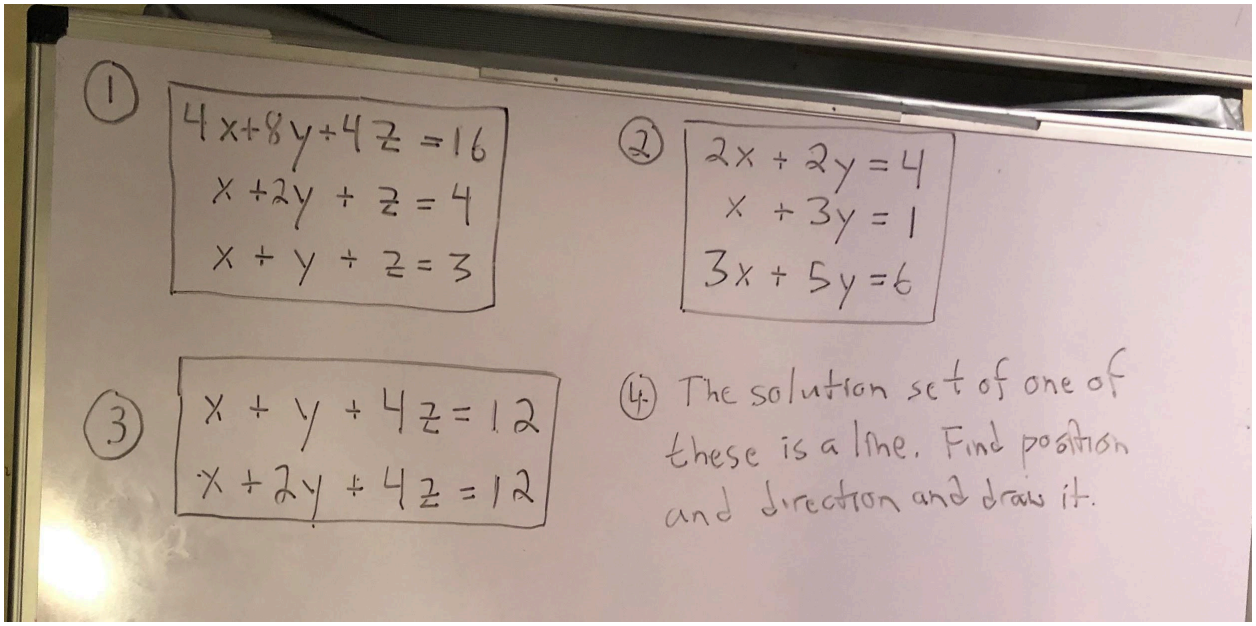
Linear Algebra Lesson 3

Solving Linear Systems with Augmented Matrices

Before starting this lesson you must have thoroughly understood lessons 1 and 2 and completed the homework. Have your homework available to consult during the lesson.

We will begin by going over the first problem in the Lesson 2 HW using a new notion called an **augmented matrix** as in the following photos.
Watch Videos 1-5 of the Playlist 313F20-3-1to7
We will also introduce a notion called **Reduced Echelon Form.**


①
4x+8y+4z=16
x+2y+z=4
x+y+z=3
②
2x+2y=4
x+3y=1
3x+5y=6
③
x+y+4z=12
x+2y+4z=12
④ The solution set of one of these is a line. Find position and direction and draw it.

① $4x+8y+4z=16$, $x+2y+z=4$, $x+y+z=3$

Check Upper Left is a 1 if not $\rho_1 \to \frac{1}{4}\rho_1$ or switch

$$\left(\begin{array}{ccc|c} 4 & 8 & 4 & 16 \\ 1 & 2 & 1 & 4 \\ 1 & 1 & 1 & 3 \end{array}\right)$$

$\rho_1 \to \frac{1}{4}\rho_1$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 1 & 2 & 1 & 4 \\ 1 & 1 & 1 & 3 \end{array}\right)$$

