Summary Calculus Workshop 2019 🌤️
University of Arizona

Calculus 3 Workshop: Multidimensional Arclength and Scalar Line Integrals

Worksheet #4 📝
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Introduction

The **line integral of a vector field** plays a crucial role in vector calculus. Out of the four **fundamental theorems** of vector calculus, three of them involve line integrals of vector fields. **Green's theorem** and **Stokes's theorem** relate line integrals around closed curves to double integrals or surface integrals. If you have a **conservative vector field**, you can relate the line integral over a curve to quantities just at the curve's two boundary points. So, it's worth the effort to develop a good understanding of line integrals! 😉

We won't have time to cover all of these fundamental theorems, but we will at least familiarize ourselves with **line integrals**. This way, you'll have a head start in understanding these more advanced theorems in vector calculus in your semester course. 👍

We'll first start with line integrals over **scalar fields** and then move on to line integrals over **vector fields**. 🤓

Arclength Review 📚

1. Consider the function

$$f(x) = \frac{1 - x^2}{2}$$

on the interval $-1 \leq x \leq 1$. What is the **arclength** of this curve? Here is the derivation for the arclength formula of a single-variable function. ✏️

📖 **Arclength of a One-Dimensional Curve**

The **arclength** of a curve over the interval $x \in [a, b]$ described by a **single-variable function** $y = f(x)$ is simply the integral of the arclength element ds:

$$\int ds = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

where

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

and we are integrating with respect to x.

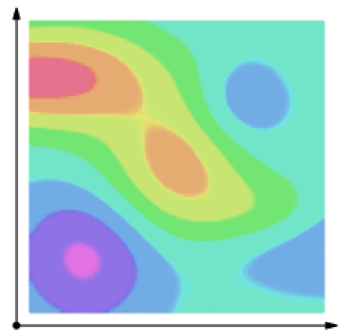
Line Integral over a Scalar Field[1]

In this Worksheet, we will continue to explore the same surfaces below we worked with in **Worksheet #2** and **Worksheet #3**. 🎨

A. $f(x,y) = y$
B. $f(x,y) = x$
C. $f(x,y) = x + y$
D. $f(x,y) = 2x + y$
E. $f(x,y) = x + 2y$
F. $f(x,y) = x^2 + y^2$
G. $f(x,y) = 4 - x^2 - y^2$
H. $f(x,y) = x^2 - y^2$

📐 **Geometric Intuition of Scalar Line Integral**

A two-dimensional **line integral over a scalar field** f is the **(signed) cross-sectional area** under the surface $z = f(x,y)$ along a two-dimensional curve C that lies in the xy-plane. (See the animation.) 👉



When the surface $z = f(x,y)$ is simple, we can compute this cross-sectional area easily using **elementary geometry**. 👍

Line Integrals using Geometry 📐

2. Compute the line integral of surfaces C, F, and H along the two-dimensional straight line C that connects… ✏️

   a. $(0,0)$ and $(10,0)$.
   b. $(0,0)$ and $(0,10)$.
   c. $(0,0)$ and $(10,10)$.

3. Compute the line integral of surface F over the two-dimensional circle C centered at the origin with radius equal to 10. ✏️

Parametrizing a Line/Curve

Recall how we determined equations in **vector form** for lines and curves in two and three dimensions in **Worksheet #1**. Refresh your memory and practice by determining the **vector function** $\vec{r}(t) = \langle x(t), y(t) \rangle$ for the two-dimensional lines and curves C from the previous problems. Determining the vector function for a line or curve is also known as **parametrizing the line or curve** or finding a **parametrization for the line or curve**. 🤓

4. Write a **vector function** of (i.e., **parametrize**) the two-dimensional straight line C that connects… ✏️

   a. $(0,0)$ and $(10,0)$.
   b. $(0,0)$ and $(0,10)$.
   c. $(0,0)$ and $(10,10)$.

[1] The animation in this section for the line integral over a scalar field is taken from the Wikipedia article on line integrals.

5. **Parametrize** the two-dimensional circle C centered at the origin with radius equal to 10. ✏️

Scalar Field and Line Integrals

📖 **Scalar Field**

A **scalar field** associates a scalar value to every point in a domain. For example, the surfaces A-H all define a **two-dimensional scalar field** that associates a height $z = f(x, y)$ for every point $(x, y) \in \mathbb{R}^2$.

Examples of scalar fields in physics include **temperature** throughout space or **pressure** in a fluid.

The **line integral** over a **two-dimensional scalar field** $f(x, y)$ along a curve C is an integral with respect to the **arclength element** ds of the **parametrized curve** C:

$$\int_C f(x, y)\, ds$$

where

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

📖 **Line Integral over a Scalar Field**

The **line integral** over a **two-dimensional scalar field** $f(x, y)$ along a curve C that starts at $t = a$ and ends at $t = b$ is

$$\int_C f(x, y)\, ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

6. Repeat Problems #2-3 using the above formula for the **line integral over a scalar field** and your parametrizations in Problems #4-5. What do you notice? ✏️

7. Redo Problem #2 but with C in the opposite **orientation**, e.g., going from $(10, 0)$ to $(0, 0)$ instead. What do you notice? ✏️

Be Creative 🎨

8. Create your own surface (i.e., scalar field) $z = f(x, y)$ and your own curve C. Compute the **line integral** over this scalar field along the curve you chose. Plot the surface $z = f(x, y)$ and your parametrized curve C using **GeoGebra**. 🖥️

Arclength of Multidimensional Curves

Recall the arclength of a curve described by a **single-variable function**.

📖 Arclength of a One-Dimensional Curve

The **arclength** of a curve over the interval $x \in [a, b]$ described by a **single-variable function** $y = f(x)$ is simply the integral of the arclength element ds:

$$\int ds = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

where

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

and we are integrating with respect to x.

The **arclength of a two-dimensional curve** is the same but with a different ds, i.e., we use the ds that we had in earlier that represents the arclength element of the parametrized curve C.

📖 Arclength of a Two-Dimensional Curve

The **arclength** of a two-dimensional curve C that starts at $t = a$ and ends at $t = b$ is, again, simply the integral of the arclength element ds:

$$\int ds = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

where

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

and we are integrating with respect to t.

Practice ✏️

9. Find a **two-dimensional parametrization of the curve** in Problem #1. Then find the arclength of this curve using the formula for the **arclength of a two dimensional curve**. What do you notice? ✏️

10. Find the **arclength** of the curve you created in Problem #8. ✏️

11. How does the **formula for the arclength of a two-dimensional curve** relate to how you found the distance traveled by the Pokéball traveling along a curve in two dimensions in [Worksheet #1]? See Problems #17 and #24 in [Worksheet #1]. Based on this, what do you suspect would be the **formula for the arclength of a three dimensional curve**? 🤔