Vector Calculus MAT226 Spring 2025
Professor Sormani
Lesson 2 Dot and Cross Products

Carefully take notes while attending class or watching the lesson videos. You will cut and paste the photos of your notes and completed classwork in a googledoc entitled:

MAT226S25-lesson2-lastname-firstname

Then share editing of that document with me sormanic@gmail.com. You will also put photos of your homework in this googledoc. If you work with any classmates, be sure to write their names on the problems you completed together.

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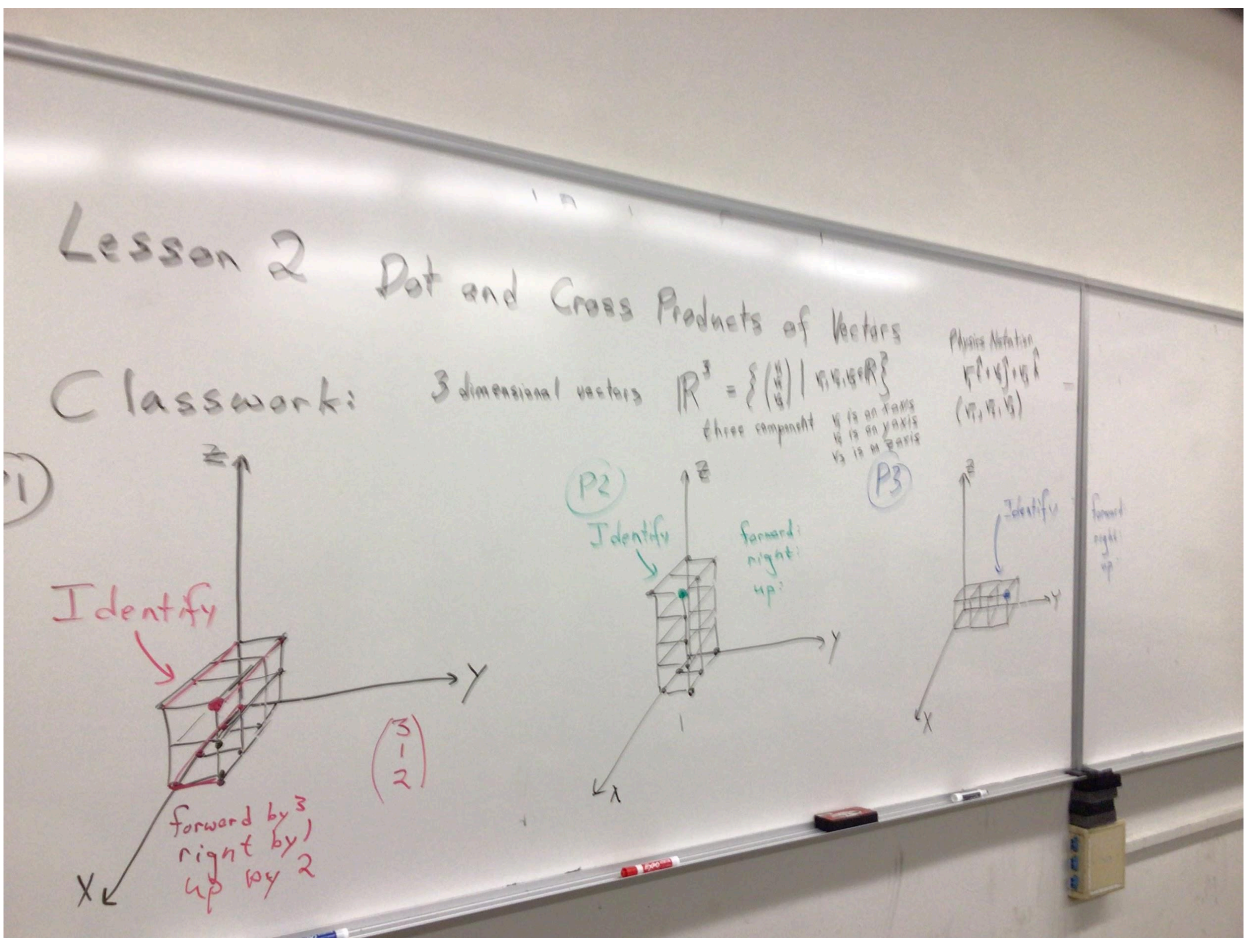
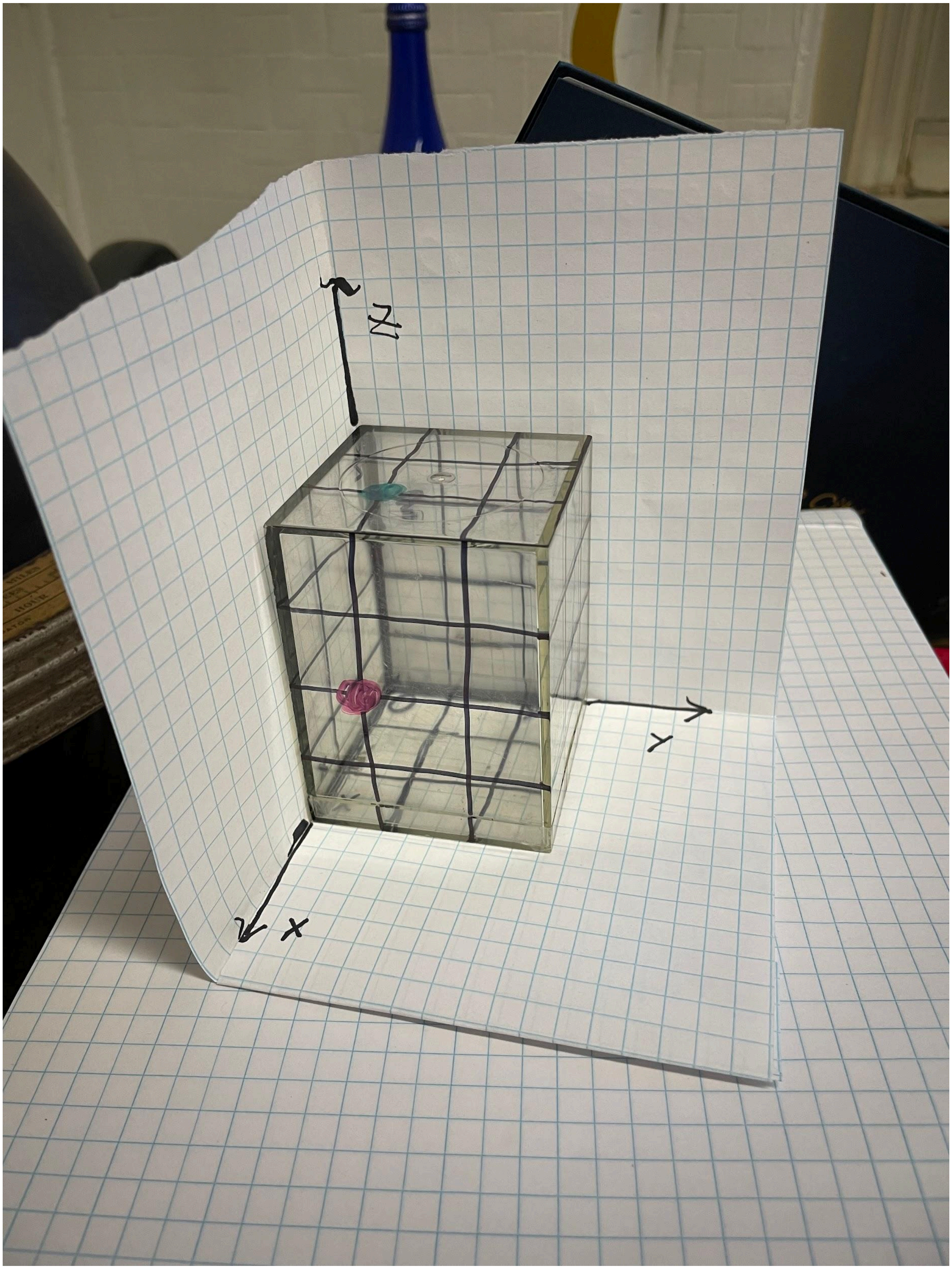
The 226F21-2 Playlist has 12 videos which you may watch if you missed class. Below are the class notes:

Quick Review of Lesson 1 (in class not in videos).

