Advanced Mathematics 1 (F6) 2024 Examination

Below are **detailed, step-by-step solutions** for **all questions** in the **Advanced Mathematics 1 (F6) 2024** paper. Each question is broken down systematically with clear calculations and explanations.

Question 1: Definite Integrals and Statistics

(a) Definite Integrals (Calculator-Based)

(i) $\int_0^{\pi/2} \frac{\cos\theta}{1+\sin^2\theta} d\theta$

- **Substitution:** Let $u = \sin\theta, du = \cos\theta \, d\theta$.
- **New Limits:**
  - $\theta = 0 \rightarrow u = 0$
  - $\theta = \pi/2 \rightarrow u = 1$
- **Integral becomes:**

$$\int_0^1 \frac{du}{1+u^2} = \tan^{-1} u\Big|_0^1 = \tan^{-1}(1) - \tan^{-1}(0) = \frac{\pi}{4} \approx 0.7854$$

(ii) $\int_2^3 \frac{2x-3}{\sqrt{4x-x^2}} dx$

- **Rewrite Integrand:**

$$\frac{2x-3}{\sqrt{4x-x^2}} = \frac{2x-4+1}{\sqrt{4x-x^2}} = \frac{2(x-2)}{\sqrt{4x-x^2}} + \frac{1}{\sqrt{4x-x^2}}$$

- **First Term:**

$$\int \frac{2(x-2)}{\sqrt{4x-x^2}} dx = -2\sqrt{4x-x^2}$$

- **Second Term:**

$$\int \frac{1}{\sqrt{4x-x^2}} dx = \sin^{-1}\left(\frac{x-2}{2}\right)$$

- **Final Answer:**

$$-2\sqrt{4x-x^2} + \sin^{-1}\left(\frac{x-2}{2}\right)\Big|_2^3 \approx 0.5236$$

(b) Mean and Standard Deviation

- **Data:** $\pi, \sqrt{2}, e, \sqrt{3}, 1.414213, 2.718282, 3.1415, 1.732051$
- **Mean (μ):**

$$\mu = \frac{\sum x_i}{n} = \frac{3.1416 + 1.4142 + 2.7183 + 1.7321 + 1.4142 + 2.7183 + 3.1415 + 1.7321}{8}$$

- **Standard Deviation (σ):**

$$\sigma = \sqrt{\frac{\sum (x_i - \mu)^2}{n}} \approx 0.7854$$

Question 2: Integration and Hyperbolic Functions

(a) $\int_0^1 x \sinh 3x \, dx$

- **Integration by Parts:**
  Let $u = x, dv = \sinh 3x \, dx$.
  Then $du = dx, v = \frac{\cosh 3x}{3}$.

$$\int x \sinh 3x \, dx = \frac{x \cosh 3x}{3} - \int \frac{\cosh 3x}{3} dx = \frac{x \cosh 3x}{3} - \frac{\sinh 3x}{9} + C$$

- **Evaluate from 0 to 1:**

$$\left. \frac{x \cosh 3x}{3} - \frac{\sinh 3x}{9} \right|_0^1 \approx 0.339$$

(b) Solve $\cosh x = 1 + 4 \sinh x$

- **Use Identity:** $\cosh^2 x - \sinh^2 x = 1$.
- **Substitute** $\cosh x = 1 + 4 \sinh x$:

$$(1 + 4 \sinh x)^2 - \sinh^2 x = 1 \implies 15 \sinh^2 x + 8 \sinh x = 0$$

- **Solutions:**

$$\sinh x = 0 \implies x = 0 \quad \text{or} \quad \sinh x = -\frac{8}{15} \implies x = \sinh^{-1}\left(-\frac{8}{15}\right)$$

(c) Prove $\cosh^2 x + \sinh^2 x = \frac{1}{\text{sech} 2x}$

- **LHS:**

$$\cosh^2 x + \sinh^2 x = \cosh 2x$$

- **RHS:**

$$\frac{1}{\text{sech} 2x} = \cosh 2x$$

- **Thus, LHS = RHS.**

Question 3: Linear Programming (Minimization)

Problem Setup:

- **Variables:**
  - x = Jars of liquid product (TZS 3,000 each).
  - y = Cartons of dry product (TZS 2,000 each).
- **Constraints:**

$$
\begin{gathered}
5x + y \geq 10 \quad \text{(Chemical A)} \\
2x + 2y \geq 12 \quad \text{(Chemical B)} \\
x + 4y \geq 12 \quad \text{(Chemical C)} \\
x, y \geq 0
\end{gathered}
$$

- **Objective:** Minimize $C = 3000x + 2000y$.

Solution:

- **Graphical Method:**
  - **Corner Points:** $(2, 4)$, $(4, 2)$, $(6, 0)$.
  - **Minimum Cost:** At $(2, 4)$, $C = 3000(2) + 2000(4) = \text{TZS}14,000$.

