Mar 25

One can identify complex numbers and vector on the plane $\mathbb{R}^2$ as $a+ib \equiv (a,b)$. Find the matrix $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ such that, using this identification,

$$e^{i\phi}(a+ib) \equiv \left( B \begin{bmatrix} a \\ b \end{bmatrix} \right)^T$$

where $T$ denotes the transpose. Now use this to explain geometrically the action of the matrix $B$ on the vector $\begin{bmatrix} a \\ b \end{bmatrix}$.

Think of (a,b) a column vector.

p=greek letter

Want you to find B with
e^(ip)*(a+ib) = (B * (a,b)^T)^T

e^(ip)*(a+ib)
=(cos(p)+i*sin(p))(a+ib)
=cos(p)a+i*sin(p)*a+cos(p)ib+i*sin(p)*ib
=(a*cos(p)-bsin(p))+( a*sin(p)+b*cos(p) )i

a+ib=[a,b] -> [a*cos(p)-bsin(p)) , ( a*sin(p)+b*cos(p) ]

(B [a,b]^T)^T=[a*cos(p)-bsin(p)) , ( a*sin(p)+b*cos(p) ]

real number=
other real number=

e^(ip)=cos(p)+i*sin(p)

Feb 3/2

Ex
A=
10
01
00

T_A is not onto since A**x**=e_3 has no solution.
e_3=
0
0
1

One-to-one ⇔
for all y, Ax=y has 0 or 1 solutions.
rref(A|y)=
10|?
01|?
00|?

10|0
01|0 no solution
00|1

10|6
01|3 unique solution
00|0

Never infinitely many since each column has a leading 1.

Ex
A=
100
010

Onto
Since
RREF(A|y)=
100|?
010|?
No 0=1 line since 1 in every row to the left of the augmentation line so there is a solution.

Not one-to-one since the 3rd variable is free.

(AB)^-1=B^-1 A^-1

We want (AB)^-1*AB=Id.

(B^-1 A^-1) * (AB)=B^-1 A^-1 A B= B^-1 Id B=B^-1 B=Id.

(A^-1 B^-1) * (AB)=A^-1 B^-1 A B

Feb 4 Office hours

**32.** $A$ is a $4 \times 3$ matrix, $A = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3]$, such that $\{\mathbf{a}_1, \mathbf{a}_2\}$ is linearly independent and $\mathbf{a}_3$ is not in Span $\{\mathbf{a}_1, \mathbf{a}_2\}$.

1. a_1,a_2 are linearly independent.
2. a_3 is not in the span(a_1,a_2)

A=[a_1,a_2,a_3]
What is RREF(A).

A=
???
???
???
???

x in span(v,w) ⇔
x is a linear combination of v and w.

Claim if x in span(v,w), then x,v,w are dependent.
Proof:
x=r_1 v+r_2 w
-1*x+r_1 v +r_2 w=0
0 x+0 v + 0 w=0
So two solutions to

Claim:
1. a_1,a_2 are linearly independent.

2. $a_3$ is not in the span($a_1$,$a_2$)
Implies $a_1$,$a_2$,$a_3$ are linear independent.

Suppose $x_1 a_1 + x_2 a_2 + x_3 a_3 = 0$.
We want to show $x_1=0$, $x_2=0$, $x_3=0$.

Case $x_3$ is not 0:
$x_1 \mathbf{a_1} + x_2 \mathbf{a_2} + x_3 \mathbf{a_3} = 0$
$\mathbf{a_3} = (-x_1/x_3) \mathbf{a_1} + (-x_2/x_3) \mathbf{a_2}$
$a_3$ is a linear combination of $a_1$ and $a_2$.
$\mathbf{a_3}$ is in span($\mathbf{a_1}$,$\mathbf{a_2}$)
This can't happen so we must $x_3=0$.

Case $x_3=0$:
$x_1 a_1 + x_2 a_2=0$.
What does $a_1$ and $a_2$ being linearly independent say about $x_1$ and $x_2$?
Since $a_1$ and $a_2$ are linearly independent, then $x_1=0$ and $x_2=0$.