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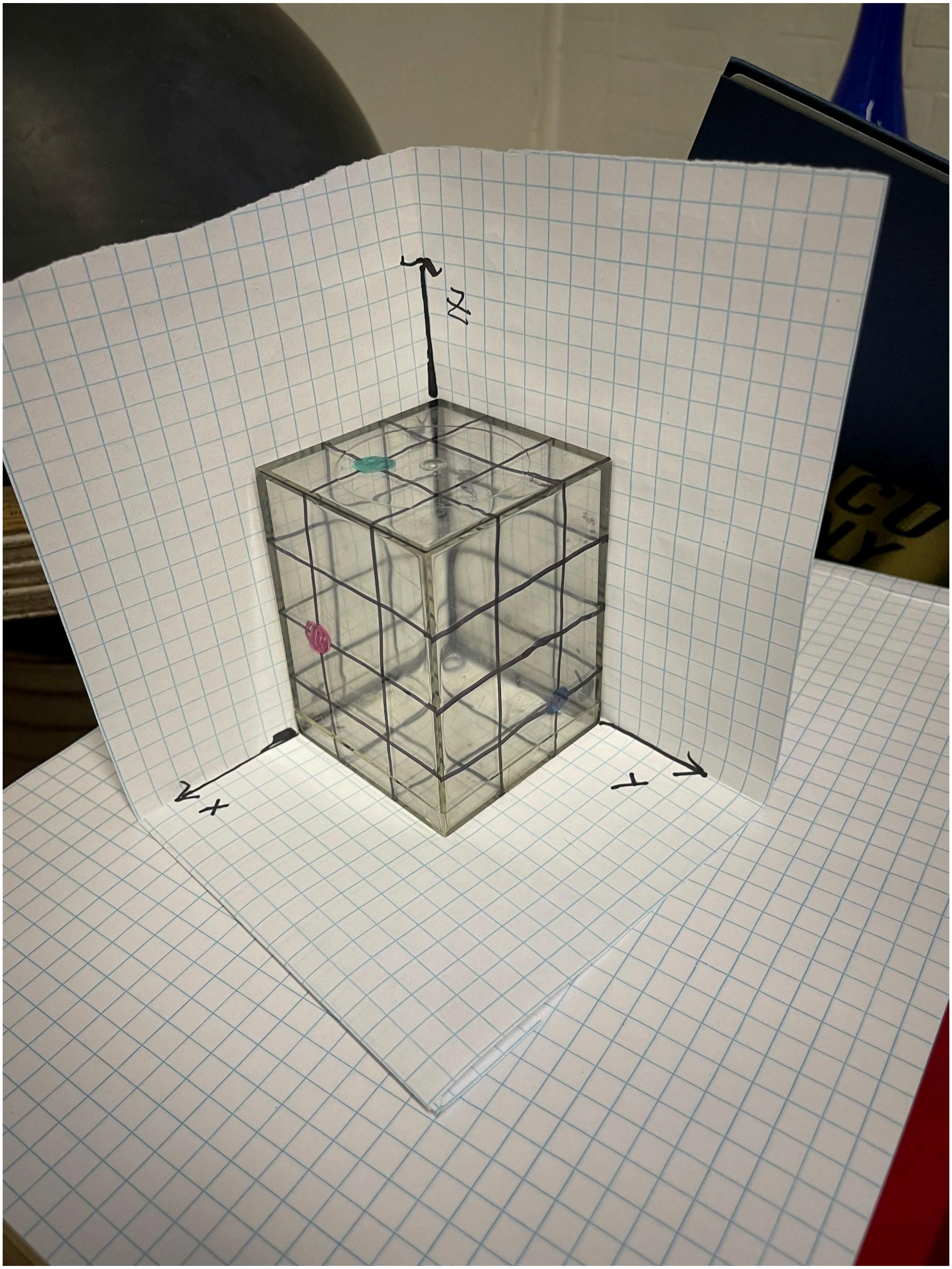
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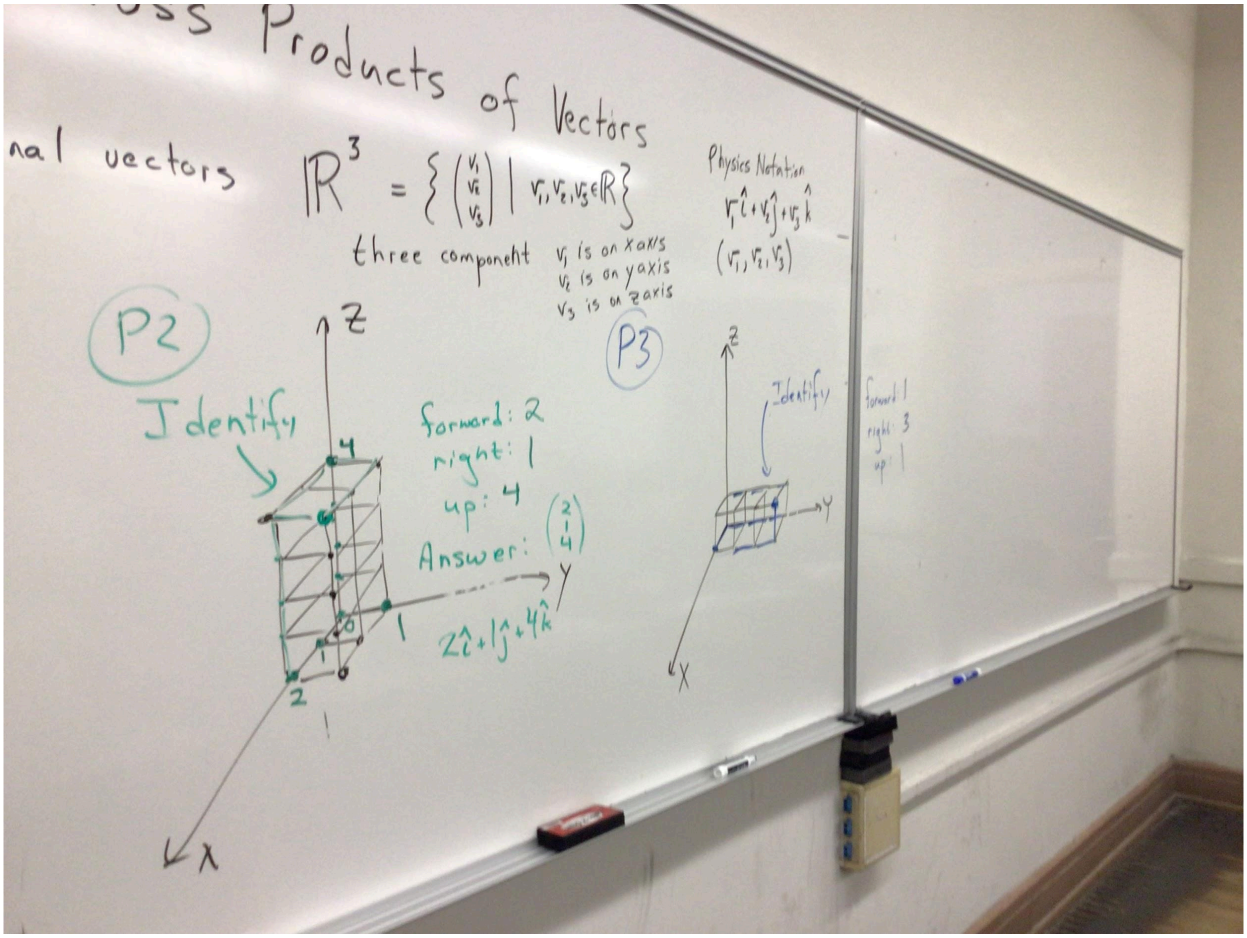