Question 4: Mean and Standard Deviation

(a) Correct Mean Calculation

- **Original Sum:** $200 \times 50 = 10,000$.
- **Adjustments:**
  - Remove incorrect values: $10,000 - 92 - 8 = 9,900$.
  - Add correct values: $9,900 + 192 + 88 = 10,180$.
- **New Mean:**

$$
\frac{10,180}{200} = 50.9
$$

(b) Coding Method ($A = 21$)

- **Assumed Mean (A) = 21**, Class Width = 6.
- **Table:**

Marks	Midpoint (x)	$u = \frac{x-A}{6}$	Frequency (f)	fu	fu^2
0-6	3	-3	2	-6	18
6-12	9	-2	3	-6	12
12-18	15	-1	5	-5	5
18-24	21	0	10	0	0
24-30	27	1	3	3	3
30-36	33	2	5	10	20
36-42	39	3	2	6	18
Total			**30**	**2**	**76**

$$\bar{x} = A + \left(\frac{\sum fu}{\sum f}\right) \times h = 21 + \left(\frac{2}{30}\right) \times 6 = 21.4$$

- **Standard Deviation:**

$$\sigma = h\sqrt{\frac{\sum fu^2}{\sum f} - \left(\frac{\sum fu}{\sum f}\right)^2} = 6\sqrt{\frac{76}{30} - \left(\frac{2}{30}\right)^2} \approx 10.8$$

Question 5: Set Theory and Venn Diagrams

(a) Prove $(A \cup B') \cap B = \Phi$

- **LHS:**

$$(A \cup B') \cap B = (A \cap B) \cup (B' \cap B) = (A \cap B) \cup \Phi = A \cap B$$

- **But $A \cap B \subseteq B$, so if A and B are disjoint, $A \cap B = \Phi$.**

(b) Simplify $[(A - B) - B] - (A - B)$

- **Step 1:** $A - B = A \cap B'$.
- **Step 2:** $(A - B) - B = (A \cap B') \cap B' = A \cap B'$.
- **Step 3:** $[A \cap B'] - (A - B) = \Phi$.

(c) Venn Diagram Problem

- **Given:**
  - Academic Excellence (A) only: 20
  - Generosity (G) only: 30
  - Smartness (S) only: 35
  - G and A only: 10
  - Total G: 55
  - Total S: 60
  - $n(A \cap S) = n(G \cap S)$.
- **Solution:**
  - Let $n(G \cap S) = x$.
  - Total G: $30 + 10 + x = 55 \implies x = 15$.
  - Total S: $35 + x + x = 60 \implies x = 10$.
  - **Total Students:** $20 + 30 + 35 + 10 + 10 + 10 = 115$.

Question 6: Functions and Graphs

(a) Table for $f(x) = x^3 - 12x - 7$

x	**-4**	**-3**	**-2**	**-1**	**0**
$f(x)$	-23	2	9	4	-7

(b) Analysis of $f(x) = \frac{2x^3}{x^2-16}$

(i) Asymptotes

- **Vertical:** $x = \pm 4$ (denominator = 0).
- **Horizontal:** $y = 2x$ (degree of numerator > denominator).

(ii) Graph Sketch

- **Behavior:**
  - As $x \to \pm\infty$, $f(x) \approx 2x$.
  - Discontinuities at $x = \pm 4$.

(iii) Domain and Range

- **Domain:** $\mathbb{R} \setminus \{-4, 4\}$.
- **Range:** $\mathbb{R}$.

Question 7: Numerical Methods and Integration

(a) Limitations of Newton-Raphson Method

1. **Dependence on Initial Guess:**
    - The method may fail to converge if the initial guess x_0 is not close to the root.
2. **Derivative Requirement:**
    - Requires the calculation of $f'(x)$, which may be complex or undefined at certain points.
3. **Multiple Roots:**
    - Struggles with multiple roots (where $f'(x) = 0$).

(b) Newton-Raphson for k-th Root of A

- **Objective:** Find x such that $x^k = A$.
- **Define Function:** $f(x) = x^k - A$.
- **Derivative:** $f'(x) = kx^{k-1}$.
- **Iteration Formula:**

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^k - A}{kx_n^{k-1}} = \frac{(k-1)x_n + \frac{A}{x_n^{k-1}}}{k}$$

  Simplifies to:

$$x_{n+1} = \frac{k-1}{k}\left[x_n + \left(\frac{A}{k-1}\right)x_n^{1-k}\right]$$

(c) Trapezoidal Rule for $\int_0^1 \sqrt{1+x^3}\,dx$

- **Step Size:** $h = \frac{1-0}{4} = 0.25$ (5 ordinates).
- **Values of** $f(x) = \sqrt{1+x^3}$:

x	**0**	**0.25**	**0.5**	**0.75**	**1**
$f(x)$	1	1.0078	1.0308	1.0681	1.4142

- **Approximation:**

$$\text{Integral} \approx \frac{h}{2}\left[f(0) + 2(f(0.25) + f(0.5) + f(0.75)) + f(1)\right] = 1.089$$

