14/12/22

Unit V

State Space Analysis

(1) **State:-** It is the group of variables which summarize the history of the given system inorder to predict the future of values.

(2) **State Variable:-** The smallest set of variables that determines the state of the system is known as state variables.

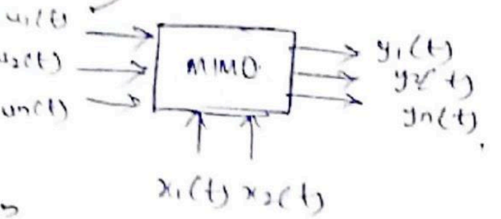
i.e. $x_1(t), x_2(t) \cdots\cdots x_n(t)$

(3) **State Vector:-** It is the vector which contains state variables.

$$[x_1(t) \cdot x_2(t) \cdots\cdots x_n(t)]$$

(4) **State Models:-**

When we are analyzing multi input and multioutput system with state space analysis. The output Eqn $y(t)$ and state Eqn $\dot{x}(t)$ or $\frac{d}{dt}(x(t))$ collectively to form state model.



State Eqn

$$\frac{d}{dt}(x_1(t)) = a_{11}x_1(t) + a_{12}x_2(t) + b_{11}u_1(t) + b_{12}u_2(t)$$

$$\dot{x}_1(t) = a_{11}x_1(t) + a_{12}x_2(t) + b_{11}u_1(t) + b_{12}u_2(t) \longrightarrow (i)$$

$$\frac{d}{dt} x_2(t) = a_{21} x_1(t) + a_{22} x_2(t) + b_{21} u_1(t) + b_{22} u_2(t)$$

$$\dot{x}_2(t) = a_{21} x_1(t) + a_{22} x_2(t) + b_{21} u_1(t) + b_{22} u_2(t) \quad \text{---} (2)$$

Eqn (1) & Eq (2) represent in Matrix form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

$$\boxed{\dot{X} = Ax + Bu}$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}; \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

O/p Eqns =

$$y_1(t) = c_{11} x_1(t) + c_{12} x_2(t) + D_{11} u_1(t) + D_{12} u_2(t)$$

$$y_2(t) = c_{21} x_1(t) + c_{22} x_2(t) + b_{21} u_1(t) + D_{22} u_2(t)$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

$$\boxed{y = Cx + Du}$$

* State transition matrix := $\phi(t)$

It is a matrix which satisfies linear homogeneous state Eqn. Then we can say state transition matrix.

$A' = A$

$\cancel{\phi(t)}$ $\boxed{\dot{X}_1 = Ax + Bu}$

$u = 0$

$\boxed{\dot{X}_1 = Ax}$

If { linear when i/p is zero }

$$\phi(t) = e^{At} = \mathcal{L}^{-1}[SI - A]^{-1}$$

* **properties of state transition Matrix := (STM)**

$$\boxed{\phi(t) = e^{At}}$$

① $\phi(0) = e^{A(0)} = 1 \cdot I$

~~② $\phi(t) =$~~

$$\boxed{\phi(0) = I}$$

② $\phi(-t) = e^{-At} = \frac{1}{e^{At}}$

$$\phi(-t) = \phi(t)^{-1}$$

③ $\phi(t)^m = e^{At^m} = e^{mAt}$

$$\phi(t)^m = \phi(tm)$$

④ $\phi(t_2 - t_1) \cdot \phi(t_1 - t_0) = e^{A(t_2 - t_1)} \times e^{A(t_1 - t_0)}$

$$= e^{A[t_2 - \cancel{t_1} + \cancel{t_1} - t_0]}$$

$$= e^{A[t_2 - t_0]}$$

$$= \phi(t_2 - t_0)$$

(5) $\phi[t_1+t_2] = e^{A(t_1+t_2)}$

$= e^{At_1} \times e^{At_2}$

$\phi(t_1+t_2) = \phi(t_1) \times \phi(t_2)$

Derivation of Transfer function from state model

$\dot{x} = Ax + BU \rightarrow (1)$

$y = Cx + DU \rightarrow (2)$

Apply L.T on b.s to Eqn (1)

$\mathcal{L}\left\{\frac{d}{dt}(x)\right\} = \mathcal{L}\{Ax + BU\}$

$sX(s) = AX(s) + BU(s)$

$IsX(s) - AX(s) = BU(s)$

$X(s)[sI - A] = BU(s)$

$X(s) = [sI-A]^{-1} BU(s)$

Apply LT on b.s to Eqn (2)