The 226F21-2 Playlist has 12 videos. Below are the class notes:

Vector Calculus Lesson 2

* Dot Products for Vectors in $\mathbb{R}^n$
* Cross Products for Vectors in $\mathbb{R}^3$

Review: Vectors in $\mathbb{R}^2$ and in $\mathbb{R}^3$
Magnitudes of Vectors in $\mathbb{R}^n$
Sums of Vectors in $\mathbb{R}^n$
Scalar Products of Vectors in $\mathbb{R}^n$

Review $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{R}^2$ $\quad \vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \in \mathbb{R}^3$

Magnitude $|\vec{v}|$ or $\|\vec{v}\|$ "norm" "length"

$$\left\| \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\| = \sqrt{v_1^2 + v_2^2} \qquad \left\| \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \right\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

Addition

$$\vec{v} + \vec{w} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ v_3 + w_3 \end{pmatrix}$$ add the components

Scalar Mult

$$r\vec{v} = r \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} rv_1 \\ rv_2 \\ rv_3 \end{pmatrix}$$

Theorems: $\|r\vec{v}\| = |r| \, \|\vec{v}\|$

$r(\vec{v} + \vec{w}) = r\vec{v} + r\vec{w}$ see textbook for more

Unit vectors $\vec{u}$ such that $\|\vec{u}\| = 1$

Thm: $\frac{1}{\|\vec{v}\|}\vec{v}$ is a unit vector in the direction of $\vec{v}$

Dot Product of two vectors

$$\vec{v} \cdot \vec{w} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = v_1 w_1 + v_2 w_2 \in \mathbb{R}$$

$$\vec{v} \cdot \vec{w} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = v_1 w_1 + v_2 w_2 + v_3 w_3 \in \mathbb{R}$$

Notice the anwer is not a vector

Theorem $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$

classwork
pause + try

Proof:

① $\vec{v} \cdot \vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ ① by defn of $\mathbb{R}^2$

② $= v_1 \cdot v_1 + v_2 \cdot v_2$ ② by defn of dot prod.

③ $= v_1^2 + v_2^2$ ③ by defn $a^2 = a \cdot a$ for $a \in \mathbb{R}$

④ $= \left(\sqrt{v_1^2 + v_2^2}\right)^2$ ④ $a = (\sqrt{a})^2$ when $a \geq 0$ here $a = v_1^2 + v_2^2 \geq 0$

⑤ $= \|\vec{v}\|^2$ ⑤ by defn of magnitude

QED

Thm: Distributative Property of the Dot Product

Given three vectors $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^n$

$$(\vec{a}+\vec{b})\cdot\vec{c} = \vec{a}\cdot\vec{c} + \vec{b}\cdot\vec{c}$$

Proof (Classwork pause + try) do for $\mathbb{R}^2$

Part I LHS

① $(\vec{a}+\vec{b})\cdot\vec{c} = \left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\right)\cdot\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ ① by defn of $\mathbb{R}^2$

② $= \begin{pmatrix} a_1+b_1 \\ a_2+b_2 \end{pmatrix}\cdot\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ ② by defn of vector add

③ $= (a_1+b_1)c_1 + (a_2+b_2)c_2$ ③ by defn of dot prod.

④ $= a_1c_1 + b_1c_1 + a_2c_2 + b_2c_2$ ④ by distribution of reals $(a+b)c = ac+bc$

Part II RHS

① $\vec{a}\cdot\vec{c} + \vec{b}\cdot\vec{c} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}\cdot\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\cdot\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ ① by defn of $\mathbb{R}^2$

② $= (a_1c_1 + a_2c_2) + (b_1c_1 + b_2c_2)$ ② defn of dot prod.

③ $= a_1c_1 + b_1c_1 + a_2c_2 + b_2c_2$ ③ by $A+B=B+A$ commutativity of addition.

want the same answer in both parts.

by Part I

$$(\vec{a}+\vec{b})\cdot\vec{c} = a_1c_1+b_1c_1+a_2c_2+b_2c_2$$
$$= \vec{a}\cdot\vec{c} + \vec{b}\cdot\vec{c} \quad \text{by Part II}$$

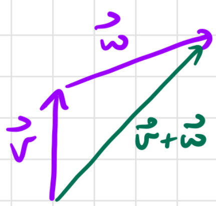
QED

LHS = left hand side $(\vec{a}+\vec{b})\cdot\vec{c}$

RHS = right hand side $\vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{c}$

Thm: $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}$ Extra credit to proof for HW.

Thm: $\|\vec{v}+\vec{w}\|^2 = \|\vec{v}\|^2 + 2\vec{v}\cdot\vec{w} + \|\vec{w}\|^2 \quad \vec{v},\vec{w}\in\mathbb{R}^n$



Proof for $\vec{v},\vec{w}\in\mathbb{R}^2$

Part I LHS:

① $\|\vec{v}+\vec{w}\|^2 =$

$=$

$=$

$=$ same

pause + try

Part II RHS

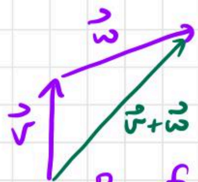
① $\|\vec{v}\|^2 + 2\vec{v}\cdot\vec{w} + \|\vec{w}\|^2 =$

$=$

$=$ same

want the same final line.

Thm: $\|\vec{v}+\vec{w}\|^2 = \|\vec{v}\|^2 + 2\vec{v}\cdot\vec{w} + \|\vec{w}\|^2 \quad \vec{v},\vec{w}\in\mathbb{R}^n$



Extra Credit

$$\|\vec{v}-\vec{w}\|^2 = \|\vec{v}\|^2 - 2\vec{v}\cdot\vec{w} + \|\vec{w}\|^2$$
$$\vec{v},\vec{w}\in\mathbb{R}^2$$

Proof for $\vec{v},\vec{w}\in\mathbb{R}^2$

Part I LHS:

① $\|\vec{v}+\vec{w}\|^2 = \left\|\binom{v_1}{v_2}+\binom{w_1}{w_2}\right\|^2$ ① by defn of $\mathbb{R}^2$

② $= \left\|\binom{v_1+w_1}{v_2+w_2}\right\|^2$ ② by defn of add.

