\,\sigma_1$ of $S_1$ and $S_2$ respectively have the same fundamental form.

6. Define second fundamental form. Find out the second fundamental form of the following:

a) Generalized cylinder $\sigma(u,v)=\gamma(u)+va$, a is a point in $R^3$

b) Unit sphere $S^2$

c) $\sigma(u,v)=(f(u)\cos v, f(u)\sin v, g(u))$ (surface of revolution)

d) the elliptic paraboloid $\sigma(u, v)=(u, v, u^2 + v^2)$.

7. If the second fundamental form of a surface patch $\sigma$ is zero everywhere, prove that $\sigma$ is planar.

8. Establish $\kappa^2=\kappa_n^2+\kappa_g^2$

9. Consider the surface of revolution $(f(u)\cos v, f(u)\sin v, g(u))$, where $u \mapsto (f(u),0,g(u))$ is a unit speed curve in $R^3$. Compute the Gaussian and mean curvatures of surface of revolution.

10. Find the normal curvature of any curve on the sphere of radius r.

11. Let $\kappa_1$ and $\kappa_2$ be the principal curvatures at a point p of a regular surface patch $\sigma$. Show that $\kappa_1$ ,$\kappa_2$ are real numbers.

12. Show that the Gaussian and mean curvatures of the surface z=f(x,y), where f is a smooth function are given by

$$K = \frac{f_{xx}f_{yy} - f_{xy}^2}{(1+f_x^2+f_y^2)^2}, H = \frac{(1+f_y^2)f_{xx} - 2f_x f_y f_{xy} + (1+f_x^2)f_{yy}}{2(1+f_x^2+f_y^2)^{3/2}}$$

13. Find the unit surface normal vector ξ for a surface given by r=(u cosv, u sinv, cv), c being constant.

14. Define Gaussian curvature on a surface. Find a relation between the Gaussian curvature and scalar curvature of a surface.

15. Define normal curvature. Show that the normal curvature of any curve on a sphere of radius r is $\pm \frac{1}{r}$ .

16. Show that the Gaussian curvature and mean curvature of a regular surface S are smooth function of S.

17. Let $S \subseteq R^3$ be a regular surface. Let (U,F,V) be a local parametrization of S. Let $W \subseteq R^2$ be a open set, $\varphi : W \to R^3$ a map with $\varphi(W) \subset S \cap V$ . Then prove that $\varphi$ considered as a map from W to $R^3$ is smooth if and only if $F^{-1} \circ \varphi : W \to U \subseteq R^2$ is smooth.

18. Show that every isometry is a conformal map. Give an example of conformal map which is not an isometry.