$y = Cx + DU$

$Y(s) = CX(s) + DU(s)$

$Y(s) = C\left[[sI-A]^{-1} BU(s) + DU(s)\right]$

$Y(s) = \left[C[sI-A]^{-1} B + D\right] U(s)$

$\frac{Y(s)}{U(s)}$

$\frac{Y(s)}{U(s)} = C[sI-A]^{-1} B + D$

pblm

The state Eqns of the time linear invariant system is given as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

and

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

find Transfer function

$$A = \begin{bmatrix} 0 & 5 \\ -1 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

$$D = [0]$$

$$\frac{Y(s)}{U(s)} = C[sI - A]^{-1} B + D$$

$$\frac{Y(s)}{U(s)} = C[sI - A]^{-1} B$$

$$[sI - A] = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 5 \\ -1 & -2 \end{bmatrix}$$

$$[sI - A] = \begin{bmatrix} s & -5 \\ 1 & s+2 \end{bmatrix}$$

$$[sI - A]^{-1} = \frac{\text{Adj } [sI - A]}{|sI - A|}$$

$$|sI - A| = s(s+2) - (1 \times -5$$

$$|sI - A| = s^2 + 2s + 5$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \text{Adj } A \quad \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\text{Adj}[sI-A] = \begin{bmatrix} s+2 & -(-5) \\ -1 & s \end{bmatrix} = \begin{bmatrix} s+2 & 5 \\ -1 & s \end{bmatrix}$$

$$[sI-A]^{-1} = \frac{\text{Adj}[sI-A]}{|sI-A|} = \frac{\begin{bmatrix} s+2 & 5 \\ -1 & s \end{bmatrix}}{s^2+2s+5}$$

$$\frac{Y(s)}{U(s)} = C[sI-A]^{-1}B$$

$$= \underset{1\times2}{\begin{bmatrix} 1 & 1 \end{bmatrix}} \underset{2\times2}{\begin{bmatrix} s+2 & 5 \\ -1 & s \end{bmatrix}} \underset{2\times1}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

$$= \frac{1}{s^2+2s+5} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} s+2+5 \\ -1+s \end{bmatrix}$$

$$= \frac{\begin{bmatrix} 1 & 1 \end{bmatrix}}{s^2+2s+5} \begin{bmatrix} s+7 \\ s-1 \end{bmatrix}$$

$$= \frac{1}{s^2+2s+5} \begin{bmatrix} 2s+6 \end{bmatrix}$$

$$\frac{Y(s)}{U(s)} = \frac{2s+6}{s^2+2s+5}$$

Obtain the state Model for the system is

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 10y = 5u(t)$$

Here y is the Output and u is the input Variables

Let $x_1(t) = y(t)$
$\dot{x}_1(t) = x_2(t)$
$\dot{x}_2(t) = x_3(t)$

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 10y(t) = 5u(t)$$

Apply L.T

$$s^2 Y(s) + 5sY(s) + 10Y(s) = 5U(s)$$

$$Y(s)\left[s^2+5s+10\right] = 5U(s)$$

$$\boxed{\frac{Y(s)}{U(s)} \Rightarrow \frac{5}{s^2+5s+10}}$$

$$Y(s) = 5x_1(t)$$

$$U(s) = s^2+5s+10$$

$$U(s) = \dot{x}_2(t) + 5\dot{x}_1(t) + 10x_1(t)$$

$$U(s) = \dot{x}_2(t) + 5x_2(t) + 10x_1(t)$$

$$\dot{x}_2(t) = -10x_1(t) - 5x_2(t) + u(t)$$

$$\begin{bmatrix} \dot{x}_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -10 & -5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} +$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

output Eqn

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 5 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u(t)$$

3) A linear time invariant system is $D^3y + 3D^2y + 3Dy + y = u$ where $D = \frac{d}{dt}$ and y = output u = input.

