Linear Algebra MAT313 Spring 2024
Professor Sormani
Lesson 8
Part I The identity matrix and permutation matrices
Part II Nonsingular matrices

Students must complete Lesson 6 including HW9 and HW10 before starting this project.

Before you start, find your team's project document here and submit one more line on the team project!

If you work with any classmates on this lesson, be sure to write their names on the problems you completed together.

You will cut and paste the photos of your notes and completed classwork in a googledoc entitled:

MAT313S24-lesson8-lastname-firstname

and share editing of that document with me sormanic@gmail.com. You will also include your homework and any corrections to your homework in this doc.

If you have a question, type QUESTION in your googledoc next to the point in your notes that has a question and email me with the subject MAT313 QUESTION. I will answer your question by inserting a photo into your googledoc or making an extra video.

****
This lesson has two parts!!!!! Each has its own videos!!!

Part I The identity matrix and permutation matrices

Part II Nonsingular matrices

There are ten homework problems in this lesson.

****
Part I The identity matrix and permutation matrices
****

Watch the Playlist 313F20-8-1to4 which includes homework as classwork that you work on in class with hints and I did not put photos of those hints below because not enough students are watching the videos.

Lesson 8

Part I

Square Matrices

* Identity Matrix

Permutation Matrices

Part II

Nonsingular Matrices

Recall

we know how to multiply a matrix by a vector.

(review Lesson 7 yourselves)

Linear Algebra

Defn A square matrix is a matrix with n rows and n columns

The set of $n \times n$ square matrices: $M_{n \times n}$

Example

$$M_{2\times 2} = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} : a_{ij} \in \mathbb{R} \right\}$$

$$= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}$$

$a_{i,j}$ is useful for $M_{n \times n}$ where n is large.

2:07 AM Sun Sep 6

Linear Algebra

Thm: If $A \in M_{n\times n}$ and $\vec{v} \in \mathbb{R}^n$ then we multiply

$$A\vec{v} \in \mathbb{R}^n$$

Pf $A\vec{v} \in \mathbb{R}^n$ where n = number of rows of A

Here $\vec{v} \in \mathbb{R}^n \leftarrow$ number of columns of A is also n.

QED

Pf when $n=2$:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} a_{11}v_1 + a_{12}v_2 \\ a_{21}v_1 + a_{22}v_2 \end{pmatrix} \in \mathbb{R}^2 \checkmark$$

QED

Linear Algebra

②

Defn An identity matrix, I, or, Id, is a square matrix with $\boxed{I_{ii} = 1}$ for $i=1$ to n and $\boxed{I_{ij} = 0}$ if $i \neq j$

In $M_{2\times 2}$: $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

In $M_{3\times 3}$: $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Try $M_{4\times 4}$

2:07 AM Sun Sep 6

Linear Algebra

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in M_{4\times4}$$

Classwork:

What is the null space of this I in $M_{4\times4}$?

Linear Algebra

Solve for Null space

$$\text{Null}(I) = \{\vec{v} \in \mathbb{R}^4 \mid I\vec{v} = \vec{0}\}$$

null space of I

Solve the homogeneous system

$$\left(\begin{array}{cccc|c} \boxed{1} & 0 & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 \end{array}\right) \rightarrow \text{reduced Echelon Form}$$

already Echelon form

already reduced echelon form

$$\boxed{\begin{aligned} 1x_1 &= 0 \\ 1x_2 &= 0 \\ 1x_3 &= 0 \\ 1x_4 &= 0 \end{aligned}}$$

$$\text{Null}(I) = \{\vec{0}\}.$$

2:09 AM Sun Sep 6

Linear Algebra

Thm: $\mathrm{Null}(I) = \{\vec{0}\}$ in all dimensions.

Proof: Solve the system

$$\sum_{j=1}^{n} I_{ij}\, x_j = 0 \text{ for } i = 1 \text{ to } n$$

is 0 when $j \neq i$ and is 1 when $j = i$

$$0 + 0 + \ldots + 1x_i + 0 \ldots + 0 = 0$$

$$1x_i = 0 \text{ for } i = 1 \text{ to } n.$$

$$\mathrm{Null}(I) = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right\}$$

$$= \{\vec{0}\} \quad \text{QED.}$$

Linear Algebra

Note that

$$\left(I \mid \vec{0} \right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right)$$

always has n leaders and no free variables and is already in reduced Echelon form

2:13 AM Sun Sep 6 | Linear Algebra

Permutation Matrices $P_{n \times n}$

Defn are square matrices where each row has a single entry that is a 1 and the rest are 0's and they same for each column.

