7.1 Integrals as Net Change

Tuesday, February 13, 2018 3:35 PM

Ex 1

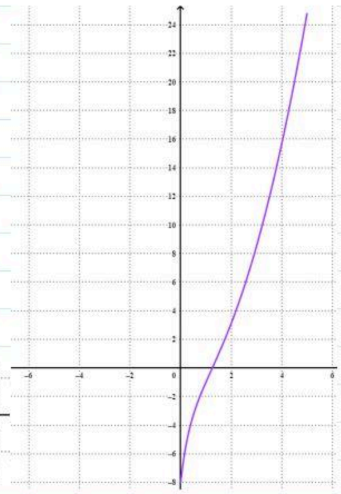
change in position

Given $\frac{ds}{dt} = v(t) = t^2 - \frac{8}{(t+1)^2}$ is the velocity of a particle moving along the $\underbrace{\text{S-axis}}_{\text{position}}$ for $0 \le t \le 5$.

The initial position of the particle is $s(0) = 9$. What is the particles position at

a) $t = 1$

b) $t = 5$



a) The position at $t=1$ is the initial position $s(0)$ plus displacement Δs.

$$\Delta S = \text{Displacement} = \underset{\substack{(\text{velocity}) \\ \uparrow \\ \text{speed}}}{\text{rate}} \times \text{time}$$

But! since velocity varies we use integration to find displacement

$$\Delta s = \int_0^1 t^2 - 8(t+1)^{-2}\,dt = \left[\frac{t^3}{3} - \frac{8(t+1)^{-1}}{-1}\right]\Big|_0^1 = \left[\frac{t^3}{3} + \frac{8}{(t+1)}\right]\Big|_0^1 = \left(\frac{1}{3} + \frac{8}{1+1}\right) - \left(\frac{0^3}{3} + \frac{8}{(0+1)}\right)$$

$\int (t+1)^{-2}dt$

let $u = t+1$

$du = dt$

$\int u^{-2}du$

$= \frac{u^{-1}}{-1} = -(t+2)^{-1}$

$$= \left(\frac{1}{3} + \frac{8}{2}\right) - (8)$$
$$= \frac{2}{6} + \frac{24}{6} - \frac{48}{6}$$
$$= \frac{26}{6} - \frac{48}{6} = \frac{-22}{6} = -\frac{11}{3}$$

net distance traveled in the first second

* Now, we need to add it to the starting position

Final Position: $s(0) + \Delta s = 9 + \left(-\frac{11}{3}\right)$

$= \frac{27}{3} + \left(-\frac{11}{3}\right)$

$= \frac{16}{3}$

b) $s(0) + \Delta s = 9 + \int_0^5 \left(t^2 - \frac{8}{(t+1)^2}\right) dt$

$$= 9 + \left[\frac{t^3}{3} + \frac{8}{t+1}\right]_0^5$$

$$= 9 + \left[\left(\frac{5^3}{3} + \frac{8}{5+1}\right) - \left(\frac{0^3}{3} + \frac{8}{0+1}\right)\right]$$

$$= 9 + \left[\frac{125}{3} + \frac{8}{6} - \frac{8}{1}\right]$$

$$= 9 + \left[\frac{125}{3} + \frac{4}{3} - \frac{24}{3}\right]$$

$$= 9 + \left[\frac{105}{3}\right]$$

$$= 9 + 35 = 44$$

Try: $v(t) = 5\cos(t) \quad 0 \leq t \leq 2\pi$

Determine the particles final position on the given interval if $s(0) = 3$

Position $\rightarrow$ $s(0) + \Delta s$

$$= 3 + \int_0^{2\pi} 5\cos(t)\,dt$$

$$= 3 + 5\left[\sin(t)\right]\Big|_0^{2\pi}$$

$$= 3 + 5(\sin(2\pi) - \sin(0))$$

$$= 3 + 5(0 - 0)$$

$$= 3 + 5(0)$$

$$= 3 \checkmark$$

Ex 2

Determine the total distance traveled by the particle in Ex 1

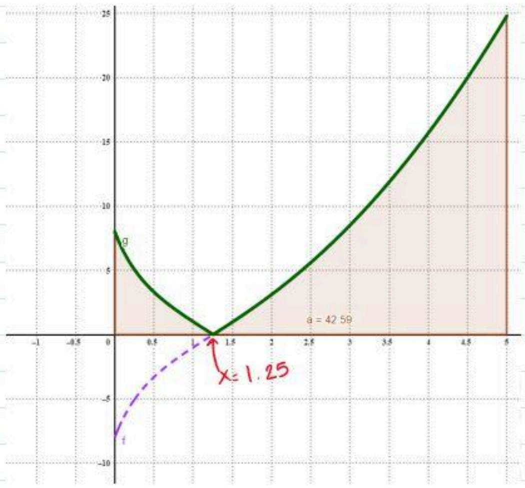
"not my ending position"
"Its how much ground I covered"

Total Distance: $\int_a^b |v(t)|\,dt$

starting point does not matter since we are not looking for ending position.

