D´Alembert's Principle and equivalent or similar principles

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Abstract

In order to be able to describe dynamics under constraints in classical mechanics, Newton's axioms must be supplemented by another axiom, namely d'Alembert's principle. This principle states that constraint forces do no work, which is equivalent to the fact that constraint forces are perpendicular to the manifold spanned by the constraint conditions.

In economic systems, constraints such as accounting identities play a very important role. Recently, to describe the dynamics of economic systems under constraints in analogy to the description of the dynamics of physical systems under constraints, the so-called General Constraint Dynamic (GCD) models have been introduced. In these models, the constraint forces are usually also defined using d'Alembert's principle. Unlike in physics, however, in economics the validity of d'Alembert's principle cannot be tested directly by experiments. In principle, other constraining forces can also occur in economics. All the more important is a deeper understanding of d'Alembert's principle. This is exactly the goal of this paper. To achieve this, similar principles are formulated and discussed under which conditions they are equivalent to d'Alembert's principle.

Keywords: d'Alembert's principle, Lagrange principle, Gauss principle, Hamilton principle, constraint dynamics, GCD-models

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1 Introduction

In <1>,<2> we introduced a new method for modelling economic models that is a natural extension of the General Equilibrium Theory in economics. It is based on the standard method in physics for modelling a dynamics under constraints. We therefore call models of this type "General Constrained Dynamic (GCD)" models. To describe dynamic systems with constraints, in physics Newton's equations must be extended by d'Alembert's principle (“No virtual work”)<3>. It states that constraint forces do no work, or equivalently, that the constraint forces are perpendicular to the manifold spanned by the constraints. This principle is an additional axiom to Newton's axioms. As to now d'Alembert's principle is consistent with experimental results in the same way as Newtonian axioms are.

Generally, d'Alembert's principle is also used in the GCD models to model the constraint forces. Whether the economy actually behaves in this way or whether other constraining forces can also occur in the economy can only be decided by comparing the GCD models with economic reality. Of course, this is not so easy to do in practice because, unlike physics, no simple experiments can be performed in economics.

For this very reason, it is necessary to obtain a deeper understanding of d'Alembert's principle. This is exactly the goal of this paper. To achieve this, similar principles are formulated and discussed under which conditions they are equivalent to d'Alembert's principle. In any case, it is important to distinguish between holonomic and non-holonomic constraints.

In physics text books usually, the notation of virtual displacements is used, whereby virtual displacements are just infinitesimal displacements in a tangent plane. As for non-physicists (and also for some mathematicians 😊) the notation with virtual displacements is very uncomfortable and needs getting used to we express the vectors of the tangent plane explicitly with the tangent condition. This description in principle is inspired by the differential geometric description but is much simpler and avoids a deep understanding of differential geometric methods and notations (which are also for some mathematicians very uncomfortable and needs getting used to 😊).

2 General notations and definitions

$Let\ M: \mathbf{R}^n \to \mathbf{R}$

$x \mapsto M(x)$

(1) $\mathbf{M} := \{x | M(x) = 0\}$.

(2) $Let\ x^0 \in \mathbf{M}$, $i.e.\ M(x^0) = 0$

(3) $Taylor\ expansion\ of\ M\ at\ x^0:\qquad M(x) = T_0M(x) + T_1M(x) + T_2M(x) + \ldots$

$T_0M(x) := M(x^0) = 0$

$$T_1M(x) := \left(\left.\frac{\partial M(x)}{\partial x_1}\right|_{x=x^0}, \ldots, \left.\frac{\partial M(x)}{\partial x_n}\right|_{x=x^0} \right) . \begin{pmatrix} (x_1 - x_1^0) \\ \ldots \\ (x_n - x_n^0) \end{pmatrix} = \left.\frac{\partial M(x)}{\partial x}\right|_{x=x^0} .(x - x^0) =$$

$$= \sum_{i=1}^{n} \frac{\partial M(x^0)}{\partial x_i}.(x_i - x_i^0) = \frac{\partial M(x^0)}{\partial x_i}.(x_i - x_i^0) \quad (with\ summing\ convention) =$$

$$= grad_x M(x^0).(x - x^0) = \frac{\partial M(x^0)}{\partial x}.(x - x^0) =$$

$$= \left\langle grad_x M(x^0), (x - x^0) \right\rangle = \left\langle \frac{\partial M(x^0)}{\partial x}, (x - x^0) \right\rangle \qquad notations\ we\ use\ are\ red$$

$$T_2M(x) := \frac{1}{2}\left((x_1 - x_1^0), \ldots, (x_n - x_n^0)\right) \begin{pmatrix} \left.\frac{\partial^2 M(x)}{\partial x_1^2}\right|_{x=x^0} & \ldots & \left.\frac{\partial^2 M(x)}{\partial x_1 \partial x_n}\right|_{x=x^0} \\ \ldots & \ldots & \ldots \\ \left.\frac{\partial^2 M(x)}{\partial x_n \partial x_1}\right|_{x=x^0} & \ldots & \left.\frac{\partial^2 M(x)}{\partial x_n^2}\right|_{x=x^0} \end{pmatrix} . \begin{pmatrix} (x_1 - x_1^0) \\ \ldots \\ (x_n - x_n^0) \end{pmatrix} =$$

$$= \frac{1}{2}(x - x^0)^T \left.\frac{\partial^2 M(x)}{\partial x^2}\right|_{x=x^0} .(x - x^0) =$$

$$= \frac{1}{2}(x - x^0)^T \frac{\partial^2 M(x^0)}{\partial x^2}.(x - x^0) = \frac{1}{2}(x - x^0)^T Hess_x M(x^0).(x - x^0) = \frac{\partial M(x^0)}{\partial x}.(x - x^0)$$