1) find state model (or) write state Eqn

2) find state transition Matrix (STM).

$$\frac{d^3y}{dt^3} + \frac{3d^2y}{dt^2} + 3\frac{dy}{dt} + y = u$$

Apply L.T on b.s.

$$S^3Y(s) + 3s^2Y(s) + 3sY(s) + Y(s) = U(s)$$

$$Y(s)\left[s^3 + 3s^2 + 3s + 1\right] = U(s)$$

$$\frac{Y(s)}{U(s)} = \frac{1}{s^3+3s^2+3s+1}$$

$Y(s) = 1$

$U(s) = s^3 + 3s^2 + 3s + 1$

Let

$y_1(t) = x_1(t)$

$s = \dot{x}_1(t) = x_2(t)$ — ①

$s^2 = \dot{x}_2(t) = x_3(t)$ — ②

$s^3 = \dot{x}_3(t) = x_4(t)$

$U(s) = s^3 + 3s^2 + 3s + 1$

$u(t) = \dot{x}_3(t) + 3\dot{x}_2(t) + 3\dot{x}_1(t) + 1\,x_1(t)$

$u(S) = \dot{x}_3(t) + 3x_3(t) + 3x_2(t) + x_1(t)$

$\dot{x}_3(t) = -x_1(t) - 3x_2(t) - 3x_3(t) + u(t)$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} u(t)$$

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} u(t)$$

Scanned with C

(ii) STM = $\phi(t) = e^{At} = L^{-1}[sI - A]^{-1}$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix}_{3\times3}$$

$$sI - A = s\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix}$$

$$[sI - A] = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 3 & s+3 \end{bmatrix}$$

$$\det|sI - A| = s[s(s+3) - (3(-1))] - (-1)[0 - (-1)]$$

$$= s[s^2 + 3s + 3] + 1$$

$$(a+b)^3 = a^3 + b^3 + 3a^2b + 3ab^2$$

$$|sI - A| = s^3 + 3s^2 + 3s + 1$$

$$|sI - A| = (s+1)^3$$

$$[sI - A]^{-1} = \frac{\text{Adj}[sI - A]}{|sI - A|}$$

$$\begin{bmatrix} \overset{+}{s} & \overset{-}{-1} & \overset{+}{0} \\ \overset{-}{0} & \overset{+}{s} & \overset{-}{-1} \\ \overset{+}{1} & \overset{-}{3} & \overset{+}{s+3} \end{bmatrix}$$

$$\text{Adj}[sI - A] = [\text{cofactor of } [sI - A]]^T$$

Cofactors of $[sI-A]$ =

$$\begin{bmatrix} +\begin{vmatrix} s & -1 \\ 3 & s+3 \end{vmatrix} & -\begin{vmatrix} 0 & -1 \\ 1 & s+3 \end{vmatrix} & +\begin{vmatrix} 0 & s \\ 1 & 3 \end{vmatrix} \\ -\begin{vmatrix} -1 & 0 \\ 3 & s+3 \end{vmatrix} & +\begin{vmatrix} s & 0 \\ 1 & s+3 \end{vmatrix} & -\begin{vmatrix} s & -1 \\ 1 & 3 \end{vmatrix} \\ +\begin{vmatrix} -1 & 0 \\ s & -1 \end{vmatrix} & -\begin{vmatrix} s & 0 \\ 0 & -1 \end{vmatrix} & +\begin{vmatrix} s & -1 \\ 0 & s \end{vmatrix} \end{bmatrix}$$

$$\begin{bmatrix} s^2+3s+3 & -1 & -s \\ s+3 & s^2+3s & -(3s+1) \\ 1 & s & s^2 \end{bmatrix}^T$$

$$\text{Adj}[sI-A] = \begin{bmatrix} s^2+3s+3 & s+3 & 1 \\ -1 & s^2+3s & s \\ -s & -(3s+1) & s^2 \end{bmatrix}$$

$$[sI-A]^{-1} = \frac{\text{Adj}(sI-A)}{|sI-A|}$$

$$= \frac{\begin{bmatrix} s^2+3s+3 & s+3 & 1 \\ -1 & s^2+3s & s \\ -s & -(3s+1) & s^2 \end{bmatrix}}{(s+1)^3}$$

$$e^{At} = \mathcal{L}^{-1}\left\{ \begin{bmatrix} \frac{s^2+3s+3}{(s+1)^3} & \frac{s+3}{(s+1)^3} & \frac{1}{(s+1)^3} \\ \frac{-1}{(s+1)^3} & \frac{s^2+3s}{(s+1)^3} & \frac{s}{(s+1)^3} \\ \frac{-s}{(s+1)^3} & \frac{-[3s+1]}{(s+1)^3} & \frac{s^2}{(s+1)^3} \end{bmatrix} \right.$$