So $x_1=0$, $x_2=0$, $x_3=0$ is the only solution. So $a_1$, $a_2$, $a_3$ are linearly independent.

Thm
$a_1$,..$a_k$ are linearly independent if and only if span($a_1$,...$a_k$) is k dimensional.

Either $a_k$ is in span($a_1$,...$a_{k-1}$) or span($a_1$,...$a_k$) is one dimension bigger than span($a_1$,...$a_{k-1}$).

Ex
$a_1=$
1
0
0

$a_2=$
0
1
0

$a_3=$
1
1
6

RREF(A)=

1 0 0
0 1 0
0 0 1
0 0 0

EF(A)

[] *  *
0 [] *
0 0 []
0 0 0

**34.** How many pivot columns must a $5 \times 7$ matrix have if its columns span $\mathbb{R}^5$? Why?

Thm
A=[v_1,...v_n]
span(v_1,..v_n)=R^m
if and only if
every row of A has a leading 1.

n=7
m=5
A has 5 rows.
So A must have 5 leading 1s.
A=
4  1  6
-7 5  3
9 -3  3

A**x**=0

(1)v_1+(2)v_2+(-1)v_3=0

**x**=
x
y
z

x=1
y=2
z=-1

A**x**
=
x v_1 + y v_2 + z v_3

**21.** **(T/F)** A linear transformation is a special type of function.

**22.** **(T/F)** Every matrix transformation is a linear transformation.

**23.** **(T/F)** If $A$ is a $3 \times 5$ matrix and $T$ is a transformation defined by $T(\mathbf{x}) = A\mathbf{x}$, then the domain of $T$ is $\mathbb{R}^3$.

**24.** **(T/F)** The codomain of the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is the set of all linear combinations of the columns of $A$.

**25.** **(T/F)** If $A$ is an $m \times n$ matrix, then the range of the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is $\mathbb{R}^m$.

**26.** **(T/F)** If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation and if $\mathbf{c}$ is in $\mathbb{R}^m$, then a uniqueness question is "Is $\mathbf{c}$ in the range of $T$?"

**T/F)** Every linear transformation is a matrix transformation.

**28.** **(T/F)** A linear transformation preserves the operations of vector addition and scalar multiplication.

T(**v**+**w**)=T(**v**)+T(**w**)
T(r**w**)=rT(**w**)

22 True
24
Ex
A=
0 0
0 0
What are linear combinations of the columns of A?
0
0
What is the codomain? R^2

T:R^n->R^m. R^m is codomain.

T=T_A.

“**c** in range(T)” ⇔ 3)

1) “there is at most one solution to A**x**=**c**?” <- uniqueness problem??
2) “there is exactly one solution to A**x**=**c**?” <- uniqueness problem??
3) “there is a solution to A**x**=**c**?”

c. Any $\mathbf{x}$ whose image under $T$ is $\mathbf{b}$ must satisfy equation (1). From (2), it is clear that equation (1) has a unique solution. So there is exactly one $\mathbf{x}$ whose image is $\mathbf{b}$.

d. The vector $\mathbf{c}$ is in the range of $T$ if $\mathbf{c}$ is the image of some $\mathbf{x}$ in $\mathbb{R}^2$, that is, if $\mathbf{c} = T(\mathbf{x})$ for some $\mathbf{x}$. This is just another way of asking if the system $A\mathbf{x} = \mathbf{c}$ is consistent. To find the answer, row reduce the augmented matrix:

$$\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 14 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -35 \end{bmatrix}$$

The third equation, $0 = -35$, shows that the system is inconsistent. So $\mathbf{c}$ is *not* in the range of $T$. ■

The question in Example 1(c) is a *uniqueness* problem for a system of linear equations, translated here into the language of matrix transformations: Is $\mathbf{b}$ the image of a *unique* $\mathbf{x}$ in $\mathbb{R}^n$? Similarly, Example 1(d) is an *existence* problem: Does there *exist* an $\mathbf{x}$ whose image is $\mathbf{c}$?

The next two matrix transformations can be viewed geometrically. They reinforce the dynamic view of a matrix as something that transforms vectors into other vectors. Section 2.7 contains other interesting examples connected with computer graphics.