(4) $T_1\mathbf{M} := \{q | T_1M(q) = 0\}$ *denotes the "tangent plane" of the manifold* $\mathbf{M}$ *at* x^0,

i.e. $T_1\mathbf{M} := \left\{q \middle| \left\langle grad_x M(x^0), q + x^0 - x^0 \right\rangle = 0\right\} = \left\{q \middle| \left\langle grad_x M(x^0), q \right\rangle = 0\right\}$

with notation $: grad_x M = \left(\left.\frac{\partial M(x)}{\partial x_1}\right|_{x=x^0}, \ldots, \left.\frac{\partial M(x)}{\partial x_n}\right|_{x=x^0} \right)$

$q \in T_1\mathbf{M}$ *is called "tangent vector" of* $\mathbf{M}$ *at* x^0 *(lies in the tangent plane)*

(5) $P_1\mathbf{M} := \{p | \langle p, q \rangle = 0\ for\ all\ q \in T_1\mathbf{M}\}$ *denotes the "cotangent plane",*

i.e. the manifold of all p *which are perpendicular to the tangent plane* $T_1\mathbf{M}$:

$p \in P_1\mathbf{M}$ *is called "cotangent vector" (or perpendicular vector) of* $\mathbf{M}$ *at* x^0.

(6) *Remark : for all* $x^0 \in \mathbf{M}\ \ \mathbf{R}^l = T_1\mathbf{M} \times P_1\mathbf{M}$

Special notation

Let $s=(s_1,...,s_n),\ v=(v_1,...,v_n)=\dot{s}=(\dot{s}_1,...,\dot{s}_n)$ (all being time dependent).

$s^0 := s(0)$

$v^0 := v(0)=\dot{s}(0)$

Let

C^k *the constraint* $C^k(s,v)=0$ *for all* $k=1,...,K$

Assume $C:=\left(C^1,...,C^K\right)$ *the set of all constraints to be regular, i.e.*

$$rank\frac{\partial C(s,v)}{\partial(s,v)}=K$$

To simplify the notation, we assume without loss of generality

$C^k(s^0,v^0)=0$ *for all* k

which always can be achieved by an affine coordinate transformation.
Let denote

C_v^k *the constraint* $C_v^k(s^0,v)=0$

$C_v:=\left\{C_v^k,\ k=1,...,K\right\}$ *set of all constraints* C_v^k

C_s^k *the constraint* $C_s^k(s,v^0)=0$

$C_s:=\left\{C_s^k,\ k=1,...,K\right\}$ *set of all constraints* C_s^k

Definition

$C^k(s,v)=0$ *is holonomic*: $\Leftrightarrow \frac{\partial C^k(s,v)}{\partial v_i}=0$ *for all* $i \Leftrightarrow C^k(s,v)$ *does not depend on* v

$\Leftrightarrow C^k$ *depends only on* $s \Leftrightarrow C^k=C^k(s)$

$C^k(s,v)=0$ *is nonholonomic* $:\Leftrightarrow$ *there exist at least one* i *with* $\frac{\partial C^k(s,v)}{\partial v_i}\neq 0$

$C^k(s,v)=0$ *is integrable*: $\Leftrightarrow$ *there exist* $B(s,v)=0$ *such that* $C^k(s,v)=\frac{d}{dt}B(s,v)=\frac{\partial B}{\partial s}.v+\frac{\partial B}{\partial v}.\dot{v}$

Remark (holonomic constraint as nonholonomic constraint)

$Let\ s(0)=s^0\ and\ C(s^0)=0$

(1) *A holonomic constraint* $C^k(s(t))=0$ *can always be transformed to an equivalent nonholonomic constraint* $\dot{C}^k(s(t),\dot{s}(t))=0$

(*which is a differential equation with initial condition* $s(0)=s^0$) *whereby*

$$\dot{C}^k(s(t),\dot{s}(t)):=\frac{d}{dt}C^k(s(t))$$

that means

$$C^k(s(t))=0 \quad \Leftrightarrow \quad \dot{C}^k(s(t),\dot{s}(t))=0$$

(2) $grad_v\dot{C}^k = grad_s C^k$

Proof

(1) $C^k=0 \Rightarrow \dot{C}^k=0$ *trivial*

$\dot{C}^k=0 \Rightarrow C^k(s(t))=a$ *by integration*

$\Rightarrow a=C^k(s(0))=C^k(s^0)=0$

$\Rightarrow C^k(s(t))=0$

(2) $$grad_v\dot{C}^k = grad_v\frac{d}{dt}C^k=\frac{\partial}{\partial v}\frac{dC^k(s)}{dt}=\frac{\partial}{\partial v}\frac{\partial C^k(s)}{\partial s}v=\frac{\partial C^k(s)}{\partial s}=grad_s C^k$$

Therefore, for sake of simplicity we use always the following notations.