above terms converted into partial fraction

$$\frac{s^2+3s+3}{(s+1)^3} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{(s+1)^3}$$

$$\frac{s^2+3s+3}{\cancel{(s+1)^3}} = \frac{A(s+1)^2 + B(s+1) + C(1)}{\cancel{(s+1)^3}}$$

$$s^2+3s+3 = A\left[s^2+2s+1\right] + Bs + B + C$$

$$s^2+3s+3 = As^2 + 2As + A + Bs + B + C$$

Compare Coefficients

$A=1$	$3 = 2A+B$	$A+B+C = 3$
	$3 = 2(1)+B$	$1+1+C = 3$
	$B = 3-2$	$2+C = 3$
	$\boxed{B=1}$	$C = 3-2$
		$C = 1$

$$\frac{s^2+3s+3}{(s+1)^3} = \frac{1}{s+1} + \frac{1}{(s+1)^2} + \frac{1}{(s+1)^3}$$

<u>2nd term</u>

$$\frac{s+3}{(s+1)^3} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{(s+1)^3}$$

$$s+1 = s^2A + s(2A+B) + (A+B+C)$$

$$A = 0 \quad \Big| \quad 1 = 2A + B, \quad B = 1 \quad \Big| \quad A + B + C = 3, \quad 0 + 1 + C = 3, \quad C = 2$$

$$\frac{s+3}{(s+1)^3} = \frac{1}{(s+1)^2} + \frac{2}{(s+1)^3}$$

3rd term

$$\frac{1}{(s+1)^3} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{(s+1)^3}$$

$$1 = s^2 A + s(2A + B) + (A + B + C)$$

$$A = 0 \quad \Big| \quad 2A + B = 0, \quad 2(0) + B = 0, \quad B = 0 \quad \Big| \quad A + B + C = 1, \quad 0 + 0 + C = 1, \quad C = 1$$

5th term

$$\frac{s^2 + 3s}{(s+1)^3} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{(s+1)^3}$$

$$s^2 + 3s = s^2 A + s(2A + B) + A + B + C.$$

$$A = 1 \quad \Big| \quad 3 = 2A + B, \quad 3 = 2 + B, \quad B = 3 - 2, \quad B = 1 \quad \Big| \quad A + B + C = 0, \quad 1 + 1 + C, \quad 2 + C = 0, \quad C = -2.$$

$$\frac{s^2+3s}{(s+1)^3} = \frac{1}{s+1} + \frac{1}{(s+1)^2} - \frac{2}{(s+1)^3}$$

6^{th} term

$$\frac{s}{(s+1)^3} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{(s+1)^3}$$

$$s = s^2 A + s(2A+B) + (A+B+C)$$

$A=0$	$1 = 2A+B$	$A+B+C=0$
	$B=1$	$0+1+C$
		$C=-1$

$$\frac{s}{(s+1)^3} = \frac{1}{(s+1)^2} - \frac{1}{(s+1)^3}$$

7^{th} term

$$-s = s^2 A + s(2A+B) + (A+B+C)$$

$A=0$	$-1 = 2A+B$	$A+B+C=0.$
	$B=-1$	$0-1+C.$
		$+1 = \cancel{+}C$
		$C=1$

$$\frac{-s}{(s+1)^3} = \frac{-1}{(s+1)^2} + \frac{1}{(s+1)^3}$$

8^{th} term

$$\frac{-(3s+1)}{(s+1)^3} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{(s+1)^3}$$

$$-3s-1 = s^2 A + s(2A+B) + (A+B+C)$$

$A=0$	$-3 = 2A+B$	$-1 = A+B+C$
		$-1 = -3+C$

$$\frac{-(3s+1)}{(s+1)^3} = \frac{-3}{(s+1)^2} + \frac{2}{(s+1)^3}$$

9th term

$$\frac{s^2}{(s+1)^3} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{(s+1)^3}$$

$$s^2 = s^2 A + s(2A+B) + (A+B+C)$$

$A=1$

$2A+B=0$
$2+B$
$B=-2$

$A+B+C=0$
$1-2+C=0$
$-1+C$
$+1 = +C \therefore C=1$

$$\frac{s^2}{(s+1)^3} = \frac{1}{(s+1)} - \frac{2}{(s+1)^2} + \frac{1}{(s+1)^3}$$

