Q.1

Classes of codes. Consider the code $\{0, 01\}$

(a) Is it instantaneous?

(b) Is it uniquely decodable?

(c) Is it nonsingular?

Solution: *Codes.*

(a) No, the code is not instantaneous, since the first codeword, 0, is a prefix of the second codeword, 01.

(b) Yes, the code is uniquely decodable. Given a sequence of codewords, first isolate occurrences of 01 (i.e., find all the ones) and then parse the rest into 0's.

(c) Yes, all uniquely decodable codes are non-singular.

Q.2

Find the Differential Entropy of the continuous Gaussian random variable with mean zero and variance σ^2.

Differential Entropy, $h(x) = -\int_{-\infty}^{\infty} f(x) \ln f(x)\, dx$

Gaussian RV, $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2}$

Thus $h(x) = -\int_{-\infty}^{\infty} f(x)\left[\frac{-x^2}{2\sigma^2} - \ln\sqrt{2\pi\sigma^2}\right] dx$

$= \frac{1}{2\sigma^2} \underbrace{\int_{-\infty}^{\infty} x^2 f(x)\, dx}_{E[X^2]} + \frac{1}{2}\ln(2\pi\sigma^2) \underbrace{\int_{-\infty}^{\infty} f(x)\, dx}_{=1}$

$= \frac{1}{2\sigma^2}\left[\sigma^2 + 0\right] + \frac{1}{2}\ln 2\pi\sigma^2$

$= \frac{1}{2} + \frac{1}{2}\ln 2\pi\sigma^2$

$h(x) = \frac{1}{2}\ln 2\pi e\sigma^2$ nats

Q.3

Shannon codes and Huffman codes. Consider a random variable X that takes on four values with probabilities $(\frac{1}{3}, \frac{1}{3}, \frac{1}{4}, \frac{1}{12})$.

(a) Construct a Huffman code for this random variable.

(b) Show that there exist two different sets of optimal lengths for the codewords; namely, show that codeword length assignments $(1, 2, 3, 3)$ and $(2, 2, 2, 2)$ are both optimal.

(a) Applying the Huffman algorithm gives us the following table

Code	Symbol	Probability			
0	1	1/3	1/3	2/3	1
11	2	1/3	1/3	1/3	
101	3	1/4	1/3		
100	4	1/12			

which gives codeword lengths of 1,2,3,3 for the different codewords.

(b) Both set of lengths 1,2,3,3 and 2,2,2,2 satisfy the Kraft inequality, and they both achieve the same expected length (2 bits) for the above distribution. Therefore they are both optimal.

Q.4

Consider a random variable X that takes six values $\{A, B, C, D, E, F\}$ with probabilities $0.5, 0.25, 0.1, 0.05, 0.05$, and 0.05, respectively.

a) Construct a quaternary Huffman code for this random variable [i.e., a code over an alphabet of four symbols (call them a, b, c and d)]. What is the average length of this code?

b) One way to construct a binary code for the random variable is to start with a quaternary code and convert the symbols into binary using the mapping a $\rightarrow$ 00, b $\rightarrow$ 01, c $\rightarrow$ 10, and d $\rightarrow$11. What is the average length of the binary code for the random variable above constructed by this process?

Solution: Since the number of symbols, i.e., 6 is not of the form $1+k(D-1)$, we need to add a dummy symbol of probability 0 to bring it to this form. In this case, drawing up the Huffman tree is straightforward.

Code	Symbol	Prob.		
a	A	0.5	0.5	1.0
b	B	0.25	0.25	
d	C	0.1	0.15	
ca	D	0.05	0.1	
cb	E	0.05		
cc	F	0.05		
cd	G	0.0		

The average length of this code is $1 \times 0.85 + 2 \times 0.15 = 1.15$ quaternary symbols.

Solution: The code constructed by the above process is $A \to 00$, $B \to 01$, $C \to 11$, $D \to 1000$, $E \to 1001$, and $F \to 1010$, and the average length is $2 \times 0.85 + 4 \times 0.15 = 2.3$ bits.

Q.5

Capacity of the carrier pigeon channel. Consider a commander of an army besieged a fort for whom the only means of communication to his allies is a set of carrier pigeons. Assume that each carrier pigeon can carry one letter (8 bits), and assume that pigeons are released once every 5 minutes, and that each pigeon takes exactly 3 minutes to reach its destination.

Assuming all the pigeons reach safely, what is the capacity of this link in bits/hour?

Solution:

The channel sends 8 bits every 5 minutes, or 96 bits/hour.

Q.6

Show that the minimum value of energy per bit, E_b/N_0 required for reliable communication through an AWGN channel is -1.6dB as channel bandwidth tends to infinity.