③ $= \left(\sqrt{(v_1+w_1)^2+(v_2+w_2)^2}\right)^2$ ③ by defn of magnitude

④ $= (v_1+w_1)^2+(v_2+w_2)^2$ ④ by $(\sqrt{a})^2 = a$ when $a \geq 0$

⑤ $= v_1^2+2v_1w_1+w_1^2+v_2^2+2v_2w_2+w_2^2$ ⑤ $(a+b)^2 = a^2+2ab+b^2$

Part II RHS

① $\|\vec{v}\|^2 + 2\vec{v}\cdot\vec{w} + \|\vec{w}\|^2 = \left\|\binom{v_1}{v_2}\right\|^2 + 2\binom{v_1}{v_2}\cdot\binom{w_1}{w_2} + \left\|\binom{w_1}{w_2}\right\|^2$

① by defn of $\mathbb{R}^2$

② $= \left(\sqrt{v_1^2+v_2^2}\right)^2 + 2\binom{v_1}{v_2}\cdot\binom{w_1}{w_2} + \left(\sqrt{w_1^2+w_2^2}\right)^2$

② by defn of magnitude

③ $= v_1^2+v_2^2+2(v_1w_1+v_2w_2)+w_1^2+w_2^2$

③ by defn of dot product

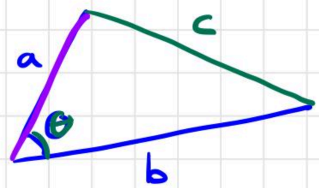
④ $= v_1^2+2v_1w_1+w_1^2+v_2^2+2v_2w_2+w_2^2$

④ by $A+B=B+A$

Step 5 of Part I = step 5 of Part II

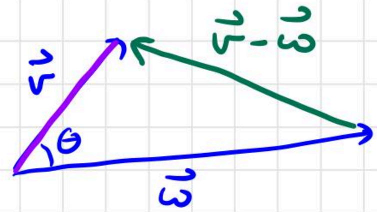
LHS = RHS QED

The Law of Cosines and $\|\vec{v}-\vec{w}\|$



$$c^2 = a^2 + b^2 - 2ab\cos(\theta)$$

proven in trigonometry



If θ = angle between $\vec{v}$ and $\vec{w}$

then $a = \|\vec{v}\|$ $b = \|\vec{w}\|$

$c = \|\vec{v}-\vec{w}\|$

by subtraction of vectors and magnitudes of vectors in Lesson I

Combine these and we have:

$$\|\vec{v}-\vec{w}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2 - 2\|\vec{v}\|\cdot\|\vec{w}\|\cos(\theta)$$

using the extra credit

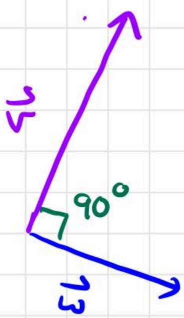
$$\|\vec{v}\|^2 - 2\,\vec{v}\cdot\vec{w} + \|\vec{w}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2 - 2\|\vec{v}\|\cdot\|\vec{w}\|\cos(\theta)$$

next subtract the $\|\vec{v}\|^2$ and $\|\vec{w}\|^2$ from both sides

$$-2\,\vec{v}\cdot\vec{w} = -2\|\vec{v}\|\cdot\|\vec{w}\|\cos\theta$$

divide both sides by -2

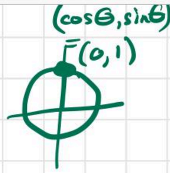
$$\boxed{\vec{v}\cdot\vec{w} = \|\vec{v}\|\cdot\|\vec{w}\|\cos\theta}$$



Application: If $\theta = 90° = \frac{\pi}{2}$

then $\cos\theta = 0$

$\vec{v} \cdot \vec{w} = \|\vec{v}\| \cdot \|\vec{w}\| \underbrace{\cos\left(\frac{\pi}{2}\right)}_{0} = 0$



If $\vec{v} \cdot \vec{w} = 0$ then either $\|\vec{v}\| = 0$
or $\|\vec{w}\| = 0$
or $\cos\theta = 0$
$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$

then $\vec{v}, \vec{w}$ are $\perp$

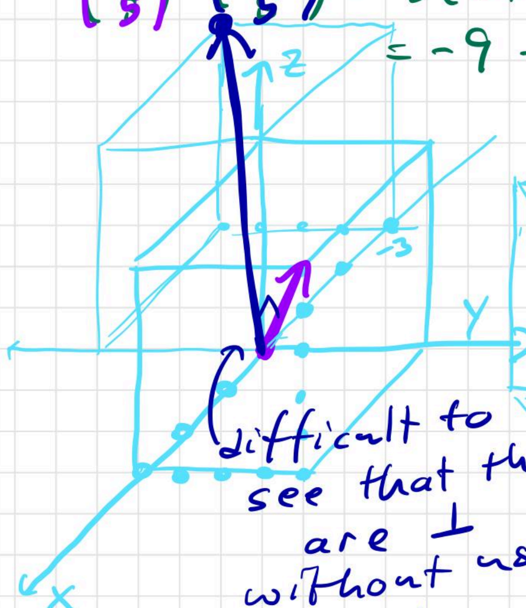
they are perpendicular

$\vec{v}$ and $\vec{w}$ are orthogonal if $\vec{v} \cdot \vec{w} = 0$

Classwork Example: Check if $\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ -4 \\ 5 \end{pmatrix}$ are orthogonal.

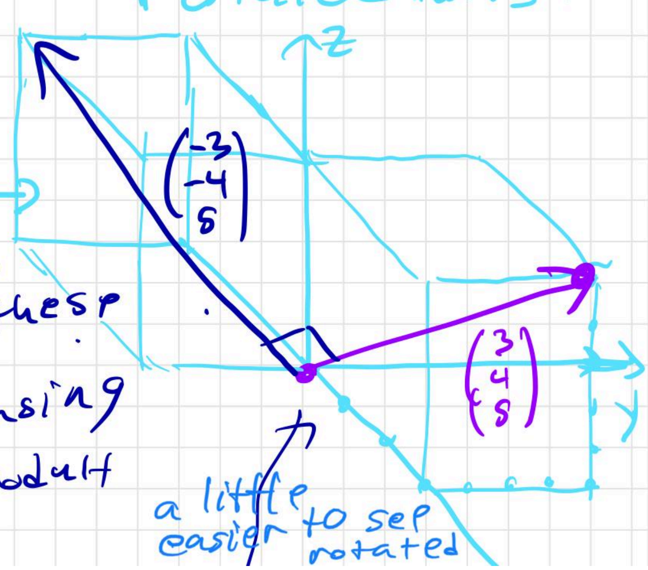
$$\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ -4 \\ 5 \end{pmatrix} = 3(-3) + 4(-4) + 5 \cdot (5)$$

$$= -9 - 16 + 25 = 0 \quad \checkmark \text{ Yes}$$



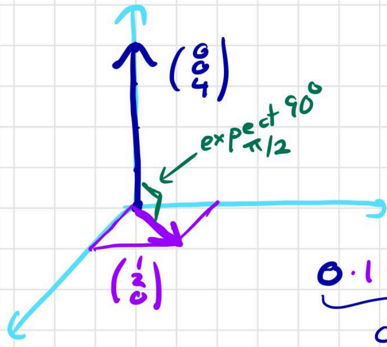
difficult to see that these are $\perp$ without using the dot product

rotated axis:



a little easier to see rotated

Example: Find the angle between vectors $\begin{pmatrix}0\\0\\4\end{pmatrix}$ and $\begin{pmatrix}1\\2\\0\end{pmatrix}$ Classwork!



$$\vec{v}\cdot\vec{w} = \|\vec{v}\|\cdot\|\vec{w}\|\cos\theta$$

$$\begin{pmatrix}0\\0\\4\end{pmatrix}\cdot\begin{pmatrix}1\\2\\0\end{pmatrix} = \left\|\begin{pmatrix}0\\0\\4\end{pmatrix}\right\|\cdot\left\|\begin{pmatrix}1\\2\\0\end{pmatrix}\right\|\cos\theta$$