$$e^{At} = \mathcal{L}^{-1}\begin{bmatrix} \frac{1}{s+1} + \frac{1}{(s+1)^2} + \frac{1}{(s+1)^3} ; & \frac{1}{(s+1)^2} + \frac{2}{(s+1)^3} ; & \frac{1}{(s+1)^3} \\ \frac{-1}{(s+1)^2} ; & \frac{1}{s+1} + \frac{1}{(s+1)^2} - \frac{2}{(s+1)^3} ; & \frac{1}{(s+1)^2} - \frac{1}{(s+1)} \\ \frac{-1}{(s+1)^2} + \frac{1}{(s+1)^3} ; & \frac{-3}{(s+1)^2} + \frac{2}{(s+1)^3} ; & \frac{1}{s+1} - \frac{2}{(s+1)^2} + \frac{1}{(s+1)} \end{bmatrix}$$

$$\mathcal{L}^{-1}\left[\frac{1}{s+a}\right] = e^{-at} ; \quad \mathcal{L}^{-1}\left[\frac{1}{(s+a)^2}\right] = te^{-at}$$

$$\mathcal{L}^{-1}\left[\frac{1}{(s+a)^3}\right] = \frac{t^2}{2} e^{-at}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s+1}+\frac{1}{(s+1)^2}+\frac{1}{(s+1)^3}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}+\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\}$$
$$+\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^3}\right\}$$

1st term :

$$e^{-t}+te^{-t}+\frac{t^2}{2}e^{-t} \Rightarrow e^{-t}\left(1+t+\frac{t^2}{2}\right)$$

2nd term

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}+\frac{2}{(s+1)^3}\right\} \Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\}+2\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^3}\right\}$$

$$= te^{-t}+\cancel{2}\frac{t^2}{\cancel{2}}e^{-t} \Rightarrow e^{-t}(t+t^2) \Rightarrow e^{-t}(t^2+t)$$

3rd term

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^3}\right\} \Rightarrow \frac{t^2}{2}e^{-t}$$

4th term :-

$$\mathcal{L}^{-1}\left\{\frac{-1}{(s+1)^3}\right\} = -\left[\frac{t^2}{2}e^{-t}\right]$$

5th term :-

$$\mathcal{L}^{-1}\left\{\frac{1}{s+1}+\frac{1}{(s+1)^2}-\frac{2}{(s+1)^3}\right]$$

$$e^{-t}+te^{-t}-\cancel{2}\frac{t^2}{\cancel{2}}e^{-t} \Rightarrow e^{-t}(1+t+t^2)$$

$$\Rightarrow e^{-t}(t^2+t+1)$$

6th term :-

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}-\frac{1}{(s+1)^3}\right\}$$

$$te^{-t}-\frac{t^2}{2}e^{-t} \Rightarrow e^{-t}\left[t-\frac{t^2}{2}\right]$$

$$= \frac{e^{-t}}{2}(2t-t^2).$$

7th term :-

$$\mathcal{L}^{-1}\left\{-\frac{1}{(s+1)^2}+\frac{1}{(s+1)^3}\right\} \Rightarrow -te^{-t}-\frac{t^2}{2}e^{-t}$$

$$e^{-t}\left[-t+\frac{t^2}{2}\right]$$

$$=\frac{e^{-t}}{2}\left(-2t+t^2\right)$$

8th term :-

$$\mathcal{L}^{-1}\left\{\frac{-3}{(s+1)^2}+\frac{2}{(s+1)^3}\right\}.$$

$$-3te^{-t}+\cancel{2}\frac{t^2}{\cancel{2}}e^{-t}$$

$$-3te^{-t}+t^2e^{-t}$$

$$e^{-t}\left[-3t+t^2\right] \Rightarrow e^{-t}\underline{\left[t^2-3t\right]}$$

9th term :-

$$\mathcal{L}^{-1}\left\{\frac{1}{s+1}-\frac{2}{(s+1)^2}+\frac{1}{(s+1)^3}\right\}$$

$$e^{-t}-2te^{-t}+\frac{t^2}{2}e^{-t} \qquad e^{-t}-2te^{-t}+\frac{t^2}{2}e^{-t}$$