$1x+2y+1z=4$, $1x+2y+1z=4$, $1x+1y+1z=3$

Make entries under the leader 0 - using skewby, leader's row

$\rho_2 \to \rho_2-\rho_1$ | $\rho_3 \to \rho_3-\rho_1$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 \end{array}\right)$$

$1x+2y+1z=4$, $0x+0y+0z=0$, $0x-1y+0z=-1$

Box next leader

row of zeroes has no leader

must switch $\rho_2 \leftrightarrow \rho_3$

$\rho_2 \leftrightarrow \rho_3$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$1x+2y+1z=4$, $0x-1y+0z=-1$, $0x+0y+0z=0$

If next leader is not 1 scale row

$\rho_2 \to -\rho_2$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$1x+2y+1z=4$, $0x+1y+0z=1$, $0x+0y+0z=0$

Echelon Form!

Sub up: Last row has no leader, free variable z

leaders x, y

Solve for y in row 2: $y=1$

Solve for x in row 1 sub in $y=1$: $x=4-2y-z$, $x=4-2\cdot 1-z=2-z$

$z=z$ free

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2-z \\ 1 \\ z \end{pmatrix} : z \in \mathbb{R} \right\}$$

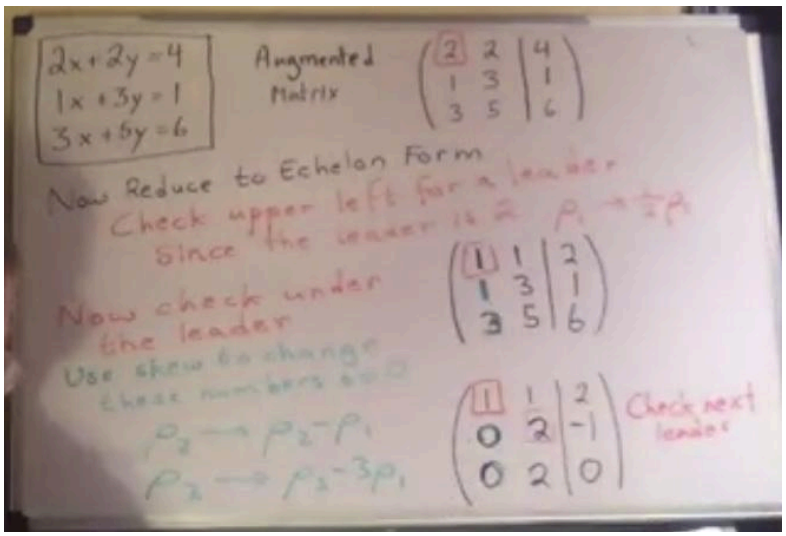
z free

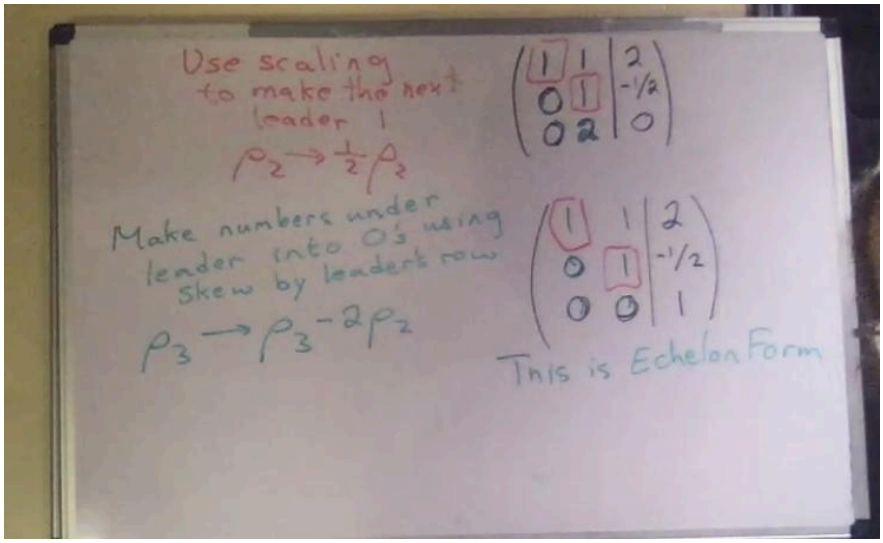
Is a line because 1 free variable

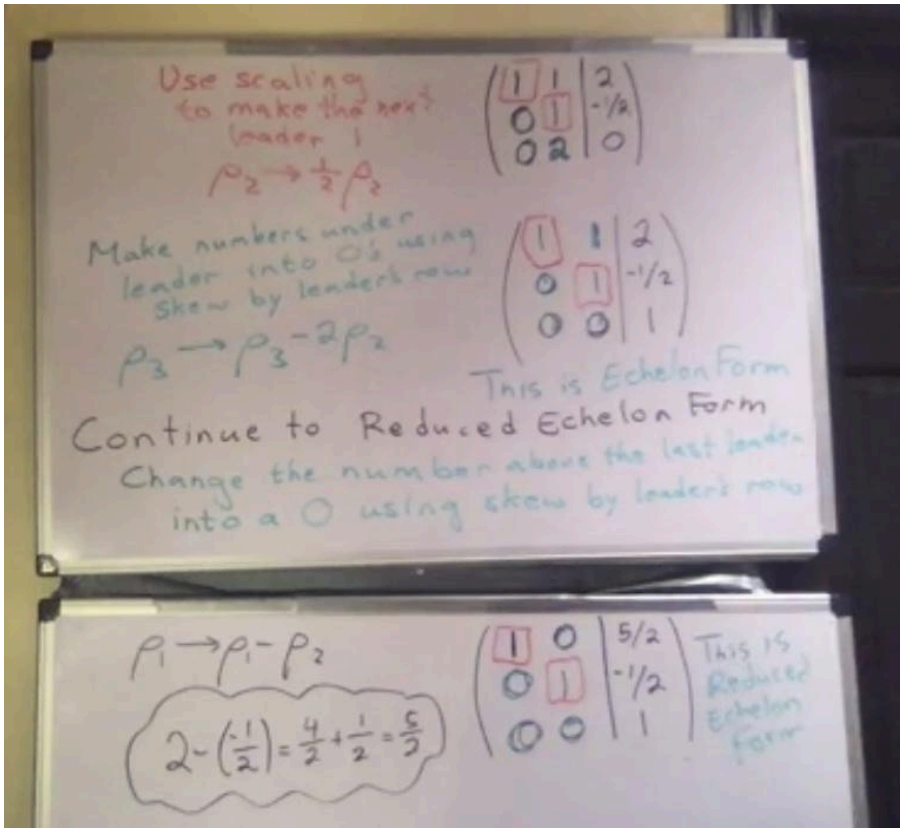
$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} : z \in \mathbb{R} \right\}$$

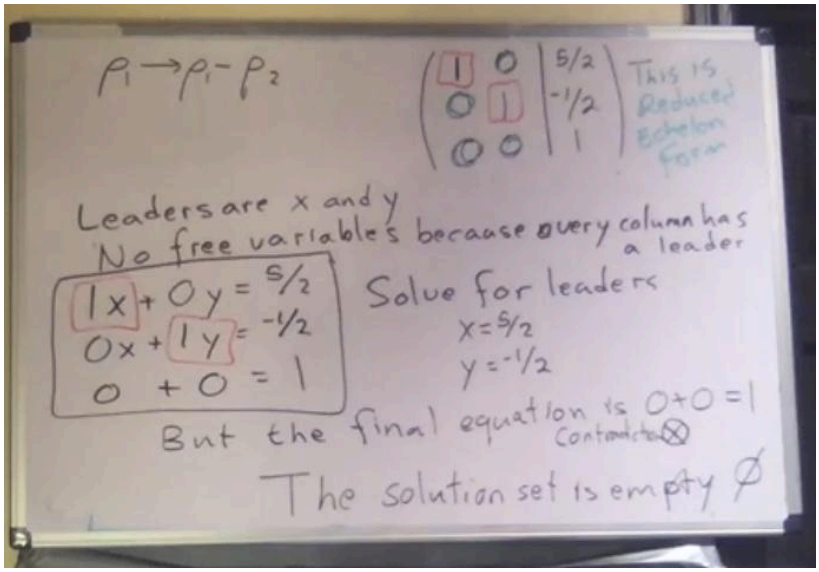
Then try doing problems 2 and 3 the same way.