Solve for θ

$$\underbrace{0\cdot1 + 0\cdot2 + 4\cdot0}_{} = \sqrt{0^2+0^2+4^2}\sqrt{1^2+2^2+0^2}\cos\theta$$

$$0+0+0 = \sqrt{4^2}\sqrt{5}\cos\theta$$

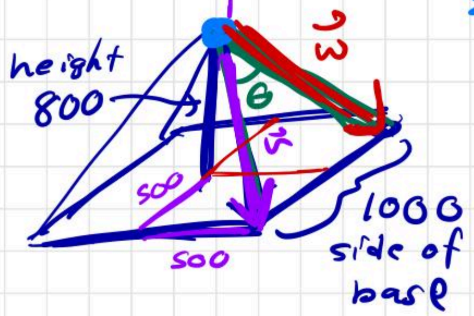
$$0 = 4\sqrt{5}\cos\theta$$

$$\cos\theta = 0$$

$$\theta = \operatorname{Arccos}(0) = \frac{\pi}{2} \checkmark$$

what we expected

Example:



Set the top to be the origin

$$\vec{v} = \begin{pmatrix} 500 \\ 500 \\ -800 \end{pmatrix} \text{ down 800}$$

$$\vec{w} = \begin{pmatrix} -500 \\ 500 \\ -800 \end{pmatrix} \begin{matrix} \text{back} \\ \text{right} \\ \text{down} \end{matrix}$$

$$\vec{v} \cdot \vec{w} = \|\vec{v}\| \cdot \|\vec{w}\| \cos\theta$$

fill in + work out solve for θ

$$\begin{pmatrix} 500 \\ 500 \\ -800 \end{pmatrix} \cdot \begin{pmatrix} -500 \\ 500 \\ -800 \end{pmatrix} = \left\| \begin{pmatrix} 500 \\ 500 \\ -800 \end{pmatrix} \right\| \cdot \left\| \begin{pmatrix} -500 \\ 500 \\ -800 \end{pmatrix} \right\| \cos\theta$$

$$500(-500) + 500(500) + (-800)(-800) = (\quad)(\quad)\cos\theta$$

$$\sqrt{(500)^2 + (500)^2 + (-800)^2}$$

$$\sqrt{(-500)^2 + (500)^2 + (-800)^2}$$

same for this problem $(500)^2 = (-500)^2$

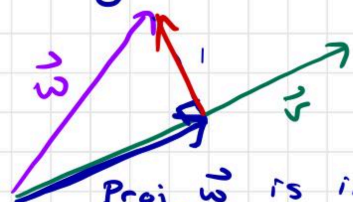
$$640000 = \quad \cos\theta$$

$$\frac{640000}{500^2 + 500^2 + 800^2} = \cos\theta$$

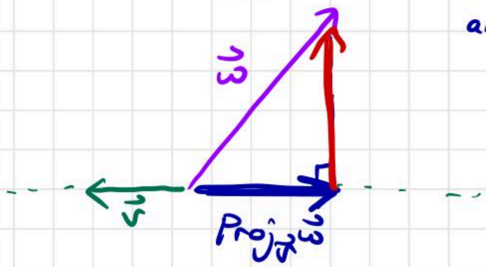
$$\theta = \operatorname{Arccos}\left(\frac{640000}{500^2 + 500^2 + 800^2}\right)$$

Projections $\text{proj}_{\vec{v}}\vec{w} = \dfrac{\vec{w}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\,\vec{v}$

projection of $\vec{w}$ onto $\vec{v}$



$\text{Proj}_{\vec{v}}\vec{w}$ is in the same or opposite direction as $\vec{v}$



and it forms a right triangle where $\vec{w}$ is the hypotenus

Drop a perpendiculan from $\vec{w}$ to the line through $\vec{v}$ to find the projection

We have a perpendicular vector

$\vec{P} = \vec{w} - \text{Proj}_{\vec{v}}\vec{w}$ is part of $\vec{w}$ $\perp$ to $\vec{v}$

$$
\begin{aligned}
\vec{P}\cdot\vec{v} &= (\vec{w} - \text{Proj}_{\vec{v}}\vec{w})\cdot\vec{v} && \text{by defn of } \vec{P}\\
&= \vec{w}\cdot\vec{v} - (\text{Proj}_{\vec{v}}\vec{w})\cdot\vec{v} && \text{by distribution}\\
&= \vec{w}\cdot\vec{v} - \left(\frac{\vec{w}\cdot\vec{v}}{\cancel{\vec{v}\cdot\vec{v}}}\right)\cancel{\vec{v}\cdot\vec{v}} && \text{by defn of Proj}\\
&= \vec{w}\cdot\vec{v} - \vec{w}\cdot\vec{v} = 0 && \text{by cancel } \frac{\vec{v}\cdot\vec{v}}{\vec{v}\cdot\vec{v}} = 1
\end{aligned}
$$

So $\vec{P}$ is perpendicular to $\vec{v}$

The textbook has examples finding these vectors.

The cross product of two vectors

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \times \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{pmatrix}$$

only defined for $\vec{v}, \vec{w} \in \mathbb{R}^3$

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Vector Calculus 226F21

The cross product of two vectors

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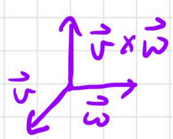
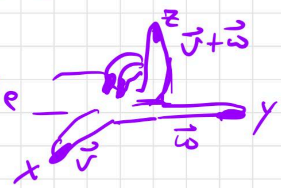
only defined for $\vec{v}, \vec{w} \in \mathbb{R}^3$

Thm: $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$ extra credit to prove
very short proof

Example: $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \cdot 0 - 0 \cdot 1 \\ 0 \cdot 0 - 1 \cdot 0 \\ 1 \cdot 1 - 0 \cdot 0 \end{pmatrix} = \begin{pmatrix} 0-0 \\ 0-0 \\ 1-0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$



right hand rule



In the book the method I taught is easier + the same.

Computing the Cross Product using determinants

2×2 determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

3×3 determinant $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$

$\begin{vmatrix} a & b & c \\ d & \boxed{e} & \boxed{f} \\ g & \boxed{h} & \boxed{i} \end{vmatrix} \quad \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \quad \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$

$$\vec{v} \times \vec{w} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$= \hat{i} \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - \hat{j} \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + \hat{k} \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}$$

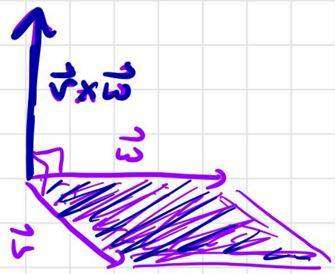
$$= (v_2 w_3 - v_3 w_2)\hat{i} - (v_1 w_3 - v_3 w_1)\hat{j} + (v_1 w_2 - v_2 w_1)\hat{k} = \begin{pmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{pmatrix}.$$

same as the defn I gave

Thm: $\vec{v} \times \vec{w}$ is $\perp$ to both $\vec{v}$ and $\vec{w}$

and $\|\vec{v} \times \vec{w}\| = \|\vec{v}\| \cdot \|\vec{w}\| \sin(\theta)$

which is the area of the parallelogram



Note that if $\vec{v} = r\vec{w}$ then there is no parallelogram and so its area $= 0$ and $\|\vec{v} \times \vec{w}\| = 0$ in that case.

