Câu 44. **[2D3-2.3-4].** **(THPT Chuyên ĐH Vinh – lần 1 - năm 2017 – 2018)** Cho hàm số $y=f(x)$ có đạo hàm liên tục trên đoạn $[0;1]$ và $f(0)+f(1)=0$. Biết $\int\limits_0^1 f^2(x)\,dx=\frac{1}{2}$, $\int\limits_0^1 f'(x)\cos(\pi x)\,dx=\frac{\pi}{2}$. Tính $\int\limits_0^1 f(x)\,dx$.

A. π. **B.** $\frac{1}{\pi}$. **C.** $\frac{2}{\pi}$. **D.** $\frac{3\pi}{2}$.

Lời giải

Chọn C.

Đặt $\begin{cases} u=\cos(\pi x) \\ dv=f'(x)\,dx \end{cases} \Rightarrow \begin{cases} du=-\pi\sin(\pi x)\,dx \\ v=f(x) \end{cases}$.

Khi đó: $\int\limits_0^1 f'(x)\cos(\pi x)\,dx=\cos(\pi x)f(x)\Big|_0^1+\pi\int\limits_0^1 f(x)\sin(\pi x)\,dx$

$=-\left(f(1)+f(0)\right)+\pi\int\limits_0^1 f(x)\sin(\pi x)\,dx=\pi\int\limits_0^1 f(x)\sin(\pi x)\,dx$

$\Rightarrow \int\limits_0^1 f(x)\sin(\pi x)\,dx=\frac{1}{2}$.

Cách 1: Ta có

Tìm k sao cho $\int\limits_0^1 \left[f(x)-k\sin(\pi x)\right]^2 dx=0$

Ta có: $\int\limits_0^1 \left[f(x)-k\sin(\pi x)\right]^2 dx=\int\limits_0^1 f^2(x)\,dx-2k\int\limits_0^1 f(x)\sin(\pi x)\,dx+k^2\int\limits_0^1 \sin^2(\pi x)\,dx$

$=\frac{1}{2}-k+\frac{k^2}{2}=0 \Leftrightarrow k=1$.

Do đó $\int\limits_0^1 \left[f(x)-\sin(\pi x)\right]^2 dx=0 \Rightarrow f(x)=\sin(\pi x)$ (do $\left[f(x)-\sin(\pi x)\right]^2\geq 0$ $\forall x\in\mathbb{R}$).

Vậy $\int\limits_0^1 f(x)\,dx=\int\limits_0^1 \sin(\pi x)\,dx=\frac{2}{\pi}$.

Cách 2: Sử dụng BĐT Holder.

$$\boxed{\left[\int\limits_a^b f(x)g(x)\,dx\right]^2\leq \int\limits_a^b f^2(x)\,dx.\int\limits_a^b g^2(x)\,dx}.$$

Dấu “ = ” xảy ra $\Leftrightarrow f(x)=k.g(x)$, $\forall x\in[a;b]$.

Áp dụng vào bài ta có $\frac{1}{4}=\left[\int\limits_0^1 f(x)\sin(\pi x)\,dx\right]^2\leq \int\limits_0^1 f^2(x)\,dx.\int\limits_0^1 \sin^2(\pi x)\,dx=\frac{1}{4}$, suy ra $f(x)=k.\sin(\pi x)$, $k\in\mathbb{Z}$.

Mà $\int\limits_0^1 f(x)\sin(\pi x)\,\mathrm{d}x=\frac{1}{2}\Leftrightarrow k\int\limits_0^1 \sin^2(\pi x)\,\mathrm{d}x=\frac{1}{2}\Leftrightarrow k=1$ $\Rightarrow f(x)=\sin(\pi x)$

Vậy $\int\limits_0^1 f(x)\,\mathrm{d}x=\int\limits_0^1 \sin(\pi x)\,\mathrm{d}x=\frac{2}{\pi}$.

Câu 46: **[2D3-2.3-4]** **(SỞ GD-ĐT NINH BÌNH -2018)** Cho $I_n=\int \tan^n x\mathrm{d}x$ với $n\in\mathbb{N}$. Khi đó $I_0+I_1+2(I_2+I_3+...+I_8)+I_9+I_{10}$ bằng

A. $\sum\limits_{r=1}^{9}\frac{(\tan x)^r}{r}+C$. **B.** $\sum\limits_{r=1}^{9}\frac{(\tan x)^{r+1}}{r+1}+C$. **C.** $\sum\limits_{r=1}^{10}\frac{(\tan x)^r}{r}+C$. **D.** $\sum\limits_{r=1}^{10}\frac{(\tan x)^{r+1}}{r+1}+C$.

Lời giải

Chọn A.

$I_n=\int \tan^{n-2}x.\tan^2 x\mathrm{d}x=\int \tan^{n-2}x.\left(\frac{1}{\cos^2 x}-1\right)\mathrm{d}x=\int \tan^{n-2}x.(\tan x)'\,\mathrm{d}x-I_{n-2}$

$=\frac{\tan^{n-1}x}{n-1}-I_{n-2}+C$

$\Rightarrow I_n+I_{n-2}=\frac{\tan^{n-1}x}{n-1}+C$.

$I_0+I_1+2(I_2+I_3+...+I_8)+I_9+I_{10}=(I_{10}+I_8)+(I_9+I_7)+...+(I_3+I_1)+(I_2+I_0)$

$=\frac{\tan^9 x}{9}+\frac{\tan^8 x}{8}+....+\frac{\tan^2 x}{2}+\tan x+C=\sum\limits_{r=1}^{9}\frac{\tan^r x}{r}+C$.