How to do on calculator:

$$\int_0^5 \left|t^2 - \frac{8}{(t+1)^2}\right| dt = 42.586$$



To do by hand:

$$\underbrace{\left|\int_0^{1.25} v(t)\,dt\right| + \left|\int_{1.25}^{5} v(t)\,dt\right|}_{\text{not impossible, but tedious}}$$

Try: $v(t) = 5\cos(t)$ $\quad 0 \le t \le 2\pi$

Determine the total distance traveled over the interval

* (starting point doesn't matter) *

$$\int_0^{2\pi} |5\cos(t)|\,dt = \boxed{20}$$

Summary: Integrating velocity gives displacement (net area between velocity curve & the time axis)

Area above − Area below

Integrating the absolute value of velocity gives total distance traveled (Total area between the velocity curve and the time axis)

Ex 3

A car moving with an initial velocity of 5mph accelerates at the rate of $a(t) = 2.4t$ mph per second for 8 seconds

a) How fast is the car traveling at the end of those 8 seconds?

b) How far did the car travel during those 8 seconds?

a) Formula for change in distance:

velocity × change in time

Formula for change in velocity:

accel × change in time

change in velocity → $dv = a(t)dt$ ← change in time

acceleration

$$\Delta v = \int a(t)dt$$

Initial velocity $= v_i = v(0) = 5$

velocity at time $t=0$

Final velocity $= v_i + \Delta v$

initial velocity

change in velocity

$$V_f = V_i + \Delta V$$
$$= 5 + \int_a^b a(t)dt$$
$$= 5 + \int_0^8 2.4t\,dt$$
$$= 5 + \left[\frac{2.4}{2}t^2\Big|_0^8\right]$$
$$= 5 + 1.2(8^2 - 0^2)$$
$$= 5 + 1.2(64)$$
$$= 5 + 76.8$$
$$= 81.8 \text{ mph}$$

b) $v(t) = V(0) + \Delta V$

Distance is given by the integral of velocity

Total Distance is given by the absolute value of the integral of velocity

$$\int_0^8 |v(t)|\,dt \quad , \quad v(t) = V(0) + \Delta V$$
$$v(t) = 5 + \int_0^t 2.4\,u\,du$$

u is a dummy variable that could be any value of t any value of t

$$v(t) = 5 + \int_0^t 2.4u\,du$$

$$= 5 + 2.4 \frac{u^2}{2}\Big|_0^t$$

$$= 5 + 1.2u^2\Big|_0^t$$

$$= 5 + 1.2t^2 \text{ mph}$$

$$= 5 + 1.2t^2 \; \frac{m}{h}$$ ← t is given in seconds, so we need to convert hours to seconds

$$= (5 + 1.2t^2)\frac{mi}{h} \cdot \frac{1\,hr}{3600\,sec}$$

So $v(t) = \frac{1}{3600}(5 + 1.2t^2)$ mi

$\int v(t)\,dt = s(t)$ ← position

$$\frac{1}{3600}\int_0^8 5 + 1.2t^2\,dt$$

$$\frac{1}{3600}\int_0^8 \underbrace{|5 + 1.2t^2|}_{\text{only positive}}\,dt$$

$$\frac{1}{3600}\left[5t + \frac{1.2}{3}t^3\Big|_0^8\right]$$

$$\frac{1}{3600}\left[5(8) + 0.4(8)^3\right]$$

$$\frac{1}{3600}\left[40 + 0.4(512)\right]$$

$$\frac{1}{3600}\left[244.8\right]$$

$$\boxed{= 0.068 \text{ miles}}$$

Try: A car accelerates from rest at $a(t) = 1 + 3\sqrt{t}$ mph/s for 9 seconds

a) What is its velocity after 9 seconds?
b) How far does it travel in those 9 seconds? (In feet)

a) $V_f = V_0 + \Delta v$

$$V_f = 0 + \int_0^9 (1 + 3\sqrt{t})\,dt$$

$$= \int_0^9 1 + 3t^{1/2}\,dt$$

$$= t + 3\left(\tfrac{2}{3}\right)t^{3/2}\Big|_0^9$$

$$= \left[t + 2t^{3/2}\right]\Big|_0^9$$

$$= 9 + 2(9)^{3/2}$$

$$= 9 + 2(27)$$

$$= 9 + 54$$

$$= 63 \text{ mph}$$

b) How far does it travel in 9 seconds?

$$\int_0^9 |v(t)|dt$$

$v(t)$ is in mph,
but t is in seconds.
so need to convert from
hours to seconds

1 hr = 3600 seconds
so
$\frac{1}{1hr} = \frac{1}{3600 \text{ sec}}$

$$v(t) = \int_0^t \underbrace{a(u)}_{\text{acceleration}} du$$

$$v(t) = \int_0^t 1 + 3\sqrt{u}\, du$$

$$v(t) = u + 2u^{3/2} \Big|_0^t$$

$$v(t) = t + 2t^{3/2}$$

$$s(t) = \int v(t)dt$$

$$s(t) = \frac{1}{3600}\int_0^9 |v(t)|dt$$

$$= \frac{1}{3600}\int_0^9 \underbrace{|t + 2t^{3/2}|}_{\text{always positive for } 0 \le t \le 9} dt$$

$$= \frac{1}{3600}\left(\frac{t^2}{2} + 2 \cdot \frac{2}{5} t^{5/2} \Big|_0^9 \right)$$

$$= \frac{1}{3600}\left(\frac{9^2}{2} + \frac{4}{5}(9)^{5/2} \right)$$

$$= \frac{1}{3600}\left(\frac{81}{2} + \frac{4}{5}(9)^{5/2} \right)$$

$$= 0.0652 \text{ mi}$$

we want it in feet

$$0.0652 \text{ mi} \cdot \frac{5280 \text{ ft}}{1 \text{ mi}}$$

$$\boxed{= 344.519 \text{ ft}}$$

"whenever you want to find cumulative affect of varying rate of change, integrate it"