Then watch 313F20-3-6a and 313F20-3-6b of the Playlist for the solutions to this classwork:









Watch 313F20-3-7 for the solution to this problem:

$$x+y+4z=12$$
$$x+2y+4z=12$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 4 & 12 \\ 1 & 2 & 4 & 12 \end{array}\right)$$

already 1

change by skew numbers under it to 0

Reduced to Echelon Form:

$\rho_2 \to \rho_2 - \rho_1$

$$\left(\begin{array}{ccc|c} 1 & 1 & 4 & 12 \\ 0 & 1 & 0 & 0 \end{array}\right)$$

next leader also already 1

Echelon Form!

Reduced Echelon Form:

change numbers above leaders into 0 using skew

$\rho_1 \to \rho_1 - \rho_2$

$$\left(\begin{array}{ccc|c} 1 & 0 & 4 & 12 \\ 0 & 1 & 0 & 0 \end{array}\right)$$

Already Echelon Form

Solution: leaders are x and y
free variable is z

Solve for leaders

$x + 0y + 4z = 12$ $\quad x = 12 - 4z$

$0x + 1y + 0z = 0$ $\quad y = 0$

$z = z$ (free)

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 12-4z \\ 0 \\ 1z \end{pmatrix} : z \in \mathbb{R}_{\text{free}} \right\}$$

a line because only one free variable

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 12 \\ 0 \\ 0 \end{pmatrix} + z \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix} : z \in \mathbb{R} \right\}$$

position direction

Check our answer: when $z=0$ $\quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 12 \\ 0 \\ 0 \end{pmatrix} + 0\begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 12 \\ 0 \\ 0 \end{pmatrix}$ $\quad x=12, y=0, z=0$

plug into original equations

$(12) + (0) + 4(0) = 12$ ✓

$(12) + 2(0) + 4(0) = 12$ ✓

when $z=1$ $\quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 12 \\ 0 \\ 0 \end{pmatrix} + 1\begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 12-4 \\ 0+0 \\ 0+1 \end{pmatrix} = \begin{pmatrix} 8 \\ 0 \\ 1 \end{pmatrix}$

$x = 8$

Homework:

Solve the following four problems using augmented matrices and reduced echelon form as outlined here. You must use the following technique as taught in this lesson.

Step 1: Write Augmented Matrix

Step 2: Row Actions to Echelon Form

- Make upper left leader into 1 using scaling (or switch if 0)
- Make zeroes beneath leader using skew by leader's row
- Move down to the next row and make sure next column has a leader which is 1 by repeating red step. Then repeat blue step.

Move down again until Echelon Form

Each leader in red box is a 1.
Below each leader are 0's.