<u>Notation</u>

$$s\,|\,v:=\begin{cases} use\ s & if\ C\ is\ holonomic \\ use\ v & if\ C\ is\ nonholonomic \end{cases}$$

$$grad_{s|v}C^k:=\begin{cases} grad_v C^k=\left(\left.\frac{\partial C^k(s^0,v)}{\partial v_1}\right|_{v=v^0},\dots,\left.\frac{\partial C^k(s^0,v)}{\partial v_n}\right|_{v=v^0}\right) & if\ C^k\ is\ nonholonomic \\ grad_v\dot{C}^k=grad_s C^k=\left(\left.\frac{\partial C^k(s)}{\partial s_1}\right|_{s=s^0},\dots,\left.\frac{\partial C^k(s)}{\partial s_n}\right|_{s=s^0}\right) & if\ C^k\ is\ holonomic \end{cases}$$

<u>Definition</u>

$$\mathbf{C}^k:=\begin{cases} \left\{x\in\mathbf{R}^n \,\middle|\, C^k(x)=0\right\} & if\ C^k\ is\ holonomic \\ \left\{x\in\mathbf{R}^n \,\middle|\, C^k(s^0,x)=0\right\} & if\ C^k\ is\ nonholonomic \end{cases}$$

$$T\mathbf{C}^k = \begin{cases} \left\{ x \in \mathbf{R}^n \middle| grad_x C^k(x) = 0 \right\} & if\ C^k\ is\ holonomic \\ \left\{ x \in \mathbf{R}^n \middle| grad_x C^k(s^0, x) = 0 \right\} & if\ C^k\ is\ nonholonomic \end{cases}$$

Note: As in the following we do not use $T_0\mathbf{C}$ *and* $T_2\mathbf{C}$ *we drop the index* 1 *and write* $T\mathbf{C}$ *instead of* $T_1\mathbf{C}$

Remark

(1) $T\mathbf{C}^k = \left\{ v \in \mathbf{R}^n \middle| \left\langle grad_{s|v} C^k, v \right\rangle = 0 \right\}$

(2) $Note: \dim T\mathbf{C}^k = n-1$

Proof

$$T\mathbf{C}^k = \begin{cases} \left\{ x \in \mathbf{R}^n \middle| grad_x C^k(x) = 0 \right\} & \text{if } C^k \text{ is holonomic} \\ \left\{ x \in \mathbf{R}^n \middle| grad_x C^k(s^0, x) = 0 \right\} & \text{if } C^k \text{ is nonholonomic} \end{cases}$$

$$= \begin{cases} \left\{ s \in \mathbf{R}^n \middle| grad_s C^k(s) = 0 \right\} & \text{if } C^k \text{ is holonomic} \\ \left\{ v \in \mathbf{R}^n \middle| grad_v C^k(s^0, v) = 0 \right\} & \text{if } C^k \text{ is nonholonomic} \end{cases}$$

$$= \left\{ v \in \mathbf{R}^n \middle| \left\langle grad_{s|v} C^k, v \right\rangle = 0 \right\}$$

Definition

$T\mathbf{C} := \bigcap_{k=1}^{K} T\mathbf{C}^k$ "all – virtual displacement manifold" at v^0 or "all – tangent manifold" at v^0

Note: $\dim T\mathbf{C} = n - K$

$P\mathbf{C}^k := \left\{ \tilde{v} \in \mathbf{R}^n \middle| \left\langle \tilde{v}, v \right\rangle = 0 \text{ for all } v \in T_1\mathbf{C}_v^k \right\}$ "cotangent (or perpendicular) manifold" at v^0

the manifold of all $\tilde{v}$ *which are perpendicular to all vectors* $v \in T\mathbf{C}^k$

$Note: P\mathbf{C}^k = \left\{ \tilde{v} \in \mathbf{R}^n \middle| \left\langle \tilde{v}, v \right\rangle = 0 \text{ for all } v \text{ with } \left\langle grad_{s|v} C^k, v \right\rangle = 0 \right\} = \left\{ \tilde{v} \in \mathbf{R}^n \middle| \tilde{v} = \lambda\, grad_{s|v} C^k \right\}$

$Note: \dim P\mathbf{C}_v^k = 1$

$P\mathbf{C} := \prod_{k=1}^{K} P\mathbf{C}^k = P\mathbf{C}^1 \times \ldots \times P\mathbf{C}^K$ "all cotangent (or all – perpendicular) manifold" at v^0

$Note: \text{if } \left\{ grad_{s|v} C^1, \ldots, grad_{s|v} C^K \right\} \text{ is a base of } P\mathbf{C} \quad (\Leftrightarrow \operatorname{rank} \frac{\partial C}{\partial(s,v)} = K) \text{ then}$

$$P\mathbf{C} = \left\{ \tilde{v} \in \mathbf{R}^n \middle| \tilde{v} = \sum_{k=1}^{K} \lambda^k grad_{s|v} C^k \right\} \text{ and}$$

$$\dim P\mathbf{C} = K$$

$Remark: \mathbf{R}^n = P\mathbf{C} \times T\mathbf{C} = \left(\prod_{k=1}^{K} P\mathbf{C}^k \right) \times \left(\bigcap_{k=1}^{K} T\mathbf{C}^k \right)$

3 Main results

Let the differential equation system

$$\dot{v} = f(s,v)$$

describe the unconstraint dynamic.

