

## Linear Algebra Lesson 3

### **Solving Linear Systems with Augmented Matrices**

Before starting this lesson you must have thoroughly understood lessons 1 and 2 and completed the homework. Have your homework available to consult during the lesson.

We will begin by going over the first problem in the Lesson 2 HW using a new notion called an **augmented matrix** as in the following photos.

Watch Videos 1-5 of the [Playlist 313F20-3-1to7](#)

We will also introduce a notion called **Reduced Echelon Form**.

①

$$\begin{aligned} 4x + 8y + 4z &= 16 \\ x + 2y + z &= 4 \\ x + y + z &= 3 \end{aligned}$$

②

$$\begin{aligned} 2x + 2y &= 4 \\ x + 3y &= 1 \\ 3x + 5y &= 6 \end{aligned}$$

③

$$\begin{aligned} x + y + 4z &= 12 \\ x + 2y + 4z &= 12 \end{aligned}$$

④ The solution set of one of these is a line. Find position and direction and draw it.

①

$$\begin{cases} 4x + 8y + 4z = 16 \\ x + 2y + z = 4 \\ x + y + z = 3 \end{cases}$$

Check Upper Left  
is a 1  
if not  $p_1 \rightarrow \frac{1}{4}p_1$   
or switch

$$\left( \begin{array}{ccc|c} 4 & 8 & 4 & 16 \\ 1 & 2 & 1 & 4 \\ 1 & 1 & 1 & 3 \end{array} \right)$$

$p_1 \rightarrow \frac{1}{4}p_1$

$$\begin{cases} 1x + 2y + 1z = 4 \\ x + 2y + z = 4 \\ x + y + z = 3 \end{cases}$$

Make entries  
under the  
leader 0 -  
using skew by  
leader's row

$$\left( \begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 2 & 1 & 4 \\ 0 & 1 & 1 & 3 \end{array} \right)$$

$p_2 \rightarrow p_2 - p_1$   $p_3 \rightarrow p_3 - p_1$

$$\begin{cases} 1x + 2y + 1z = 4 \\ 0x + 0y + 0z = 0 \\ 0x - 1y + 0z = -1 \end{cases}$$

Box next leader  
row of zeroes has  
no leader  
must switch  $p_2 \leftrightarrow p_3$

$p_2 \rightarrow p_2 - p_1$   $p_3 \rightarrow p_3 - p_1$

$$\left( \begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 \end{array} \right)$$

$p_2 \leftrightarrow p_3$

$$\begin{cases} 1x + 2y + 1z = 4 \\ 0x - 1y + 0z = -1 \\ 0x + 0y + 0z = 0 \end{cases}$$

If next leader  
is not 1  
scale row

$$\left( \begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$p_2 \rightarrow -p_2$

$$\begin{cases} 1x + 2y + 1z = 4 \\ 0x + 1y + 0z = 1 \\ 0x + 0y + 0z = 0 \end{cases}$$

Echelon  
Form!

$$\left( \begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

Sub up: Last row has no leader  
leaders  $x, y$  free variable  $z$

Solve for  $y$  in row 2

$$y = 1$$

Solve for  $x$  in row 1

$$x = 4 - 2y - z$$

Sub in  $y = 1$

$$x = 4 - 2 \cdot 1 - z = 2 - z$$

$$z = z \text{ free}$$

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 - z \\ 1 \\ z \end{pmatrix} : z \in \mathbb{R} \right\}$$

Is a line because 1 free variable

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \cdot z : z \in \mathbb{R} \right\}$$

Then try doing problems 2 and 3 the same way.

Then watch 313F20-3-6a and 313F20-3-6b of the Playlist for the solutions to this classwork:

$$\begin{cases} 2x + 2y = 4 \\ 1x + 3y = 1 \\ 3x + 5y = 6 \end{cases}$$

Augmented Matrix  $\left( \begin{array}{cc|c} 2 & 2 & 4 \\ 1 & 3 & 1 \\ 3 & 5 & 6 \end{array} \right)$

Now Reduce to Echelon Form  
Check upper left for a leader  
Since the leader is 2  $P_1 \rightarrow \frac{1}{2}P_1$

Now check under the leader  
Use skew to change these numbers to 0  
 $P_2 \rightarrow P_2 - P_1$   
 $P_3 \rightarrow P_3 - 3P_1$

$$\left( \begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 2 & -1 \\ 0 & 2 & 0 \end{array} \right)$$
 Check next leader