List all 2x2 permutation matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$ ← each row has a 1 and a 0

each column has a 1 and a 0.

Linear Algebra

$$\begin{pmatrix} \boxed{1} & 0 \\ 0 & 1 \end{pmatrix}$$ is only one that ~~s~~ starts with 1 up left

$$\begin{pmatrix} \boxed{0} & 1 \\ 1 & 0 \end{pmatrix}$$ ← row also needs a 1

this column also needs a 1

this forces a 0 here

All 2x2 permutation matrices

$$P_{2\times 2} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

2:18 AM Sun Sep 6

Linear Algebra

$P_{3\times 3}$

start with a 1 here

Case I

$$\begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & & \\ 0 & & \end{bmatrix}$$

rest of row must be 0

what can be here?

rest of column must be 0

Case II

$$\begin{bmatrix} 0 & & \\ & & \\ & & \end{bmatrix}$$

later check starting with 0

pause + try

80%

Linear Algebra

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

if we put 1 here

must be 0

must be 0

must be 1

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \boxed{0} & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

if we put a 0 here.

must be a 1

must be 1

must have 0 here

2:21 AM Sun Sep 6 — Linear Algebra | Linear Algebra — 80%

$P_{3\times 3}$

start with a 1 here

Case I

$$\begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & & \\ 0 & & \end{bmatrix}$$

rest of row must be 0

what can be here?

rest of column must be 0

Case II

$$\begin{bmatrix} 0 & 0 & 1 \\ & & \\ & & \end{bmatrix}$$

later check starting with 0

or

$$\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$$

even more cases!

pause + try

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

if we put 1 here

must be 0

must be 0

must be 1

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \boxed{0} & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

if we put a 0 here.

must be a 1

must be 1

must have 0 here

$$\begin{bmatrix} 0 & 0 & 1 \\ & & 0 \\ & & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 & 1 \\ \boxed{1} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 0 & 1 \\ \boxed{0} & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ & 0 & \\ & 0 & \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ \boxed{1} & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 1 & 0 \\ \boxed{0} & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$P_{3\times 3} = \left\{ \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}, \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}, \begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}, \begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}, \begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}, \begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix} \right\}$$

$\boxed{\text{HW1}}$ Find $P_{4\times 4}$

Hint

Case I
$$\begin{bmatrix}1&0&0&0\\ & & & \\ & & & \end{bmatrix}$$

Case II
$$\begin{bmatrix}0&1&0&0\\ & & & \\ & & & \end{bmatrix}$$

What the follows

Case III
$$\begin{bmatrix}0&0&1&0\\ & & & \\ & & & \end{bmatrix}$$

Case IV
$$\begin{bmatrix}0&0&0&1\\ & & & \\ & & & \end{bmatrix}$$

Each case will lead to a few more.

Linear Algebra

Case I

$$\begin{bmatrix}1&0&0&0\\ & & & \\ & & & \\ & & & \end{bmatrix}$$

what is forced on us?