Let the differential-algebraic equation system

$$\dot{v} = f(s,v) + c(s,v)$$

$$0 = C(s,v)$$

describe the constraint dynamic.

Given the external force f at first not only the n coordinate variables of v are unknown but also the n coordinate variables of the constraint force c are unknown, i.e. we have $2n$ unknown variables but only $(n+K)$ equations. To solve the differential-algebraic equation system we need $(n-K)$ further equations. These equations cannot be derived from Newtonian axioms. Therefore, Newton axioms are to be completed by what is usually called d'Alembert's principle (no virtual work). That d'Alembert's principle describes what in physics (real nature) actually happens can only be verified by experiments. As to now d'Alembert's principle is consistent with experimental results in the same way as Newtonian axioms are. Originally d'Alembert's principle was formulated for holonomic constraints. The following theorem states principles which are equivalent to d'Alembert's principle.

Definitions

(A) "d'Alembert's principle" ("no virtual work principle")

$\langle c,v\rangle = 0$ *for all* $v \in T\mathbf{C}$ *with* $T\mathbf{C} = \left\{ v \in \mathbf{R}^n \middle| \left\langle grad_{s|v} C^k, v \right\rangle = 0 \textit{ for all } k \right\}$

Note: d'Alembert#s principle (A) was historically the first

Note: $\dim T\mathbf{C} = n - K$, therefore the condition reduces to a set of $n-K$ equations, for a set of $n-K$ linear independent $v \in T\mathbf{C}$. Thus, d'Alembert's principal yields the missing $n-K$ equations to solve

(A1) "perpendicular principle"

$c \in P\mathbf{C}$

Note: Obviously the perpendicular principle (A1) is the most fundamental principle

(A2) "Gaussian least constraint principle"

$(x+c) \in T\mathbf{C}$ and

$\|c\|$ is minimal under condition $(x+c) \in T\mathbf{C}$

Note: The Gaussian principle typically is formulated in context of a constraint dynamic system by setting $x = f$ but it holds without connection to a constraint dynamic system.

Note: The principles (A), (A1), (A2) and the following (L), (C), (FK) are pure geometric principles and hold without any connection to a constraint dynamic system in contrary to the following "unnamed principle" (A3) and "least deviation principle" (A4)

(A3) "unnamed principle"

If $v \textit{ and } c$ are solutions of the constraint dynamic system

$$\dot{v} = f(s,v) + c(s,v)$$
$$0 = C(s,v)$$

Then c fulfils the unnamed principle iff

$$\frac{d\|v\|}{dt} = \frac{\langle v,f\rangle}{\|v\|}$$

Note: (A3) $\Rightarrow$ the constraining force c has no influence on $\|v\|$ but only on the direction of v

But: the inversion $\Leftarrow$ does not hold. For this the additional condition is necessary that c is minimal among all constraining forces which do not influence $\|v\|$. Therefore, we formulate the following principle (A4)

(A4) "least deviation principle"

$v \; and \; c$ are solutions of the constraint dynamic system

$$\dot{v} = f(s,v) + c(s,v)$$
$$0 = C(s,v)$$

Define: $V0 := \left\{ c \in \mathbb{R}^n \middle| x + c \in T\mathbf{C} \; and \; c \; does \; not \; influence \; \|v\| \right\}$

Then c fulfils the least deviation principle iff

$c \in V0$ and

$\|c\|$ is minimal among all $c \in V0$

(L) "Lagrange principle"

$$there \; exist \; \lambda(t) \in \mathbf{R}^K \; such \; that \; c = \sum_{k=1}^{K} \lambda^k grad_s \, C^k = \sum_{k=1}^{K} \lambda^k \frac{\partial C^k(s)}{\partial s}$$

(C) "Chetaev principle"<4>

$$there \; exist \; \lambda(t) \in \mathbf{R}^K \; such \; that \; c = \sum_{k=1}^{K} \lambda^k grad_v \, C^k = \sum_{k=1}^{K} \lambda^k \frac{\partial C^k(s,v)}{\partial v}$$

(FK) "Flannery/Krupkova principle"<5>,<6>

$$there \; exist \; \lambda(t) \in \mathbf{R}^K \; such \; that \; c = \sum_{k=1}^{K} \lambda^k grad_v \, C^k = \sum_{k=1}^{K} \lambda^k \frac{\partial C^k(s,v,\dot{v})}{\partial \dot{v}}$$

__**Theorem**__

For constraints $C^k(s,v) = 0 \; with \; k = 1,...,K$ with regularity condition $rank \frac{\partial C}{\partial (s,v)} = K$ the following holds:

(1) the following principles are equivalent:

(A) $\Leftrightarrow$ **(A1)** $\Leftrightarrow$ **(A2)** $\Leftrightarrow$ **(A3)** $\Leftrightarrow$ **(A4)**

(2) if $C = C(s)$ is holonomic then

(A) $\Leftrightarrow$ **(L)**

(3) if $C = C(s,v)$ is nonholonomic

(A) $\Leftrightarrow$ **(C)**

(4) Note: if $C = C(s,v,\dot{v})$ then

(A) $\Leftrightarrow$ **(FK)**

Proof:

$(A1) \Leftrightarrow (A) \Leftrightarrow (L) \Leftrightarrow (C):$

$$(A1) \quad \Leftrightarrow c \in P\mathbf{C}$$

$$(A) \quad \Leftrightarrow \langle c,v\rangle = 0 \ \text{for all } v \in T\mathbf{C} \qquad \text{by definition}$$

$$\Leftrightarrow \langle c,v\rangle = 0 \ \text{for all } v \text{ with } \left\langle grad_{s|v} C^k, v \right\rangle = 0 \ \text{for all } k = 1,\dots,K$$

$$\Leftrightarrow \langle c,v\rangle = 0 \ \text{for all } v \text{ with } \left\langle \lambda^k grad_{s|v} C^k, v \right\rangle = 0 \ \text{for all } k = 1,\dots,K$$

$$\Leftrightarrow \langle c,v\rangle = 0 \ \text{for all } v \text{ with } \left\langle \sum_{k=1}^{K} \lambda^k grad_{s|v} C^k, v \right\rangle = 0$$

$$\Leftrightarrow c = \sum_{k=1}^{K} \lambda^k grad_{s|v} C^k \qquad \text{because } \left\{ grad_{s|v} C^1, \dots, grad_{s|v} C^K \right\} \text{ is a base of } P\mathbf{C}$$

$$\text{because } rank \frac{\partial C}{\partial (s,v)} = K$$

$$\Leftrightarrow (L),(C)$$

1.*Proof of equivalence of* (*A*2)

(*A*2) ⇔ (*L*), (*C*):

$$(A2) \quad \Leftrightarrow \|c\| \text{ is minimal under condition } (x+c) \in T\mathbf{C}$$

$$\Leftrightarrow \|c\| \text{ is minimal under } k = 1,\dots,K \text{ conditions } \left\langle grad_{s|v} C^k, (x+c) \right\rangle = 0$$

$$\Leftrightarrow L(c,\beta) := \left(\|c\| + \sum_{k=1}^{K} \beta^k \left\langle grad_{s|v} C^k, (x+c) \right\rangle \right) \text{ is minimal}$$

$$\Leftrightarrow grad_c L(c,\beta) = grad_c \left(\|c\| + \sum_{k=1}^{K} \beta^k \left\langle grad_{s|v} C^k, (x+c) \right\rangle \right) \equiv 0 \text{ and}$$

$$grad_\beta L(c,\beta) = \left(\left\langle grad_{s|v} C^1, (x+c) \right\rangle, \dots, \left\langle grad_{s|v} C^K, (x+c) \right\rangle \right) \equiv 0$$

$$\Leftrightarrow \frac{1}{\|c\|} c_i + \sum_{k=1}^{K} \beta^k \frac{\partial C^k}{\partial (s_i | v_i)} = 0 \text{ for all } i = 1,\dots,n, \text{ and } (x+c) \in T\mathbf{C}$$

$$\Leftrightarrow c = -\|c\| \sum_{k=1}^{K} \beta^k grad_{s|v} C^k \text{ and } (x+c) \in T\mathbf{C}$$

$$\Leftrightarrow c = \sum_{k=1}^{K} \lambda^k grad_{s|v} C^k \qquad \text{with } \lambda^k = -\|c\| \beta^k \text{ and } (x+c) \in T\mathbf{C}$$

$$\Leftrightarrow (L), (C)$$

2.*Proof of equivalence of* (*A*2) *wich is a little more intuitive*

(*A*2) ⇔ (*A*1):

(*a*) *Split* $x = x_T + x_P$ *with* $x_T \in T\mathbf{C}$ *and* $x_P \in P\mathbf{C}$

(*b*) *Split* $c = c_T + c_P$ *with* $c_T \in T\mathbf{C}$ *and* $c_P \in P\mathbf{Cp} \Rightarrow c_T \perp c_P$

then

(*c*) $x + c = x_T + x_P + c_T + c_P \in T\mathbf{C} \Leftrightarrow x_P + c_P = 0$