Proof idea:

Useful Trig Fact $\cos^2\theta + \sin^2\theta = 1$

$\sin^2\theta = 1 - \cos^2\theta$

$\sin\theta = \sqrt{1-\cos^2\theta}$ when $\sin\theta \geq 0$

Thus

$$\|\vec{v}\| \cdot \|\vec{\omega}\| (\sin\theta) = \|\vec{v}\| \cdot \|\vec{\omega}\| \sqrt{1-\cos^2\theta}$$

Recall that $\vec{v} \cdot \vec{\omega} = \|\vec{v}\| \|\vec{\omega}\| \cos\theta$

So $\cos\theta = \dfrac{\vec{v}\cdot\vec{\omega}}{\|\vec{v}\| \cdot \|\vec{\omega}\|}$

Thus

$$\|\vec{v}\| \cdot \|\vec{\omega}\| \sin\theta = \|\vec{v}\| \cdot \|\vec{\omega}\| \sqrt{1 - \left(\frac{\vec{v}\cdot\vec{\omega}}{\|\vec{v}\| \cdot \|\vec{\omega}\|}\right)^2}$$

$$= \sqrt{\|\vec{v}\|^2 \|\vec{\omega}\|^2 \left(1 - \left(\frac{\vec{v}\cdot\vec{\omega}}{\|\vec{v}\| \cdot \|\vec{\omega}\|}\right)^2\right)}$$

$$= \sqrt{\|\vec{v}\|^2 \|\vec{\omega}\|^2 \cdot 1 - (\vec{v}\cdot\vec{\omega})^2}$$

$$= \sqrt{(v_1^2+v_2^2+v_3^2)(\omega_1^2+\omega_2^2+\omega_3^2) - (v_1\omega_1+v_2\omega_2+v_3\omega_3)^2}$$

now simplify and check

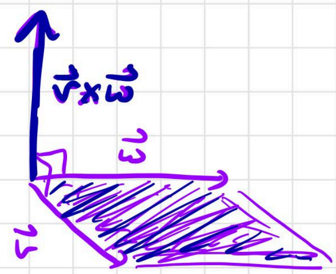
$\|\vec{v} \times \vec{\omega}\|^2 = \ldots =$ the same thing.

QED

<u>Thm</u>: $\vec{v} \times \vec{w}$ is $\perp$ to both $\vec{v}$ and $\vec{w}$

and $\|\vec{v} \times \vec{w}\| = \|\vec{v}\| \cdot \|\vec{w}\| \sin(\theta)$

which is the <u>area of</u> <u>the parallelogram</u>



Note that if $\vec{v} = r\vec{w}$
then there is no parallelogram
and so its area $= 0$
and $\|\vec{v} \times \vec{w}\| = 0$ in that case.

$\|(r\vec{w}) \times \vec{w}\| = \left\| \begin{pmatrix} rw_2w_3 - rw_3w_2 \\ rw_3w_1 - rw_1w_3 \\ rw_1w_2 - rw_2w_1 \end{pmatrix} \right\| = \left\| \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\| = 0$

<u>Proof idea</u>:

Useful Trig Fact $\cos^2\theta + \sin^2\theta = 1$

$\sin^2\theta = 1 - \cos^2\theta$

$\sin\theta = \sqrt{1-\cos^2\theta}$

Useful Trig Fact $\cos^2\theta + \sin^2\theta = 1$

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$\sin\theta = \sqrt{1-\cos^2\theta}$

when $\sin\theta \geq 0$

Thus

$LHS = \|\vec{v}\| \cdot \|\vec{w}\| (\sin\theta) = \|\vec{v}\| \cdot \|\vec{w}\| \sqrt{1-\cos^2\theta}$

Recall that $\vec{v} \cdot \vec{w} = \|\vec{v}\| \|\vec{w}\| \cos\theta$

So $\cos\theta = \dfrac{\vec{v}\cdot\vec{w}}{\|\vec{v}\| \cdot \|\vec{w}\|}$

Thus

$LHS = \|\vec{v}\| \cdot \|\vec{w}\| \sin\theta = \|\vec{v}\| \cdot \|\vec{w}\| \sqrt{1 - \left(\dfrac{\vec{v}\cdot\vec{w}}{\|\vec{v}\| \cdot \|\vec{w}\|}\right)^2}$

$= \sqrt{\|\vec{v}\|^2 \|\vec{w}\|^2 \left(1 - \left(\dfrac{\vec{v}\cdot\vec{w}}{\|\vec{v}\| \cdot \|\vec{w}\|}\right)^2\right)}$

$$= \sqrt{\|\vec{v}\|^2 \|\vec{w}\|^2 \cdot 1 - (\vec{v}\cdot\vec{w})^2}$$

$$= \sqrt{(v_1^2+v_2^2+v_3^2)(w_1^2+w_2^2+w_3^2) - (v_1w_1+v_2w_2+v_3w_3)^2}$$

= simplify this

show the same

$$RHS = \|(\vec{v}\times\vec{w})\| = \left\| \begin{pmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{pmatrix} \right\|$$

$$= \sqrt{(v_2w_3-v_3w_2)^2+(v_3w_1-v_1w_3)^2+(v_1w_2-v_2w_1)^2}$$

= simplify this

RHS = LHS

Thus ~~and~~

$$\|\vec{v}\times\vec{w}\| = \|\vec{v}\|\cdot\|\vec{w}\|\sin(\theta)$$

Next show:

$\vec{v}\times\vec{w}$ is $\perp$ to both $\vec{v}$ and $\vec{w}$

To do that show

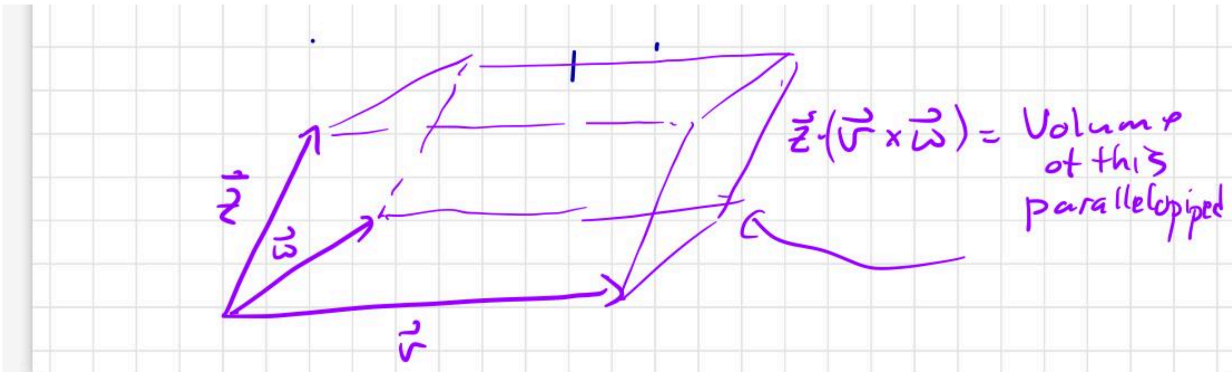
$$(\vec{v}\times\vec{w})\cdot\vec{v} = 0$$

Extra credit.