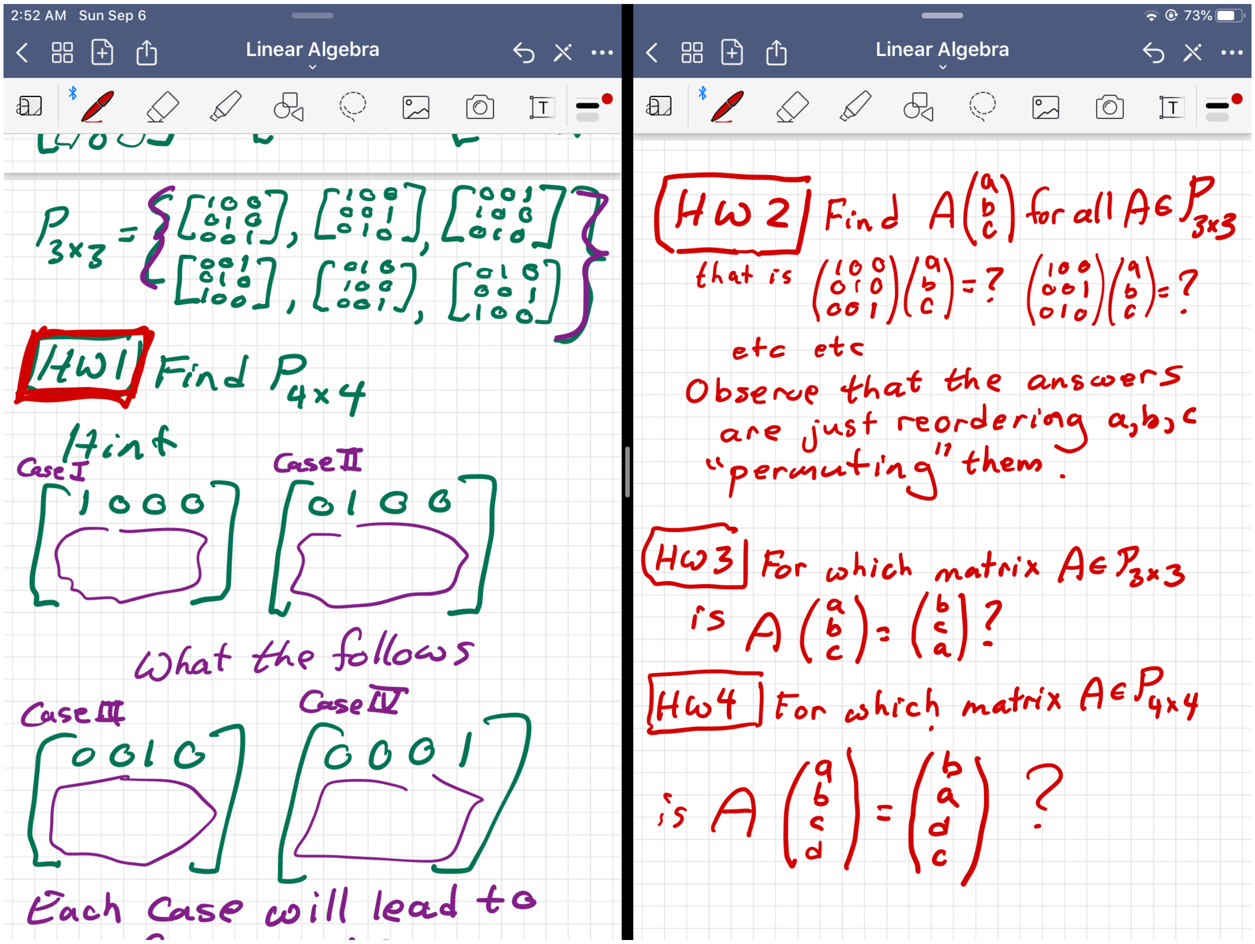
the rest of this column must be 0

$$\begin{bmatrix}1&0&0&0\\0& & & \\0& & & \\0& & & \end{bmatrix}$$

what possibilities fit here?

a 3x3 matrix with a single 1 in each row and a single 1 in each column.

6 possible ways to finish: each of the $P_{3\times 3}$ work!



HW1 has a significant amount done within the videos.
HW2-4 have no hints in the videos.
Do all this homework before watching Part 2.

****

Part II Nonsingular matrices

****

Watch the two videos in [Playlist 313F21-8-P2]
Note this is a little confusing if you skipped Lesson 7 where we learned Null(A) but the main point is that you are checking for free variables: a nonsingular matrix is a square matrix with no free variables.

Lesson 8 Part II

Defn: A nonsingular matrix is a square matrix $A \in M_{m \times n}$ with $m=n$ such that $Null(A) = \{\vec{0}\}$.