Step 3: Row actions to Reduced Echelon Form

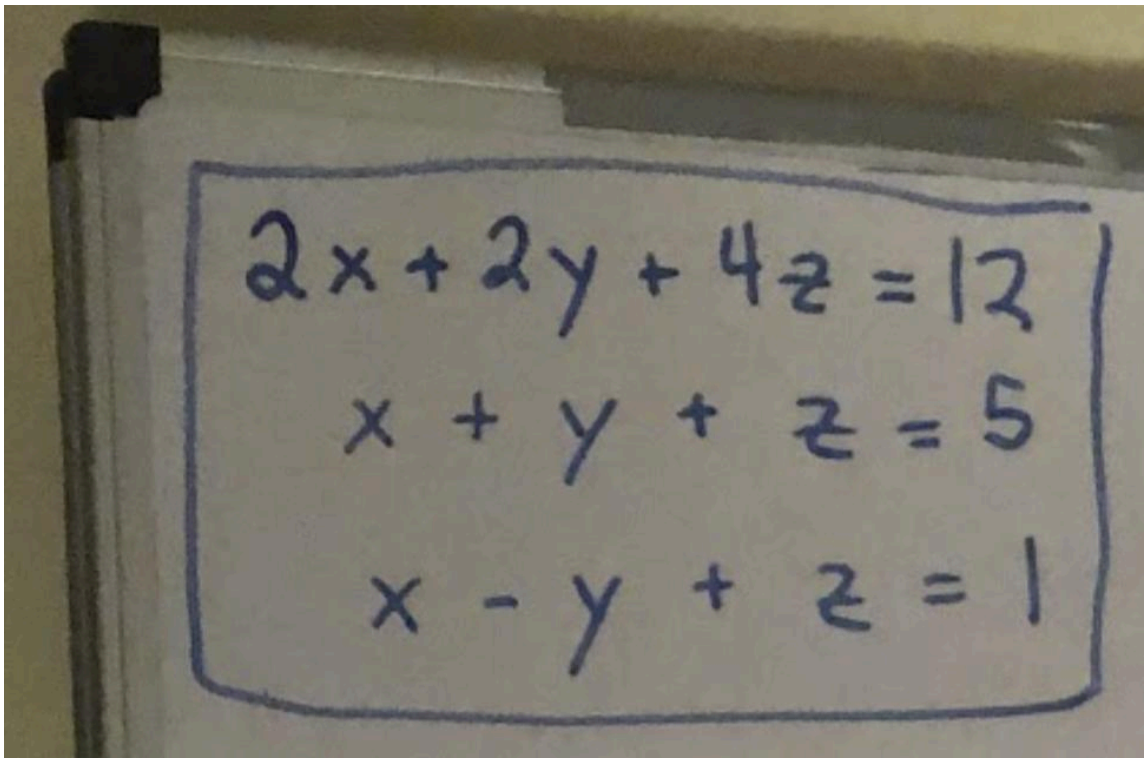
- start with bottom leader Make sure 0's above it using skew by the leader's row.

Repeat with next to last leader until 0's above all leaders

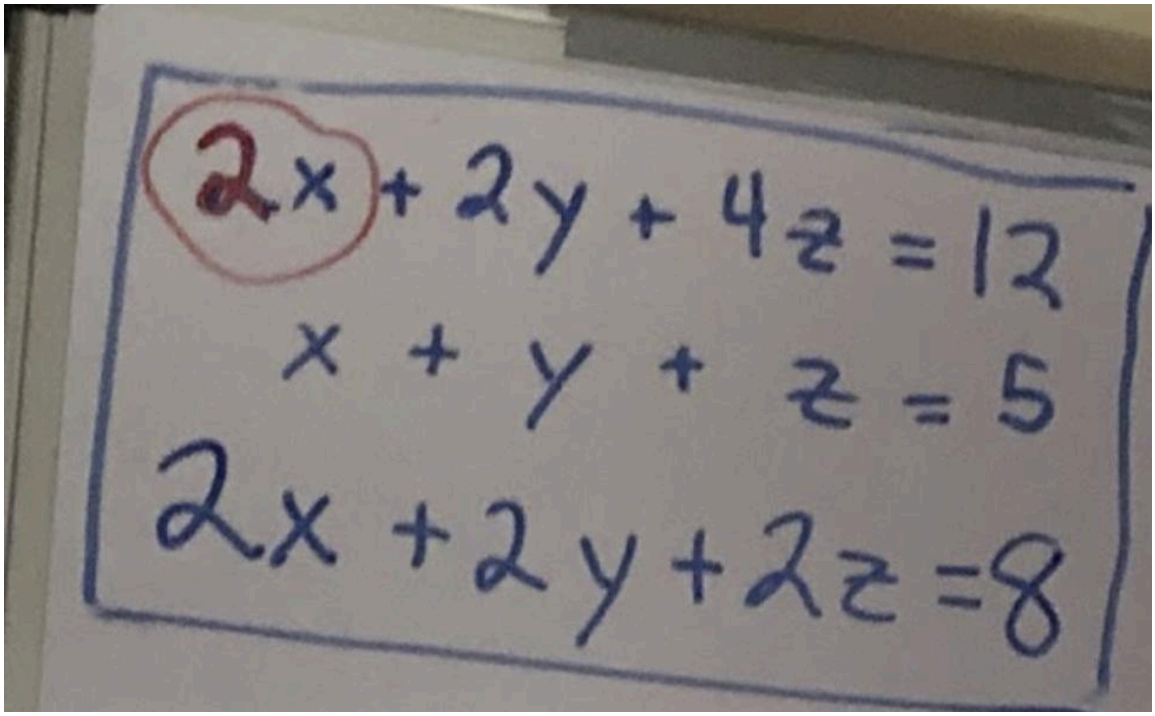
Step 4: Write as equations and solve for leaders and set free variables equal to themselves