Câu 34. **[2D3-2.3-4]** **(CHUYÊN HÀ TĨNH -LẦN 1-2018)** Cho $I=\int\limits_0^m (2x-1)\mathrm{e}^{2x}\mathrm{d}x$. Tập hợp tất cả các giá trị của tham số m để $I<m$ là khoảng $(a;b)$. Tính $P=a-3b$.

A. $P=-3$. **B.** $P=-2$. **C.** $P=-4$. **D.** $P=-1$.

Lời giải

Chọn A.

$I=\int\limits_0^m (2x-1)\mathrm{e}^{2x}\mathrm{d}x$

Đặt $\begin{cases}u=2x-1\\ \mathrm{d}v=\mathrm{e}^{2x}\mathrm{d}x\end{cases}\Rightarrow\begin{cases}\mathrm{d}u=2\mathrm{d}x\\ v=\frac{\mathrm{e}^{2x}}{2}\end{cases}$.

$I=\int\limits_0^m (2x-1)\mathrm{e}^{2x}\mathrm{d}x=\left.\frac{(2x-1)\mathrm{e}^{2x}}{2}\right|_0^m-\int\limits_0^m \mathrm{e}^{2x}\mathrm{d}x=\frac{(2m-1)\mathrm{e}^{2m}}{2}+\frac{1}{2}-\left.\frac{1}{2}\mathrm{e}^{2x}\right|_0^m=m\mathrm{e}^m-\mathrm{e}^{2m}+1$

$I<m\Leftrightarrow m\mathrm{e}^{2m}-\mathrm{e}^{2m}+1<m\Leftrightarrow(m-1)(\mathrm{e}^{2m}-1)<0\Leftrightarrow 0<m<1$.

Suy ra $a=0,b=1\Rightarrow a-3b=-3$.

Câu 42: **[2D3-2.3-4]** **(CHUYÊN ĐẠI HỌC VINH -LẦN 1-2018)** Cho hàm số $y=f(x)$ có đạo hàm liên tục trên đoạn $[0;1]$ và $f(0)+f(1)=0$. Biết $\int_0^1 f^2(x)\,\mathrm{d}x=\frac{1}{2}$, $\int_0^1 f'(x)\cos(\pi x)\,\mathrm{d}x=\frac{\pi}{2}$. Tính $\int_0^1 f(x)\,\mathrm{d}x$.

A. π. **B.** $\frac{1}{\pi}$. **C.** $\frac{2}{\pi}$. **D.** $\frac{3\pi}{2}$.

Lời giải

Chọn C.

Đặt $\begin{cases} u=\cos(\pi x) \\ \mathrm{d}v=f'(x)\,\mathrm{d}x \end{cases} \Leftrightarrow \begin{cases} \mathrm{d}u=-\pi\sin(\pi x)\,\mathrm{d}x \\ v=f(x) \end{cases}$. Khi đó:

$$\int_0^1 f'(x)\cos(\pi x)\,\mathrm{d}x=\cos(\pi x)f(x)\Big|_0^1+\pi\int_0^1 f(x)\sin(\pi x)\,\mathrm{d}x$$

$$=-\left(f(1)+f(0)\right)+\pi\int_0^1 f(x)\sin(\pi x)\,\mathrm{d}x=\pi\int_0^1 f(x)\sin(\pi x)\,\mathrm{d}x \Rightarrow \int_0^1 f(x)\sin(\pi x)\,\mathrm{d}x=\frac{1}{2}.$$

Cách 1: Ta có $\int_0^1\left[f(x)-k\sin(\pi x)\right]^2\mathrm{d}x=\int_0^1 f^2(x)\,\mathrm{d}x-2k\int_0^1 f(x)\sin(\pi x)\,\mathrm{d}x+k^2\int_0^1\sin^2(\pi x)\,\mathrm{d}x$

$$=\frac{1}{2}-k+\frac{k^2}{2}=0\Leftrightarrow k=1.$$

Do đó $\int_0^1\left[f(x)-\sin(\pi x)\right]^2\mathrm{d}x=0\Rightarrow f(x)=\sin(\pi x)$. Vậy $\int_0^1 f(x)\,\mathrm{d}x=\int_0^1\sin(\pi x)\,\mathrm{d}x=\frac{2}{\pi}$.

Cách 2: Sử dụng BĐT Holder.

$$\boxed{\left[\int_a^b f(x)g(x)\,\mathrm{d}x\right]^2\le\int_a^b f^2(x)\,\mathrm{d}x.\int_a^b g^2(x)\,\mathrm{d}x}.$$

Dấu "=" xảy ra $\Leftrightarrow f(x)=kg(x),\forall x\in[a;b]$.

Áp dụng vào bài ta có $\frac{1}{4}=\left[\int_0^1 f(x)\sin(\pi x)\,\mathrm{d}x\right]^2\le\int_0^1 f^2(x)\,\mathrm{d}x.\int_0^1\sin^2(\pi x)\,\mathrm{d}x=\frac{1}{4}$,

suy ra $f(x)=k\sin(\pi x)$.

Mà $\int_0^1 f(x)\sin(\pi x)\,\mathrm{d}x = \frac{1}{2} \Leftrightarrow k\int_0^1 \sin^2(\pi x)\,\mathrm{d}x = \frac{1}{2} \Leftrightarrow k = 1 \Rightarrow f(x) = \sin(\pi x)$.

Vậy $\int_0^1 f(x)\,\mathrm{d}x = \int_0^1 \sin(\pi x)\,\mathrm{d}x = \frac{2}{\pi}$.