Ex 4

Over 5 years, the rate of potato consumption was $C(t) = 2.2 + 1.1^t$ million bushels per year. How many bushels were consumed between the second and fourth year?

$$\underbrace{\int_2^4 2.2 + 1.1^t \, dt}$$

If you can use calculator, plug it in! $\longrightarrow$ 7.066 million bushels

... If not...

$$2.2t + \frac{1.1^t}{\ln(1.1)} \Bigg|_2^4$$

$$\left(2.2(4) + \frac{1.1^4}{\ln(1.1)}\right) - \left(2.2(2) + \frac{1.1^2}{\ln(1.1)}\right)$$

$= 7.066$ million bushels

Try: pg. 386 #21

U.S. Oil Consumption The rate of consumption of oil in the United States during the 1980s (in billions of barrels per year) is modeled by the function $C = 27.08 \cdot e^{t/25}$, where t is the number of years after January 1, 1980. Find the total consumption of oil in the United States from January 1, 1980 to January 1, 1990.

$t=10$

$t=0$

$$\int_0^{10} 27.08 e^{\frac{t}{25}} dt$$

$$27.08 \int_0^{10} e^{\frac{t}{25}} dt$$

$u = \frac{t}{25}$ $\quad$ $t = 0$ $\quad$ $t = 10$

$u = \frac{1}{25}t$ $\quad$ $u = \frac{0}{25} = 0$ $\quad$ $u = \frac{10}{25} = \frac{2}{5}$

$du = \frac{1}{25}dt$

$25du = dt$

$$27.08(25) \int_0^{\frac{2}{5}} e^u du$$

$$(27.08)(25) \left[e^u\right]\Big|_0^{2/5}$$

$$(27.08)(25)\left(e^{2/5} - e^0\right)$$

$$\boxed{= 332.965 \text{ barrels}}$$

Ex 5

The data shows the amount of gallons pumped over an hour long period.

Time (min)	Rate (gal/min)
0	58
5	60
10	65
15	64
20	58
25	57
30	55
35	55
40	59
45	60
50	60
55	63
60	63

Since we don't have a formula we need another way to represent "area under the curve", – to integrate –

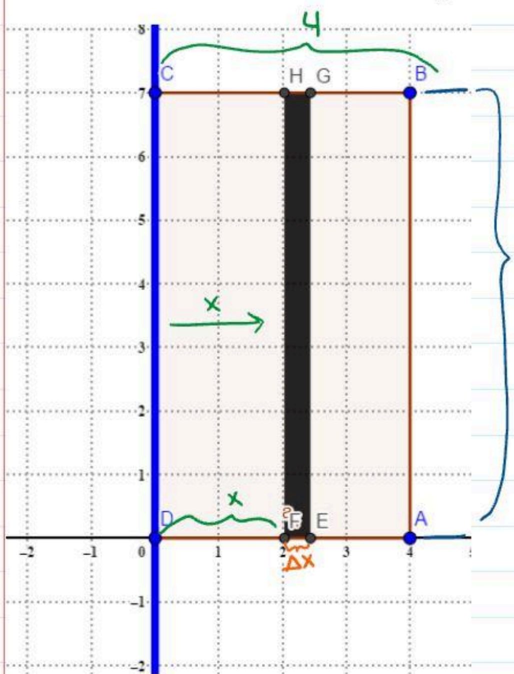
$$\text{Gallons pumped} = \int_0^{60} R(t)dt$$

The best way is to use the trapezoid method. $h = 5$

$$\int_0^{60} R(t)dt = \frac{5}{2}\left[58 + 2(60) + 2(65) + 2(64) + \ldots + 2(63) + 63\right]$$

$$= 3582.5 \text{ gallons}$$

Ex 6 Population Density



A city located beside a river has a rectangular boundary as shown. The population density of the city at any point along a thin strip x miles from the rivers edge is $f(x)$ persons per square mile.

a) what is the area of the thin strip x miles from the rivers edge?

$7 \cdot \Delta x$

b) what is the population of the thin strip?

Population density $= \dfrac{\text{population}}{\text{Area}} = f(x)$

so the population $\rightarrow$ $f(x) \cdot \text{Area}$

$f(x) \cdot 7\Delta x$

c) what is the total population of the city:

$$\text{Total Population} = \int_0^4 f(x) \cdot 7\,dx = 7\int_0^4 f(x)\,dx$$

integrating over the x-values