Write as $\left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} \equiv \\ \equiv \\ \equiv \end{pmatrix} ; \text{ free} \in \mathbb{R} \right\}$

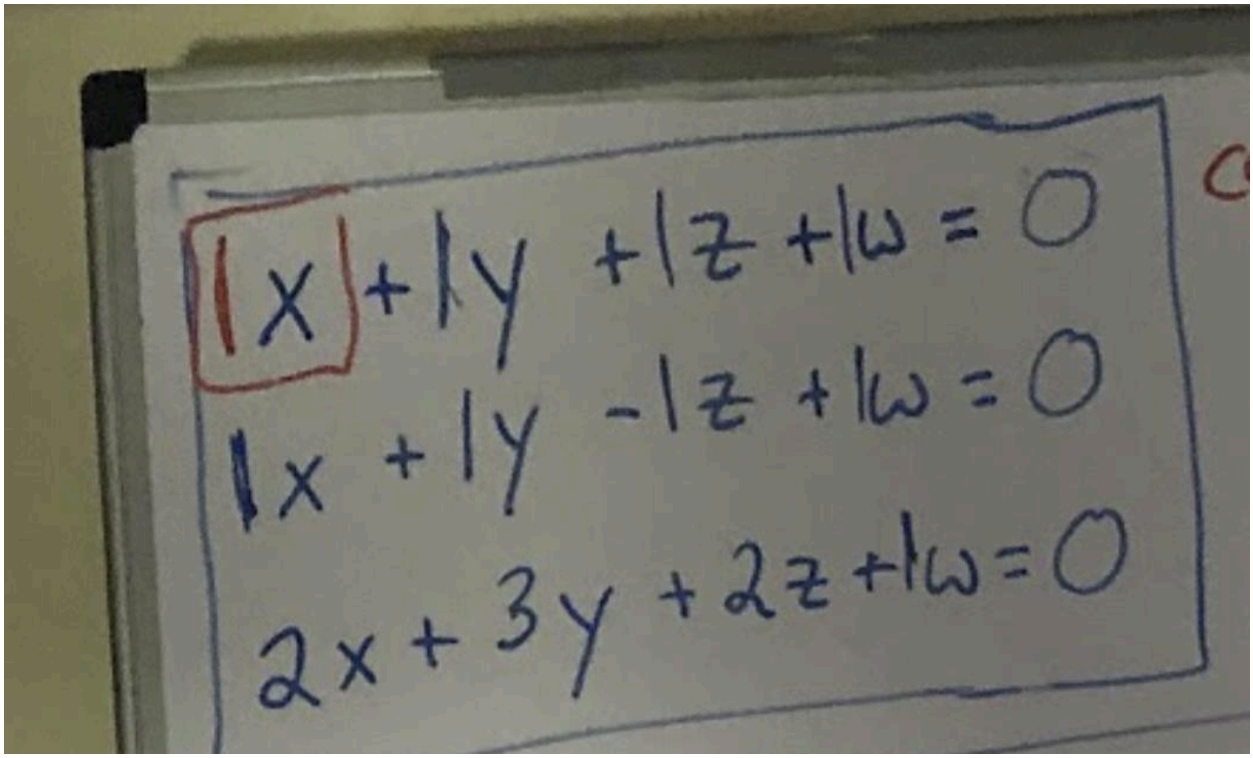
HW1:



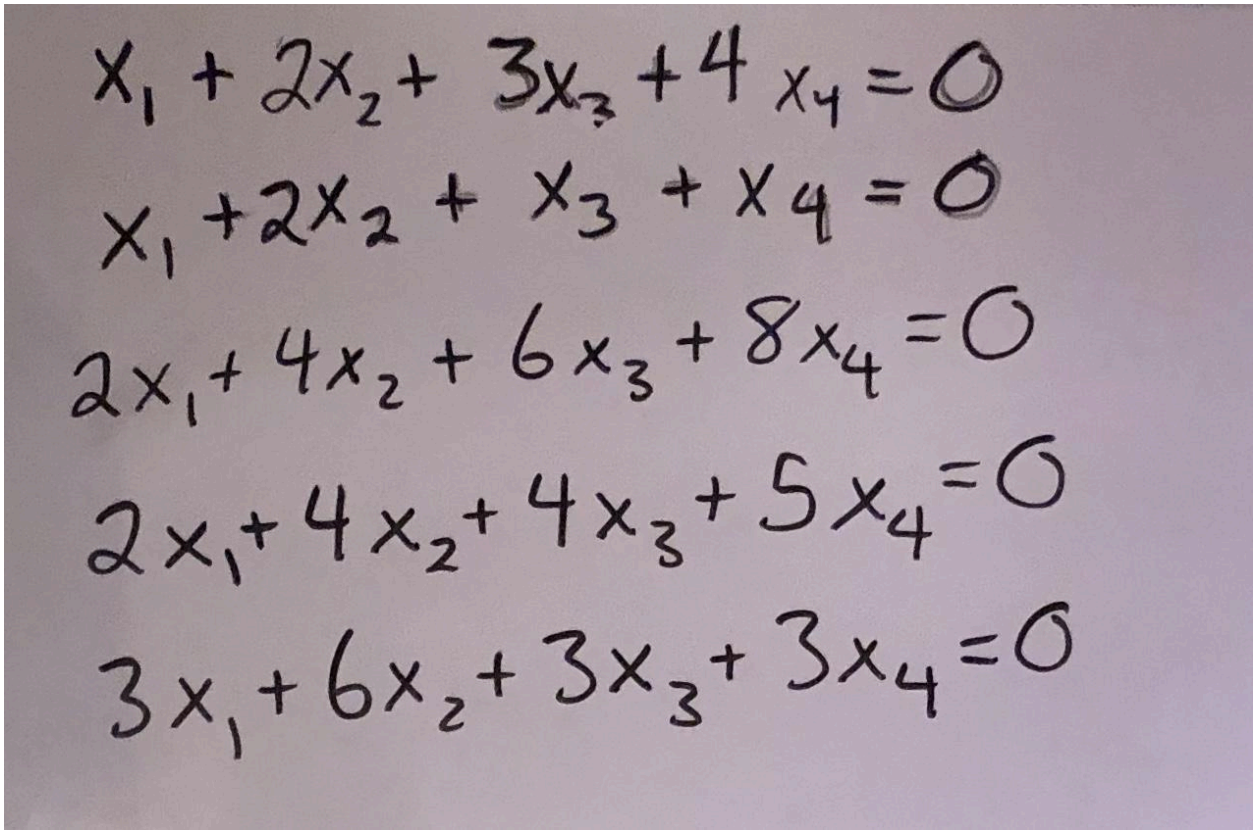
HW2:



HW3:



HW4:



Esteban is the grader for these homework problems. Share your google doc with him esteban.alcantara780@gmail.com when you are ready for him to check it. He is only responsible to tell you if you are right or wrong, not tell you the answers. He will confirm that you are using the correct row actions in the correct order and that you are completing the actions correctly. It is not considered correct if you do extra row actions that don't follow the algorithm taught in class. Be sure to write "MAT313 Lesson3" in the subject of your email when you contact him.