$$e^{-t}\left[1-2t+\frac{t^2}{2}\right] \qquad e^{-t}\left[1-2t+\frac{t^2}{2}\right]$$

$$\frac{e^{-t}}{2}\left(2-4t+t^2\right)$$

$$\frac{e^{-t}}{2}\left[2-\cancel{2}4t+t^2\right] \Rightarrow \frac{e^{-t}}{2}\left[t^2-\cancel{2}4t+2\right] \quad 9$$

$$e^{At}=\begin{bmatrix} \frac{e^{-t}}{2}(t^2+2t+2); & e^{-t}(t^2+t); & \frac{t^2}{2}e^{-t} \\ -\frac{t^2}{2}e^{-t}; & e^{-t}(-t^2+t+1); & \frac{e^{-t}}{2}(2t-t^2) \\ \frac{e^{-t}}{2}[t^2-2t]; & \frac{e^{-t}}{2}(2t-t^2); & e^{-t}(t^2-3t) \end{bmatrix}$$

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(4) Given $\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$

find STM & find the Total Response for unit step Response.

When $x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

$$\cancel{SI - A} \Rightarrow S \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$SI - A \Rightarrow S \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

$$SI - A = \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

$$SI - A = \begin{bmatrix} S & -1 \\ 2 & S+3 \end{bmatrix}$$

$$\det |SI - A| = s(s+3) - (-1)(2)$$

$$\det |SI - A| = s^2 + 3s + 2.$$

$$SI - A = s^2 + 3s + 2$$

$$[SI - A]^{-1} = \frac{Adj\,[SI - A]}{|SI - A|}$$

$$Adj \text{ of } |SI - A| = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\begin{bmatrix} S+3 & -(-1) \\ -2 & S \end{bmatrix} \Rightarrow \begin{bmatrix} S+3 & 1 \\ -2 & S \end{bmatrix}$$

$$\therefore \ Adj\,|sI-A| = \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

$$[sI-A]^{-1} = \frac{Adj\,(sI-A)}{|sI-A|}$$

$$[sI-A]^{-1} = \frac{\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}}{s^2+3s+2}$$

$$[sI-A]^{-1} = L^{-1}\left\{ \begin{bmatrix} \frac{s+3}{s^2+3s+2} & \frac{1}{s^2+3s+2} \\ \frac{-2}{s^2+3s+2} & \frac{s}{s^2+3s+2} \end{bmatrix} \right.$$

$(s+1)(s+2)$

1st term :-

$\frac{1}{(s+a)(s+b)}$

$$\frac{s+3}{s^2+3s+2} = \frac{A}{s+1} + \frac{B}{s+2} \quad -(1)$$

$$\frac{s+3}{\cancel{s^2+3s+2}} = \frac{A(s+2)+B(s+1)}{\cancel{(s^2+3s+2)}}$$

$$s+3 = A(s+2) + B(s+1) \quad -(2)$$

put $s=-1$ in (2)

$$-1+3 = A(-1+2) + B(-1+1)$$

$$2 = A+0$$

$$\boxed{A=2}$$

$$s+3 = A(s+2) + B(s+1)$$

put $s=-2$ in (2)

$$-2+3 = A(-2+2) + B(-2+1)$$

$$1 = 0 - B$$

$$\boxed{B=-1}$$

sub A and B values.

$$\frac{s+3}{s^2+3s+2} = \frac{2}{s+1} - \frac{1}{s+2}$$

$$\frac{s+3}{s^2+3s+2} = 2(s+2) < s+1$$

2nd term :=

$$\frac{1}{s^2+3s+2} = \frac{A}{s+1} + \frac{B}{s+2}. \quad -(1)$$

$$1 = A(s+2) + B(s+1) \quad -(2)$$

put $s=-1$ in (2)

$$1 = A(-1+2) + B(-1+1)$$

$$1 = A(1) + 0$$

$$\boxed{A=1}$$

$$1 = A(s+2) + B(s+1)$$

put $s=-2$

$$1 = A(-2+2) + B(-2+1)$$

$$1 = 0 - B$$

$$\boxed{B=-1}$$

$$\frac{1}{s^2+3s+2} = \frac{1}{s+1} - \frac{1}{s+2}$$

3rd term :=

$$\frac{-2}{s^2+3s+2} = \frac{A}{(s+1)} + \frac{B}{(s+2)} \quad -(1)$$

$$-2 = A(s+2) + B(s+1) \quad -(2)$$

put $s=-1$ in (2)

$$-2 = A(-1+2) + B(-1+1)$$

$$-2 = A \Rightarrow \boxed{A=-2}$$

$-\cancel{2} = 1$

put $s = -2$.

