Assignment 4

PM1/04 (Group B)

(Symbols have their usual meanings)

1. (a) What do you mean by an orientable surface?

   (b) Show that Mobius band is not orientable.

   (c) Find the equation of the tangent plane to the surface patch given by $\sigma(u,v)=(u,v,u^2-v^2)$ at the point (1,1,0).

2. Define smooth surface. Show that $S=\left\{(x,y,z)\in \mathrm{R}^3 : \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1\right\}$, where a,b,c are non-zero constant is a smooth surface.

3. Define first fundamental form of a surface S at any point p in R^3. Find out the first fundamental form of the following:

a) Generalized cylinder $\sigma(u,v)=\gamma(u)+va$, a is a point in R^3

b) Unit sphere S^2

c) $\sigma(u,v)=(f(u)\cos v, f(u)\sin v, g(u))$ (surface of revolution)

d) $\sigma(u, v) = (\sinh u \sinh v, \sinh u \cosh v, \sinh u)$.

4. When two surfaces S_1 and S_2 are locally isometry? Prove that a smooth map $f:S_1\rightarrow S_2$ is local isometry if and only if the symmetric bilinear form $<\ ,\ >_p$ and $f^*<\ ,\ >_p$ on T_pS_1 are equal for all p in S_1.

5. Prove that a local diffeomorphism $f:S_1\rightarrow S_2$ is a local isometry if and only if for any surface patch σ_1 of S_1, the patches σ_1 and $f_o\,\sigma_1$ of S_1 and S_2 respectively have the same fundamental form.

6. Define second fundamental form. Find out the second fundamental form of the following:

a) Generalized cylinder $\sigma(u,v)=\gamma(u)+va$, a is a point in R^3

b) Unit sphere S^2

c) $\sigma(u,v)=(f(u)\cos v, f(u)\sin v, g(u))$ (surface of revolution)

d) the elliptic paraboloid $\sigma(u, v)=(u, v, u^2 + v^2)$.

7. If the second fundamental form of a surface patch σ is zero everywhere, prove that σ is planar.

8. Establish $\kappa^2=\kappa_n^2+\kappa_g^2$

9. Consider the surface of revolution (f(u)cosv, f(u) sinv, g(u)), where $u \mapsto (f(u),0,g(u))$ is a unit speed curve in R^3. Compute the Gaussian and mean curvatures of surface of revolution.

10. Find the normal curvature of any curve on the sphere of radius r.

11. Let κ_1 and κ_2 be the principal curvatures at a point p of a regular surface patch σ. Show that κ_1 ,κ_2 are real numbers.

12. Show that the Gaussian and mean curvatures of the surface z=f(x,y), where f is a smooth function are given by

$$K = \frac{f_{xx}f_{yy} - f_{xy}^2}{(1+f_x^2+f_y^2)^2}, H = \frac{(1+f_y^2)f_{xx} - 2f_xf_yf_{xy} + (1+f_x^2)f_{yy}}{2(1+f_x^2+f_y^2)^{3/2}}$$

13. Find the unit surface normal vector ξ for a surface given by r=(u cosv, u sinv, cv), c being constant.

14. Define Gaussian curvature on a surface. Find a relation between the Gaussian curvature and scalar curvature of a surface.

15. Define normal curvature. Show that the normal curvature of any curve on a sphere of radius r is $\pm \frac{1}{r}$.

16. Show that the Gaussian curvature and mean curvature of a regular surface S are smooth function of S.

17. Let $S \subseteq R^3$ be a regular surface. Let (U,F,V) be a local parametrization of S. Let $W \subseteq R^2$ be a open set, $\varphi : W \to R^3$ a map with $\varphi(W) \subset S \cap V$. Then prove that φ considered as a map from W to R^3 is smooth if and only if $F^{-1} \circ \varphi : W \to U \subseteq R^2$ is smooth.

18. Show that every isometry is a conformal map. Give an example of conformal map which is not an isometry.