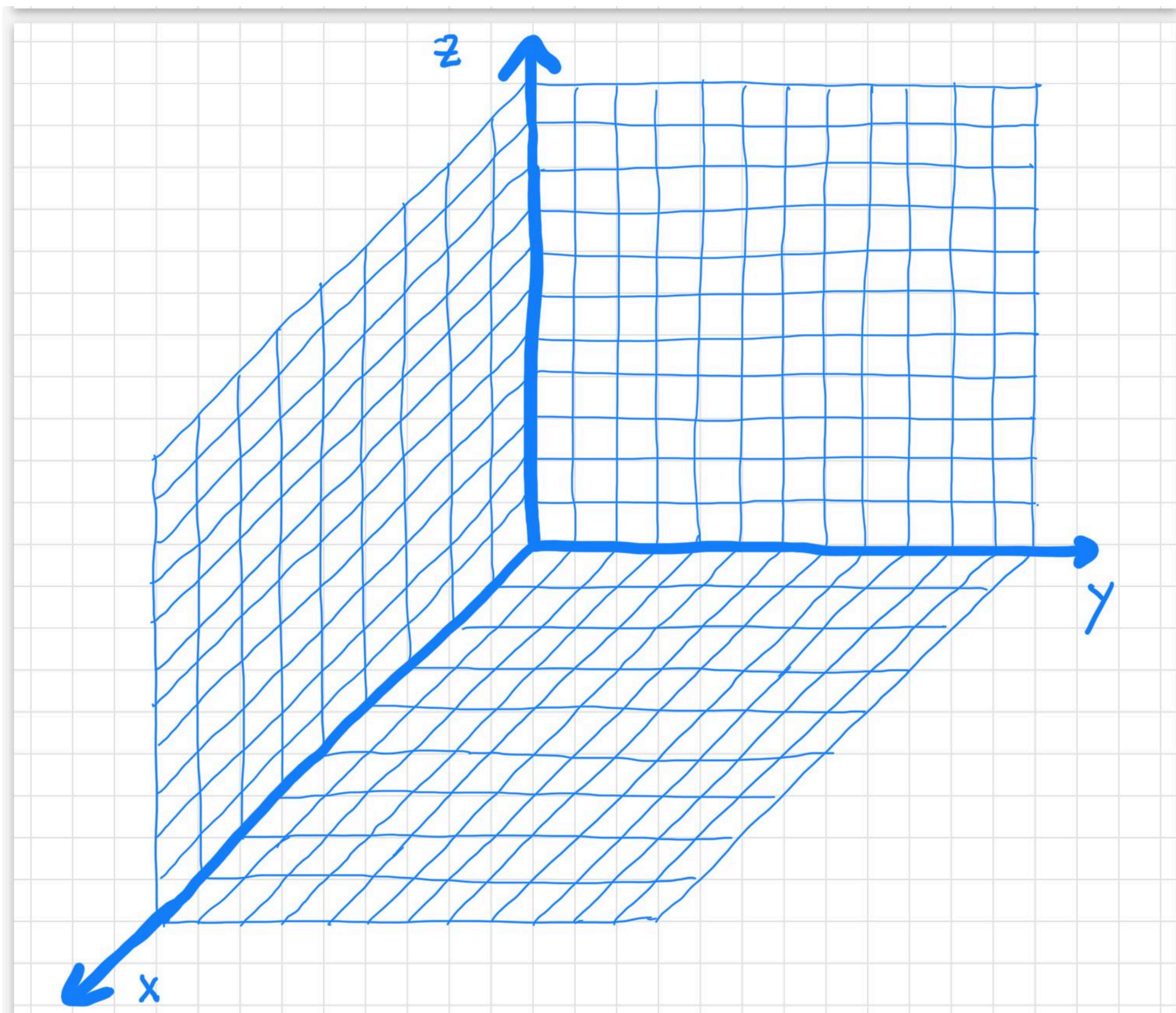
also show

$$(\vec{v}\times\vec{w})\cdot\vec{w} = 0$$

QED



A chart you can print out and use as 3D graph paper:



Reference Sheet:

Useful facts $r,s,t \in \mathbb{R}$ $\vec{a},\vec{b},\vec{c} \in \mathbb{R}^3$

Magnitude

$\|\vec{v}\| \in \mathbb{R}$

$$\left\|\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}\right\| = \sqrt{v_1^2+v_2^2+v_3^2}$$

a unit vector $\vec{u}$ has $\|\vec{u}\|=1$

Vector Addition and Subtraction

$\vec{v}+\vec{w} \in \mathbb{R}^3$

$$\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}+\begin{pmatrix}w_1\\w_2\\w_3\end{pmatrix}=\begin{pmatrix}v_1+w_1\\v_2+w_2\\v_3+w_3\end{pmatrix}$$

$\vec{v}+\vec{0}=\vec{v}$

$\vec{v}-\vec{w} \in \mathbb{R}^3$

$$\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}-\begin{pmatrix}w_1\\w_2\\w_3\end{pmatrix}=\begin{pmatrix}v_1-w_1\\v_2-w_2\\v_3-w_3\end{pmatrix}$$

$\vec{v}-\vec{v}=\vec{0}$

Scalar Product

$r\vec{a} \in \mathbb{R}^3$

$$r\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}=\begin{pmatrix}ra_1\\ra_2\\ra_3\end{pmatrix}$$

$r(\vec{a}+\vec{b})=r\vec{a}+r\vec{b}$

$(r+s)\vec{a}=r\vec{a}+s\vec{a}$

$r(s\vec{a})=(rs)\vec{a}$

$\vec{a}-\vec{b}=\vec{a}+(-1)\vec{b}$

$\|r\vec{a}\|=|r|\,\|\vec{a}\|$

$\frac{\vec{a}}{\|\vec{a}\|}=\left(\frac{1}{\|\vec{a}\|}\right)\vec{a}$ is a unit vector

Dot Product

$\vec{a}\cdot\vec{b} \in \mathbb{R}$

$$\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\cdot\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}=a_1b_1+a_2b_2+a_3b_3$$

$\vec{a}\cdot(\vec{b}+\vec{c})=\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}$

$(\vec{a}+\vec{b})\cdot\vec{c}=\vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{c}$

$\vec{a}\cdot\vec{b}=|\vec{a}|\cdot|\vec{b}|\cos\theta$

$\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$

$\vec{a}\cdot\vec{a}=\|\vec{a}\|^2$

$\text{Proj}_{\vec{a}}\vec{b}=\frac{\vec{b}\cdot\vec{a}}{\vec{a}\cdot\vec{a}}\vec{a}$

Cross Product

$\vec{a}\times\vec{b} \in \mathbb{R}^3$

$$\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\times\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}=\begin{pmatrix}a_2b_3-a_3b_2\\a_3b_1-a_1b_3\\a_1b_2-a_2b_1\end{pmatrix}$$

$\vec{a}\times(\vec{b}+\vec{c})=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$

$(\vec{a}+\vec{b})\times\vec{c}=\vec{a}\times\vec{c}+\vec{b}\times\vec{c}$

$\|\vec{a}\times\vec{b}\|=|\vec{a}|\cdot|\vec{b}|\sin\theta$

$(\vec{a}\times\vec{b})\cdot\vec{b}=0$

$(\vec{a}\times\vec{b})\cdot\vec{a}=0$

$\vec{a}\times\vec{b}=-\vec{b}\times\vec{a}$

Response to student question on vector notation: ([video])

Notation for Vectors in $\mathbb{R}^3$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $\quad \langle x, y, z \rangle$

(easiest notation to use) $\quad$ textbook notation (easier to type)

In physics $\quad x\hat{i} + y\hat{j} + z\hat{k}$

$$2\hat{i} + 3\hat{j} + 5\hat{k} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$$

Warning $\quad \hat{i} = 1\hat{i} + 0\hat{j} + 0\hat{k} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$$\hat{j} = 0\hat{i} + 1\hat{j} + 0\hat{k} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\hat{k} = 0\hat{i} + 0\hat{j} + 1\hat{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Addition:

$$(2\hat{i} + 5\hat{j}) + (4\hat{i} + 8\hat{k})$$

$$= \begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix} + \begin{pmatrix} 4 \\ 0 \\ 8 \end{pmatrix} = \begin{pmatrix} 2+4 \\ 5+0 \\ 0+8 \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \\ 8 \end{pmatrix}$$

$$= 6\hat{i} + 5\hat{j} + 8\hat{k}$$

Scalar Multiplication

$$5\left(1\hat{i}+8\hat{j}+2\hat{k}\right) = 5\begin{pmatrix}1\\8\\2\end{pmatrix} = \begin{pmatrix}5\\40\\10\end{pmatrix} = 5\hat{i}+40\hat{j}+10\hat{k}$$

Dot product

$$\left(2\hat{i}+4\hat{j}\right)\cdot\left(5\hat{j}+2\hat{k}\right) = \begin{pmatrix}2\\4\\0\end{pmatrix}\cdot\begin{pmatrix}0\\5\\2\end{pmatrix}$$

$$= 2\cdot 0 + 4\cdot 5 + 0\cdot 2 = 20$$

Cross product

$$\left(2\hat{i}+4\hat{j}\right)\times\left(5\hat{j}+2\hat{k}\right) = \begin{pmatrix}2\\4\\0\end{pmatrix}\times\begin{pmatrix}0\\5\\2\end{pmatrix} =$$

$$= \begin{pmatrix}4\cdot 2-0\cdot 5\\0\cdot 0-2\cdot 2\\2\cdot 5-4\cdot 0\end{pmatrix} = \begin{pmatrix}8\\-4\\10\end{pmatrix} = 8\hat{i}-4\hat{j}+10\hat{k}$$

Check to be sure you watched all the videos and did all the classwork before doing homework.