For reliable Communication,

$R < C$, which for band-limited AWGN channel is

$$R < W \log_2\left(1 + \frac{P}{N_0 W}\right)$$

$$\frac{R}{W} < \log_2\left(1 + \frac{P}{N_0 W}\right)$$

$$r \triangleq \frac{R}{W} \triangleq \text{BW efficiency}$$

and $$\mathcal{E}_b = \frac{\mathcal{E}}{\log_2 M} = \frac{P T_s}{\log_2 M} = \frac{P}{R}$$

So

$$r < \log_2\left(1 + \frac{\mathcal{E}_b \cdot R}{N_0 W}\right)$$

$$r < \log_2\left(1 + r \cdot \frac{\mathcal{E}_b}{N_0}\right)$$

or $$\boxed{\frac{\mathcal{E}_b}{N_0} > \frac{2^r - 1}{r}}$$

SNR/bit

(A measure of Power efficiency of a system)

This relation states a condition for reliable communication in terms of BW efficiency, r and power efficiency, $\frac{E_b}{N_0}$. The minimum value of E_b/N_0 for which reliable commⁿ is possible is obtained by letting $r \to 0$ (BW becomes ∞).

$$\frac{E_b}{N_0} > \lim_{r \to 0} \frac{2^{r}-1}{r}$$

$$\frac{E_b}{N_0} > \ln 2 = 0.693$$

$$\sim -1.6 \text{ dB}$$

let $2^{r}-1 = x$

$r = \log_2(1+x)$

$$\lim_{r \to 0} \frac{2^{r}-1}{r} = \lim_{x \to 0} \frac{x}{\log_2(1+x)}$$

$$= \ln 2$$

This is the minimum value of E_b/N_0 for any communication system. No system can transmit reliably below this limit ~~[illegible]~~

Q.7

Unused symbols. Show that the capacity of the channel with probability transition matrix

$$P_{y|x} = \begin{bmatrix} 2/3 & 1/3 & 0 \\ 1/3 & 1/3 & 1/3 \\ 0 & 1/3 & 2/3 \end{bmatrix} \tag{7.43}$$

is achieved by a distribution that places zero probability on one of input symbols. What is the capacity of this channel? Give an intuitive reason why that letter is not used.

Solution: *Unused symbols* Let the probabilities of the three input symbols be p_1, p_2 and p_3. Then the probabilities of the three output symbols can be easily calculated to be $(\frac{2}{3}p_1 + \frac{1}{3}p_2, \frac{1}{3}, \frac{1}{3}p_2 + \frac{2}{3}p_3)$, and therefore

$$
\begin{aligned}
I(X;Y) &= H(Y) - H(Y|X) && (7\\
&= H(\frac{2}{3}p_1 + \frac{1}{3}p_2, \frac{1}{3}, \frac{1}{3}p_2 + \frac{2}{3}p_3) - (p_1+p_3)H(\frac{2}{3}, \frac{1}{3}) - p_2 \log 3 && (7\\
&= H(\frac{1}{3} + \frac{1}{3}(p_1 - p_3), \frac{1}{3}, \frac{1}{3} - \frac{1}{3}(p_1 - p_3)) - (p_1+p_3)H(\frac{2}{3}, \frac{1}{3}) - (1 - p_1 - p_3)(7
\end{aligned}
$$

where we have substituted $p_2 = 1 - p_1 - p_3$. Now if we fix $p_1 + p_3$, then the second and third terms are fixed, and the first term is maximized if $p_1 - p_3 = 0$, i.e., if $p_1 = p_3$. (The same conclusion can be drawn from the symmetry of the problem.)

Now setting $p_1 = p_3$, we have

$$
\begin{aligned}
I(X;Y) &= H(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) - (p_1+p_3)H(\frac{2}{3}, \frac{1}{3}) - (1 - p_1 - p_3) \log 3 && (7.47)\\
&= \log 3 - (p_1+p_3)H(\frac{2}{3}, \frac{1}{3}) - (1 - p_1 - p_3) \log 3 && (7.48)\\
&= (p_1 + p_3)(\log 3 - H(\frac{2}{3}, \frac{1}{3})) && (7.49)
\end{aligned}
$$

which is maximized if $p_1 + p_3$ is as large as possible (since $\log 3 > H(\frac{2}{3}, \frac{1}{3})$). Therefore the maximizing distribution corresponds to $p_1 + p_3 = 1$, $p_1 = p_3$, and therefore $(p_1, p_2, p_3) = (\frac{1}{2}, 0, \frac{1}{2})$. The capacity of this channel $= \log 3 - H(\frac{2}{3}, \frac{1}{3}) = \log 3 - (\log 3 - \frac{2}{3}) = \frac{2}{3}$ bits.