Use scaling to make the next leader 1  
 $P_2 \rightarrow \frac{1}{2}P_2$

Make numbers under leader into 0's using skew by leader's row  
 $P_3 \rightarrow P_3 - 2P_2$

$$\left( \begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \end{array} \right)$$

This is Echelon Form



Use scaling to make the next leader 1  
 $p_2 \rightarrow \frac{1}{2}p_2$

$$\left( \begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -1/2 \\ 0 & 2 & 0 \end{array} \right)$$

Make numbers under leader into 0's using skew by leader's row  
 $p_3 \rightarrow p_3 - 2p_2$

$$\left( \begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \end{array} \right)$$

This is Echelon Form

Continue to Reduced Echelon Form  
 Change the number above the last leader into a 0 using skew by leader's row

$p_1 \rightarrow p_1 - p_2$

$$\left( \begin{array}{cc|c} 1 & 0 & 5/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \end{array} \right)$$

This is Reduced Echelon Form

$2 - (-1/2) = \frac{4}{2} + \frac{1}{2} = \frac{5}{2}$

$p_1 \rightarrow p_1 - p_2$

$$\left( \begin{array}{cc|c} 1 & 0 & 5/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \end{array} \right)$$

This is Reduced Echelon Form

Leaders are x and y  
 No free variables because every column has a leader

Solve for leaders

$$\begin{array}{l} 1x + 0y = 5/2 \\ 0x + 1y = -1/2 \\ 0 + 0 = 1 \end{array}$$

$x = 5/2$   
 $y = -1/2$

But the final equation is  $0 + 0 = 1$   
 Contradiction

The solution set is empty  $\emptyset$

Watch 313F20-3-7 for the solution to this problem:

$$\begin{cases} x+y+4z=12 \\ x+2y+4z=12 \end{cases}$$

Reduced  
to  
Echelon  
Form :

Reduced  
Echelon  
Form :

$$\left( \begin{array}{ccc|c} 1 & 1 & 4 & 12 \\ 1 & 2 & 4 & 12 \end{array} \right)$$

already 1  
change by skew  
numbers under  
it to 0

$$p_2 \rightarrow p_2 - p_1$$

$$\left( \begin{array}{ccc|c} 1 & 1 & 4 & 12 \\ 0 & 1 & 0 & 0 \end{array} \right)$$

next leader  
also already 1  
Echelon Form!

change numbers above leaders  
into 0 using skew

$$p_1 \rightarrow p_1 - p_2$$

$$\left( \begin{array}{ccc|c} 1 & 0 & 4 & 12 \\ 0 & 1 & 0 & 0 \end{array} \right)$$

Already Echelon Form

Solution: leaders are  $x$  and  $y$   
free variable is  $z$

Solve for leaders

$$\begin{cases} x+0y+4z=12 \\ 0x+1y+0z=0 \end{cases}$$

$$x=12-4z$$

$$y=0$$

$$z=z \text{ (free)}$$

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 12-4z \\ 0 \\ z \end{pmatrix} : z \in \mathbb{R} \right\} \quad \text{a line because only one free variable}$$

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 12 \\ 0 \\ 0 \end{pmatrix} + z \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix} : z \in \mathbb{R} \right\}$$

position      direction

Check our answer: when  $z=0$   $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 12 \\ 0 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 12 \\ 0 \\ 0 \end{pmatrix}$   $\begin{matrix} x=12 \\ y=0 \\ z=0 \end{matrix}$

plug into original equations

$$(12) + (0) + 4(0) = 12 \checkmark$$

$$(12) + 2(0) + 4(0) = 12 \checkmark$$

when  $z=1$   $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 12 \\ 0 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 12-4 \\ 0+0 \\ 0+1 \end{pmatrix} = \begin{pmatrix} 8 \\ 0 \\ 1 \end{pmatrix}$   $\begin{matrix} x=8 \\ y=0 \\ z=1 \end{matrix}$

Homework:

Solve the following four problems using augmented matrices and reduced echelon form as outlined here. You must use the following technique as taught in this lesson.



Step 1: Write Augmented Matrix

Step 2: Row Actions to Echelon Form

- Make upper left leader into 1 using scaling (or switch if 0)
- Make zeroes beneath leader using skew by leader's row
- Move down to the next row and make sure next column has a leader which is 1 by repeating red step.

Then repeat blue step.

Move down again until Echelon Form

Each leader in red box is a 1.

Below each leader are 0's.