Let $x + c \in T\mathbf{C}$

$$(A2) \Leftrightarrow \|c\| = \sqrt{\|c_T\|^2 + \|c_P\|^2} \text{ minimal under condition } x + c \in T\mathbf{C} \qquad \text{because of } (b)$$

$$\Leftrightarrow \|c\| = \sqrt{\|c_T\|^2 + \|c_P\|^2} \text{ minimal under condition } x_P + c_P = 0 \quad \text{because of } (c)$$

$$\Leftrightarrow \|c_T\| = 0$$

$$\Leftrightarrow c \in P\mathbf{C}$$

$$\Leftrightarrow (A1)$$

(*A*1) ⇔ (*A*3):

$$\text{because of} \quad \frac{d\|v\|}{dt} = \frac{\langle v, \dot{v} \rangle}{\|v\|} = \frac{\langle v, f + c \rangle}{\|v\|} = \frac{\langle v, f \rangle}{\|v\|} + \frac{\langle v, c \rangle}{\|v\|}$$

$$(A3) \Leftrightarrow \frac{d\|v\|}{dt} = \frac{\langle v, f \rangle}{\|v\|} \Leftrightarrow \langle v, c \rangle = 0 \Leftrightarrow (A1)$$

$(A1) \Rightarrow (A4)$: *obvious*

$(A4) \Rightarrow (A1)$: *similar to 2.Proof of equivalence of* $(A2)$

(a) define $V0 := \{ c \in \mathbb{R}^n \mid x + c \in T\mathbf{C} \text{ and } c \text{ does not influence } \|v\| \}$

(a) Split $x = x_T + x_P$ with $x_T \in T\mathbf{C}$ and $x_P \in P\mathbf{C}$

(b) Split $c = c_T + c_P$ with $c_T \in T\mathbf{C}$ and $c_P \in P\mathbf{Cp}\!\!> c_T \perp c_P$

(c) Split $c_T = c_T^1 + c_T^2$ with c_T^1 influence $\|v\|$ and with c_T^2 does not influence $\|v\|$

$\Rightarrow c_T^1 \perp c_T^2$

then

(d) $c \in V0 \Rightarrow c_T^1 = 0$ because c_T^1 influence $\|v\|$

(f) $x + c = x_T + x_P + c_T^1 + c_T^2 + c_P \in T\mathbf{C} \Leftrightarrow x_P + c_P = 0$

Let $c \in V0$

$(A4) \Leftrightarrow \|c\| = \sqrt{\|c_T^1\|^2 + \|c_T^2\|^2 + \|c_P\|^2}$ minimal under condition $c \in V0$ because of (b) and (c)

$\Leftrightarrow \|c\| = \sqrt{\|c_T^2\|^2 + \|c_P\|^2}$ minimal under condition $c \in V0$ because of (d)

$\Leftrightarrow \|c_T^2\| = 0$ because of (f)

$\Leftrightarrow \|c_T\| = 0$ because of $c_T = c_T^1 + c_T^2$ and (d)

$\Leftrightarrow c \in P\mathbf{C}$

$\Leftrightarrow (A1)$

$(A1) \Leftrightarrow (FK)$: straightforward as $(A1) \Leftrightarrow (L),(C)$ by differentiating C twice: use $\ddot{C}$ *instead of* $\dot{C}$ and $grad_{s|v|v}$ *instead of* $grad_{s|v}$

Hamiltons variational principle under holonomic constraint

Assume the special case when $f(s)$ is a gradient force of potential $U(s)$ depending only on s, which means $(f_1(s),...,f_n(s)) = (\frac{\partial U(s)}{\partial s_1},...,\frac{\partial U(s)}{\partial s_n})$ and assume $C(s)$ to be holonomic.

Definition

(Hv) Hamiltons variational principle under holonomic constraint:

$$\textit{define constraint Lagrange function } L^C(s,\lambda,\dot{s},\dot{\lambda}) := \frac{1}{2}\langle \dot{s},\dot{s}\rangle + U(s) + \lambda C(s)$$

$$\textit{then Hamiltons variational principle} :\Leftrightarrow \int_{t_0}^{t_1} L^C(s,\lambda,\dot{s},\dot{\lambda})dt \rightarrow minimal$$

Remark: In General Equilibrium Models of economics the stationary points ($\dot{x}=0$, which implies $\dot{x}=0$) are typically unique and equal the points which maximise the constraint Lagrangefunktion L_0^C , whereby index 0 refers to $\dot{x}=0$.

$L_0^C(x) := L^C(x,0) = U(x) + \lambda C(x)$

Therefore, the equations describing the equilibrium are

$$0 = \frac{d}{dt}\frac{\partial L_0^C}{\partial \dot{x}} - \frac{\partial L_0^C}{\partial x} = 0 - \frac{\partial U(x)}{\partial x} - \lambda\frac{\partial C(x)}{\partial x}$$

$$0 = \frac{d}{dt}\frac{\partial L_0^C}{\partial \dot{x}} - \frac{\partial L_0^C}{\partial \lambda} = 0 - C(x)$$

i.e.

$$\frac{\partial U(x)}{\partial x} + \lambda\frac{\partial C(x)}{\partial x} = 0$$

$$C(x) = 0$$

Proposition:

(1) If $C(s,\dot{s})$ is holonomic or is integrable or is affin linear with respect to $\dot{s}$ then Hamiltons variational principle implies the above principles.

(2) If $C(s,\dot{s})$ is neither holonomic nor is integrable nor is linear with respect to $\dot{s}$ Hamiltons variational principle does not imply the above principles because in the Euler equations occurs an additional term.