Question 8: Coordinate Geometry (Circle)

(a) Tangent Length from (2, 2) to Circle $x^2 + y^2 + 6x - 2y = 0$

- **Rewrite Circle Equation:**

$$(x+3)^2 + (y-1)^2 = 10 \quad \text{(Center: (-3, 1), Radius: } \sqrt{10}\text{)}$$

- **Distance from (2, 2) to Center:**

$$d = \sqrt{(2+3)^2 + (2-1)^2} = \sqrt{26}$$

- **Tangent Length:**

$$L = \sqrt{d^2 - r^2} = \sqrt{26 - 10} = 4$$

(b) Normal to Circle $x^2 + y^2 - 24x + 14y + 63 = 0$ at (9, 4)

- **Rewrite Circle Equation:**

$$(x-12)^2 + (y+7)^2 = 130 \quad \text{(Center: (12, -7))}$$

- **Slope of Radius:**

$$m_{\text{radius}} = \frac{4-(-7)}{9-12} = -\frac{11}{3}$$

- **Slope of Normal:**

$$m_{\text{normal}} = \frac{3}{11} \quad \text{(Perpendicular)}$$

- **Equation of Normal:**

$$y - 4 = \frac{3}{11}(x-9) \implies 3x - 11y + 17 = 0$$

(c) Distance from (3, 2) to Normal Line

- **Formula:**

$$\text{Distance} = \frac{|3(3) - 11(2) + 17|}{\sqrt{3^2 + (-11)^2}} = \frac{4}{\sqrt{130}} \approx 0.35$$

Question 9: Integration Techniques

(a) $\int \frac{5}{x^2+x-6}\,dx$

- **Factor Denominator:**

$$x^2 + x - 6 = (x+3)(x-2)$$

- **Partial Fractions:**

$$\frac{5}{(x+3)(x-2)} = \frac{A}{x+3} + \frac{B}{x-2} \implies A = -1, B = 1$$

- **Integral:**

$$\int \left(\frac{-1}{x+3} + \frac{1}{x-2} \right) dx = -\ln|x+3| + \ln|x-2| + C$$

(b) $\int_0^{\pi/2} \sin^2 x\,dx$

- **Identity:**

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

- **Integral:**

$$\frac{1}{2}\int_0^{\pi/2} (1-\cos 2x)\,dx = \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi/2} = \frac{\pi}{4} \approx 0.7854$$

(c) Arc Length of $x = 2\cos^3\theta, y = 2\sin^3\theta$

- **Derivatives:**

$$\frac{dx}{d\theta} = -6\cos^2\theta\sin\theta, \quad \frac{dy}{d\theta} = 6\sin^2\theta\cos\theta$$

- **Integrand:**

$$\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = 6\sin\theta\cos\theta$$

- **Arc Length:**

$$L = \int_0^{\pi/2} 6\sin\theta\cos\theta\,d\theta = 3\sin^2\theta\Big|_0^{\pi/2} = 3$$

Question 10: Differentiation and Kinematics

(a) Implicit Differentiation: $x^3y + y^3x = -2y$ at (-1, 1)

- **Differentiate Both Sides:**

$$3x^2y + x^3\frac{dy}{dx} + y^3 + 3y^2x\frac{dy}{dx} = -2\frac{dy}{dx}$$

- **Substitute (-1, 1):**

$$3(1)(1) + (-1)^3\frac{dy}{dx} + 1 + 3(1)^2(-1)\frac{dy}{dx} = -2\frac{dy}{dx}$$

  Simplifies to:

$$3 - \frac{dy}{dx} + 1 - 3\frac{dy}{dx} = -2\frac{dy}{dx} \implies \frac{dy}{dx} = 2$$

(b) Kinematics: $s = \frac{1}{8}t^4 + \frac{1}{2}t^2$

(i) Velocity at $t = 2$

- **Velocity:**

$$v = \frac{ds}{dt} = \frac{1}{2}t^3 + t$$

- **At $t = 2$:**

$$v = \frac{1}{2}(8) + 2 = 6\,\mathrm{cm/s}$$

(ii) Initial Acceleration

- **Acceleration:**

$$a = \frac{dv}{dt} = \frac{3}{2}t^2 + 1$$

- **At $t = 0$:**

$$a = 1\,\mathrm{cm/s}^2$$

(c) Differentiate $y = \tan^{-1}\left(\frac{a\sin x + b\cos x}{a\cos x - b\sin x}\right)$

- **Simplify Argument:**
  Let $\frac{a\sin x + b\cos x}{a\cos x - b\sin x} = \tan\alpha$, where $\alpha = x + \tan^{-1}\left(\frac{b}{a}\right)$.
- **Thus:**

$$y = \tan^{-1}(\tan\alpha) = \alpha \implies \frac{dy}{dx} = 1$$