Notice that when we do row reduction to find the null space of matrix:

$$\left[A \,\middle|\, \begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \end{matrix} \right] \longrightarrow \left[\quad \,\middle|\, \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \end{matrix} \right] \rightarrow \cdots \rightarrow \left[E \,\middle|\, \begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \end{matrix} \right]$$

Reduced Echelon Form

If $Null(A) = \{\vec{0}\}$ that means $\left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right\}$ there are no free variables all variables are leaders

so the Reduced Echelon Form has a 1 in every column.

Since A is a square matrix, then so is E
we don't have any extra rows
so we have a 1 in every row.
Since E has zeroes above and below leaders

$$E = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & 1 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = I \quad \text{Identity Matrix.}$$

So a nonsingular matrix is a square matrix whose reduced Echelon Form is $\left[I \,\middle|\, \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \right]$

Examples of Nonsingular Matrices

$\boxed{\text{Classwork}}$ check if each of the following matrices is nonsingular

This is the identity

$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ Find Null(A) $\left[\begin{array}{cc|c} \boxed{1} & 2 & 0 \\ 0 & \boxed{1} & 0 \end{array}\right] \xrightarrow{\rho_1 \to \rho_1 - 2\rho_2} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$ $x_1 = 0$, $x_2 = 0$ $\text{Null}(A) = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$

Echelon form

So A is nonsingular.

square matrix $\in M_{2\times 2}$

We could notice A is nonsingular at the Echelon Form because we see that all columns have leaders $\rightarrow$ eventually the reduced echelon form is $[I \mid \vec{0}]$

$B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{\rho_1 \leftrightarrow \rho_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$ Nonsing?

All permutation matrices are Nonsingular because they are square and after enough switches they have $[I \mid \vec{0}]$.

$C = \begin{bmatrix} \boxed{1} & 2 & 3 & 4 \\ 0 & \boxed{5} & 6 & 7 \\ 0 & 0 & \boxed{8} & 9 \\ 0 & 0 & 0 & \boxed{5} \end{bmatrix}$ ← Is C nonsingular?

some scaling will give $\left[\begin{array}{cccc|c} \boxed{1} & * & * & * & 0 \\ 0 & \boxed{1} & * & * & 0 \\ 0 & 0 & \boxed{1} & * & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 \end{array}\right]$

C is nonsingular

Defn

A singular matrix is a matrix that is not nonsingular.

Warning: To check if a square matrix is nonsingular you will usually need to do more row reduction than we have here!

Remember: a nonsingular matrix is a square matrix with no free variables. You only need to go to Echelon Form to see if there are free variables! Be sure to follow the algorithm for row reduction to Echelon Form that we learned in Lesson 3.

Homework: Check if the following matrices are nonsingular

HW5
$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 0 & 0 & 1 \end{pmatrix}$$

HW6
$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 5 & 0 \end{pmatrix}$$

HW7
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

HW8
$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 2 & 3 \end{pmatrix}$$

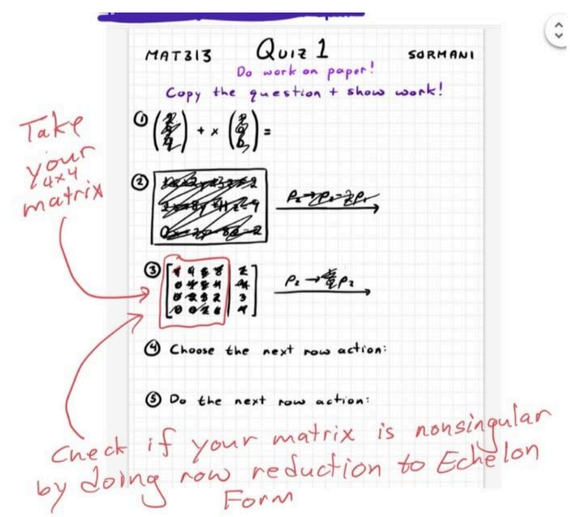
HW9
$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

ExtraCredit

Find all values of a and b s.t $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ is nonsing.

HW10 Copy the 4x4 matrix that is the left part of the augmented matrix on your Quiz 1 (3) and check if it is a singular matrix.

Where do I find this HW10 matrix?



Complete HW5-HW10 stated above.

When you are done, find your team's project document and submit one line on the team project!