Proof of (1) for holonomic constraints:

$$(Hv) \Leftrightarrow \qquad \int_{t_0}^{t_1} L(s,\lambda,\dot{s},\dot{\lambda})dt \rightarrow minimal$$

$$\Leftrightarrow \qquad \int_{t_0}^{t_1} \left(\frac{1}{2}\langle \dot{s},\dot{s} \rangle + U(s) + \lambda C(s) \right) dt \rightarrow minimal \qquad \Rightarrow$$

$$\Rightarrow \qquad \frac{\delta}{\delta(s,\lambda)} \int_{t_0}^{t_1} \left(\frac{1}{2}\langle \dot{s},\dot{s} \rangle + U(s) + \lambda C(s) \right) dt = 0 \qquad \Leftrightarrow$$

$$\Leftrightarrow \qquad \frac{\partial U(s)}{\partial(s)} + \lambda \frac{\partial C(s)}{\partial(s)} - \ddot{s} = 0 \qquad Euler - Lagrange\ equations$$

$$C(s) = 0$$

$$\Leftrightarrow \qquad \dot{v} = \frac{\partial U(s)}{\partial(s)} + \lambda \frac{\partial C(s)}{\partial(s)} \qquad with\ \dot{s} = v$$

$$C(s) = 0$$

$$\Leftrightarrow \qquad Lagrange\ equations\ with\ c = \lambda \frac{\partial C(s)}{\partial(s)}$$

$$\Leftrightarrow \qquad (3)\ Lagrange\ \ principle$$

Proof of (1) for a simple linear nonholonomic constraint

Let $C(s,v) = a\,v \;\; with\; a \in \mathbf{R}$ (which is a simple version of a linear nonholonomic constraint) then

$$
\begin{aligned}
&Euler\ equation \quad \Leftrightarrow \\
&\dot{v} = \frac{\partial U(s)}{\partial s} + \lambda \frac{\partial C(s,v)}{\partial s} - \frac{d}{dt}(\lambda \frac{\partial C(s,v)}{\partial v}) \qquad see\ below \\
&\Leftrightarrow \\
&\dot{v} = \frac{\partial U(s)}{\partial s} + \lambda \frac{\partial C(s,v)}{\partial s} - \frac{d}{dt}(\lambda a) \\
&= \frac{\partial U(s)}{\partial s} + \lambda \frac{\partial C(s,v)}{\partial s} - \dot{\lambda} a \\
&= \frac{\partial U(s)}{\partial s} + \lambda \frac{\partial C(s,v)}{\partial s} - \dot{\lambda} \frac{\partial C(s,v)}{\partial v} \\
&= \frac{\partial U(s)}{\partial s} + \lambda \frac{\partial \frac{dC(s,v)}{dt}}{\partial \frac{ds}{dt}} - \dot{\lambda} \frac{\partial C(s,v)}{\partial v} \\
&= \frac{\partial U(s)}{\partial s} + \lambda \frac{\partial \left(\frac{\partial C(s,v)}{\partial s}\dot{s} + \frac{\partial C(s,v)}{\partial v}\dot{v} \right)}{\partial v} - \dot{\lambda} \frac{\partial C(s,v)}{\partial v} \\
&= \frac{\partial U(s)}{\partial s} + \lambda \frac{\partial \left(0 + a\dot{v} \right)}{\partial v} - \dot{\lambda} \frac{\partial C(s,v)}{\partial v} \\
&= \frac{\partial U(s)}{\partial s} + 0 - \dot{\lambda} \frac{\partial C(s,v)}{\partial v} \\
&= \frac{\partial U(s)}{\partial s} + \lambda^{*} \frac{\partial C(s,v)}{\partial v} \qquad rename\ \lambda^{*} := \dot{\lambda} \\
&\Leftrightarrow \\
&Lagrange - Chetaev\ equations\ with\ \lambda^{*}
\end{aligned}
$$

In a similar way the equivalence of Euler equations with Lagrange-Chetaev equations can be shown for a more general $C(s,v)$ which is linear in v.

Proof of (2):

Let $C(s,v) = 0$ which is nonlinear in v.
Then in Euler equations in general occur an additional term to Lagrange equations.

Euler equations:

$$0=\frac{\delta L^{C}(s,\lambda,\dot{s},\dot{\lambda})}{\delta(s,\lambda)}=\frac{\partial L^{C}}{\partial(s,\lambda)}-\frac{d}{dt}\left(\frac{\partial L^{C}}{\partial(\dot{s},\dot{\lambda})}\right)=\frac{\partial U(s)}{\partial s}+\lambda\frac{\partial C(s,\dot{s})}{\partial s}-\frac{d}{dt}\left(\dot{s}+\lambda\frac{\partial C(s,\dot{s})}{\partial \dot{s}}\right)=$$

$$=\frac{\partial U(s)}{\partial s}+\lambda\frac{\partial C(s,\dot{s})}{\partial s}-\ddot{s}-\frac{d}{dt}\left(\lambda\frac{\partial C(s,\dot{s})}{\partial \dot{s}}\right)$$

$$0=C(s,\dot{s})$$

$\Leftrightarrow$

$$\dot{v}=\frac{\partial U(s)}{\partial s}+\lambda\frac{\partial C(s,v)}{\partial s}-\frac{d}{dt}\left(\lambda\frac{\partial C(s,v)}{\partial v}\right)$$

$$0=C(s,v)$$

which is typically not equivalent to Lagrange – Chetaev equations with general $C(s,v)=0$

$$\dot{v}=\frac{\partial U(s)}{\partial s}+\lambda\frac{\partial C(s,v)}{\partial v}$$

$$0=C(s,v)$$

Thus, typically Euler equations are not equal to Lagrange-Chetaev equations because of the additional term $-\frac{d}{dt}\left(\lambda\frac{\partial C(s,v)}{\partial v}\right)$