Old Lehman Homework Read 11.3-11.4 and do all odd problems from
Calculus with Early Transcendentals by Larson, Hostetler, and Edwards Ed4
I copied the questions below so you do not need the book, but might find it useful to have as a resource (its only $20 used).

HW 11.3 odd problems, 11.4 odd problems

$\boxed{11.3}$ (1) – (7) find (a) $\vec{u}\cdot\vec{v}$ (b) $\vec{u}\cdot\vec{u}$ (c) $\|\vec{u}\|^2$ (d) $(\vec{u}\cdot\vec{v})\vec{v}$ (e) $\vec{u}\cdot(2\vec{v})$

(1) $\vec{u} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ $\vec{v} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$

Show all work then check:

Answer (a) -6 (b) 25 (c) 25 (d) $\begin{pmatrix} -12 \\ 18 \end{pmatrix}$ (e) -12

(2) $\vec{u} = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$ $\vec{v} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$

Show all work then check:

Answer (a) -17 (b) 26 (c) 26 (d) $\begin{pmatrix} 51 \\ -34 \end{pmatrix}$ (e) -34

(5) $\vec{u} = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$ $\vec{v} = \begin{pmatrix} 0 \\ 6 \\ 5 \end{pmatrix}$

Show all work, then check

Answers (a) 2 (b) 29 (c) 29 (d) $\begin{pmatrix} 0 \\ 12 \\ 10 \end{pmatrix}$ (e) 4

(7) $\vec{u} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ $\vec{v} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

Show all work, then check

Answers (a) 1 (b) 6 (c) 6 (d) $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ (e) 2

(11)–(15) Find the angle between the vectors

Show work!

(11) $\vec{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ answer: 90° or $\frac{\pi}{2}$

(13) $\vec{u} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ $\vec{v} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$ answer $\text{Arccos}\left(\frac{-6+4}{\sqrt{10}\sqrt{20}}\right) = \cos^{-1}\left(\frac{-\sqrt{2}}{10}\right)$

(15) $\vec{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\vec{v} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ answer $\text{Arccos}\left(\frac{2}{\sqrt{3}\sqrt{6}}\right) = \cos^{-1}\left(\frac{\sqrt{2}}{3}\right)$

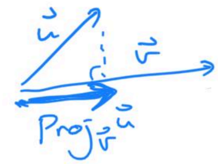
(19)–(25) Are $\vec{u}$ and $\vec{v}$ parallel, orthogonal, or neither?

(19) $\vec{u} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$ $\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ answer: neither (write why)

(21) $\vec{u} = \begin{pmatrix} 0 \\ 2 \\ 6 \end{pmatrix}$ $\vec{v} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$ answer: parallel (write why)

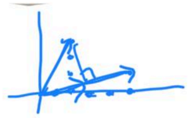
(23) $\vec{u} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$ $\vec{v} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$ answer: orthogonal (write why)

(47) - (49) Find $\text{Proj}_{\vec{v}} \vec{u} = \dfrac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v}$ and Plot:



(47) $\vec{u} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ $\vec{v} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$

Show all work answer: $\begin{pmatrix} 5/2 \\ 1/2 \end{pmatrix}$



(49) $\vec{u} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ $\vec{v} = \begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix}$ answer $\begin{pmatrix} 0 \\ 33/25 \\ 44/25 \end{pmatrix} \approx \begin{pmatrix} 0 \\ 1.32 \\ 1.76 \end{pmatrix}$

Chapter 11.4 (odd problems) $\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \times \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{pmatrix}$

(1) $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ Show work answer $\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$

(3) $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ answer $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

(5) $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ answer $\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$

(7) - (9) Find (a) $\vec{u} \times \vec{v}$ (b) $\vec{v} \times \vec{u}$ (c) $\vec{v} \times \vec{v}$

(7) $\vec{u} = \begin{pmatrix} -2 \\ 3 \\ 4 \end{pmatrix}$ $\vec{v} = \begin{pmatrix} 3 \\ 7 \\ 2 \end{pmatrix}$ Answer (a) $\begin{pmatrix} -22 \\ 16 \\ -23 \end{pmatrix}$ (b) $\begin{pmatrix} 22 \\ -16 \\ 23 \end{pmatrix}$ (c) $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

(9) $\vec{u} = \begin{pmatrix} 7 \\ 3 \\ 2 \end{pmatrix}$ $\vec{v} = \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix}$ Answer (a) $\begin{pmatrix} 17 \\ -33 \\ 10 \end{pmatrix}$ (b) $\begin{pmatrix} -17 \\ 33 \\ -10 \end{pmatrix}$ (c) $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

(11)-(15) Find $\vec{u}\cdot(\vec{u}\times\vec{v})$ and $\vec{v}\cdot(\vec{u}\times\vec{v})$

Show all work.

Your answers should be 0

because $\vec{u}\times\vec{v}$ is $\perp$ to both $\vec{u}$ and $\vec{v}$

(11) $\vec{u}=\begin{pmatrix}2\\-3\\1\end{pmatrix}$ $\vec{v}=\begin{pmatrix}1\\-2\\1\end{pmatrix}$

(13) $\vec{u}=\begin{pmatrix}12\\-3\\0\end{pmatrix}$ $\vec{v}=\begin{pmatrix}-2\\5\\0\end{pmatrix}$

(15) $\vec{u}=\begin{pmatrix}1\\1\\1\end{pmatrix}$ $\vec{v}=\begin{pmatrix}2\\1\\-1\end{pmatrix}$

(21) - (23) Find the unit vector $\perp$ to $\vec{u}$ and $\vec{v}$ by taking $\vec{u} \times \vec{v}$ and then $\dfrac{\vec{u} \times \vec{v}}{\|\vec{u} \times \vec{v}\|}$

(21) $\vec{u} = \begin{pmatrix} -4 \\ -3.5 \\ 7 \end{pmatrix}$ $\vec{v} = \begin{pmatrix} -1 \\ 8 \\ 4 \end{pmatrix}$ answer $\begin{pmatrix} -140/\sqrt{24965} \\ -46/\sqrt{24965} \\ 52/\sqrt{24965} \end{pmatrix}$

(23) $\vec{u} = \begin{pmatrix} -3 \\ 2 \\ -5 \end{pmatrix}$ $\vec{v} = \begin{pmatrix} 1/2 \\ -3/4 \\ 1/10 \end{pmatrix}$ answer $\begin{pmatrix} -71/\sqrt{7602} \\ -44/\sqrt{7602} \\ 25/\sqrt{7602} \end{pmatrix}$

(41) - (43) Find the volume of the parallelopiped

$\vec{u} \cdot (\vec{v} \times \vec{w})$

(41) $\vec{u} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$ $\vec{v} = \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix}$ $\vec{w} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ Answer: 10

(43) $\vec{u} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ $\vec{v} = \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}$ $\vec{w} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ Answer: 6

Review Differentiation from your Calculus I notes or text before continuing to Lesson 3.