Step 3: Row actions to Reduced Echelon Form

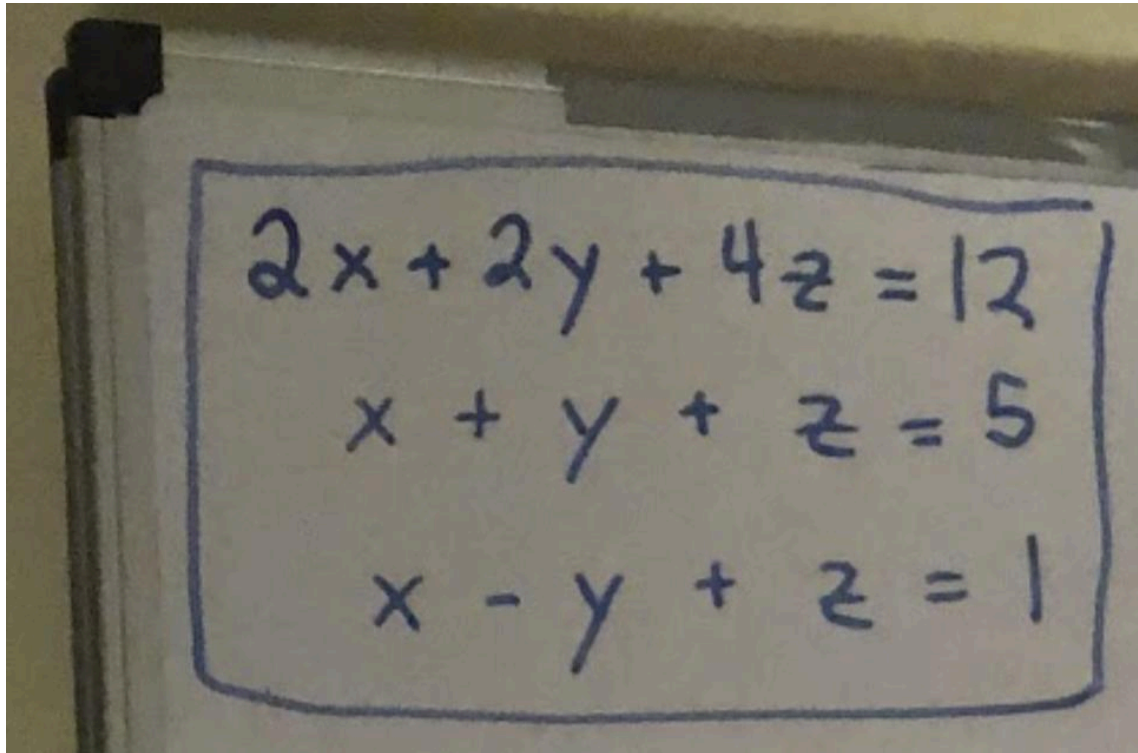
- Start with bottom leader
- Make sure 0's above it using skew by the leader's row.
- Repeat with next to last leader until 0's above all leaders

Step 4: Write as equations and solve for leaders and set free variables equal to themselves

$$\text{Write as } \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} \text{---} \\ \text{---} \\ \text{---} \end{pmatrix}, \text{ free} \in \mathbb{R} \right\}$$



HW1:



A photograph of a piece of paper with three linear equations written in blue ink. The equations are enclosed in a hand-drawn rectangular border. The equations are:

$$\begin{aligned}2x + 2y + 4z &= 12 \\x + y + z &= 5 \\x - y + z &= 1\end{aligned}$$

HW2:

$$\begin{aligned} 2x + 2y + 4z &= 12 \\ x + y + z &= 5 \\ 2x + 2y + 2z &= 8 \end{aligned}$$

HW3:

$$\begin{aligned} 1x + 1y + 1z + 1w &= 0 \\ 1x + 1y - 1z + 1w &= 0 \\ 2x + 3y + 2z + 1w &= 0 \end{aligned}$$

HW4:

$$x_1 + 2x_2 + 3x_3 + 4x_4 = 0$$

$$x_1 + 2x_2 + x_3 + x_4 = 0$$

$$2x_1 + 4x_2 + 6x_3 + 8x_4 = 0$$

$$2x_1 + 4x_2 + 4x_3 + 5x_4 = 0$$

$$3x_1 + 6x_2 + 3x_3 + 3x_4 = 0$$

Esteban is the grader for these homework problems. Share your google doc with him [esteban.alcantara780@gmail.com](mailto:esteban.alcantara780@gmail.com) when you are ready for him to check it. He is only responsible to tell you if you are right or wrong, not tell you the answers. He will confirm that you are using the correct row actions in the correct order and that you are completing the actions correctly. It is not considered correct if you do extra row actions that don't follow the algorithm taught in class. Be sure to write "MAT313 Lesson3" in the subject of your email when you contact him.