Remark

In the case of general nonholonomic constraints (i.e. when Euler equations differ from Lagrange-Chetaev equations) the Euler equations

$$\dot{v}=\frac{\partial U(s)}{\partial s}+\lambda\frac{\partial C(s,v)}{\partial s}-\frac{d}{dt}\left(\lambda\frac{\partial C(s,v)}{\partial v}\right)$$

$$0=C(s,v)$$

are called **vakonomic equations.**

In general, the following holds.

Definitions
(Hv) "Hamilton variational principle"
(Hd) "Hamilton differential principle"
(NHv) "nonholonomic Hamilton variational principle"
(NHd) "nonholonomic Hamilton differential principle"

Theorem
(1) if C is holonomic then

(A) ⇔ **(L)** ⇔ **(Hv)** ⇔ **(Hd)**

(2) if C is nonholonomic and integrable (or linear)?

(A) ⇔ **(C)** ⇔ **(Hv)** ⇔ **(Hd)**

(3) if C is nonholonomic and nonintegrable

(A) ⇔ **(C)** ⇔ **(NHv)** ⇔ **(NHd)**

4 Conclusion

The dynamics resp. the constraint forces of a constraint system in classical mechanics is determined by d'Alembert's principle (or a lot of equivalent principles), which must be considered as axiom and which is verified by experiments like Newtonian laws.
There is no such axiom for constrained systems in economics. The structure of the constraint forces in economics depend on the special model. For some models it is plausible to use d'Alembert's principle in full analogy to classical mechanics, some may be described by vakonomic equations and others may be described in a different way.

5 Appendix (Example of non-d'Alembert constraint force)

A type of constraint force, which does **not satisfy d'Alembert's principle,** can occur in particular with a constraint force describing a limited resource is a constraint force that is centrally directed to the origin. We therefore refer to this as a "central constraint force" <2>,.

$$c(x_1,x_2) = \begin{pmatrix} c_1(s_1,s_2) \\ c_2(s_1,s_2) \end{pmatrix} = \lambda \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}$$

A model for this is constraining forces such as those that occur in theoretical biology in the derivation of the so-called replicator equation. <7>. This model assumption of a central constraint force is equivalent in biology to the assumption that in the struggle for limited resources equally high death rates are triggered for all species.

Let us illustrate this with an example. A typical dynamic in biology is the initially independent exponential growth of 2 species A and B with birth rates b_A, b_B . $s_1 = n_A, s_2 = n_B$ denote the number of individuals of A, B .

$$\begin{aligned} \dot{n}_A &= b_A\, n_A & \qquad & b_{AA} \text{ "growth rate" of } A \\ \dot{n}_B &= b_B\, n_B & & b_{BB} \text{ "growth rate" of } B \end{aligned}$$

A constraint typical of biology is, for example, the assumption of limited resources. This can be given, for example, by a limitation of the food supply or also by a limitation of the habitat. This leads to the fact that the sum of the number of absolute frequency of the different species remains constant. This is formally described by the constraint condition

$$C(n_1, n_2, \ldots) = \sum_i n_i - constant = 0$$

Assuming that the same death rates are triggered by the constraint condition in both species, the differential following differential-algebraic equation system is obtained

$$\begin{aligned} \dot{n}_A &= b_{AA}\, n_A - \lambda\, n_A \\ \dot{n}_B &= b_{BB} n_B - \lambda\, n_b \\ C(n_A, n_B) &= n_A + n_B - n = 0 \qquad\qquad n \text{ constant} \end{aligned}$$

Assuming that A is twice as successful ("powerful") in the struggle for resources, the death rate for A would be half as large, yielding the system of equations

$$\dot{n}_A = b_{AA}\, n_A - \lambda \frac{1}{2} n_A$$
$$\dot{n}_B = b_{BB}\, n_B - \lambda\ n_b$$
$$C(n_A, n_B) = n_A + n_B - n = 0$$

Applied to economic constraints, this can be interpreted as follows. Agents may have different power to oppose constraints. For example, if raw materials are limited in sum, it may be easier for some countries to get the necessary raw materials anyway than for others.

In the most general case, different types of constraining forces can occur. For the modelling it is only essential that for constraints exactly K linearly independent constraint forces have to be used, which have to be multiplied by the respective Lagrange multiplier.

Note: If under the constraint conditions $C_k,\ k = 1,2,...,K$ not all constraint forces are vertical, $s = (s_1, s_2, ...)$ typically do not converge to the maximum value of a master utility function, even if the master utility function is convex.

6 References

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