$-2 = A(-2+2) + B(-2+1)$

$+2 = 0 + B$

$\boxed{B = 2}$

Sub A & B values

$$\frac{-2}{s^2+3s+2} = \frac{-2}{s+1} + \frac{2}{s+2}$$

4th term :-

$$\frac{s}{s^2+3s+2} = \frac{A}{s+1} + \frac{B}{s+2} \quad \text{--- (1)}$$

$s = A(s+2) + B(s+1) \quad \text{--- (2)}$

put $s = -1$

$-1 = A(-1+2) + B(-1+1)$

$-1 = A$

$\boxed{A = -1}$

put $s = -2$

$-2 = A(-2+2) + B(-2+1)$

$+2 = 0 + B(+1)$

$\boxed{B = 2.}$

$$\frac{s}{s^2+3s+2} = \frac{-1}{s+1} + \frac{2}{s+2}$$

$$\mathcal{L}^{-1}\left\{\begin{bmatrix} \frac{2}{s+1} - \frac{1}{s+2} & \frac{1}{s+1} - \frac{1}{s+2} \\ \frac{-2}{s+1} + \frac{2}{s+2} & \frac{-1}{s+1} + \frac{2}{s+2} \end{bmatrix}\right\}$$

$$[sI - A]^{-1} = \begin{bmatrix} \mathcal{L}^{-1}\left\{\frac{2}{s+1} - \frac{1}{s+2}\right\} & \mathcal{L}^{-1}\left[\frac{1}{s+1} - \frac{1}{s+2}\right] \\ \mathcal{L}^{-1}\left[\frac{-2}{s+1} + \frac{2}{s+2}\right] & \mathcal{L}^{-1}\left[\frac{-1}{s+1} + \frac{2}{s+2}\right] \end{bmatrix}$$

$$[sI - A]^{-1} = \begin{bmatrix} 2e^{-t} + e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

1.	Zero initial response $(ZIR) = e^{At} x(0)$
2.	Zero state Response $(ZSR) = \mathcal{L}^{-1}\{\phi(s) B U(s)\}$
	Total Response = ZIR + ZSR.

① $ZIR = e^{At} X(0)$

$$ZIR = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2e^{-t} - e^{-2t} + e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} + (-e^{-t} + 2e^{-2t} \end{bmatrix}$$

$2 \times \cancel{2}$ $\cancel{2} \times 1$

$$= \begin{bmatrix} 3e^{-t} - 2e^{-2t} \\ -3e^{-t} + 4e^{-2t} \end{bmatrix}$$

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(2) $ZSR = \mathcal{L}^{-1}\{\phi(s)\cdot B\,U(s)\}$

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad U(s) = \frac{1}{s}$$

$$\phi(s) = \begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix}$$

$$ZSR = \mathcal{L}^{-1}\left\{ \begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{1}{s} \right\}$$

$$ZSR = \mathcal{L}^{-1}\left\{ \begin{bmatrix} \frac{1}{(s+1)(s+2)} \\ \frac{s}{(s+1)(s+2)} \end{bmatrix} \left[\frac{1}{s} \right] \right\}$$

$$\mathcal{L}^{-1}\left\{ \begin{bmatrix} \frac{1}{s(s+1)(s+2)} \\ \frac{\cancel{s}}{\cancel{s}(s+1)(s+2)} \end{bmatrix} \right\}$$

$$ZSR = \mathcal{L}^{-1}\left\{ \begin{matrix} \frac{1}{s(s+1)(s+2)} \\ \frac{1}{(s+1)(s+2)} \end{matrix} \right\}$$

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$$\frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2} \quad -(1)$$

$$\frac{1}{s(s+1)(s+2)} = \frac{A(s+1) + B(s+2) + C(s)}{s(s+1)(s+2)} \quad -(2)$$

$$1 = As + A + Bs + 2B + Cs$$

$A = 1$ | x'
$A + 2B = 1$ |

$$\frac{1}{\cancel{s(s+1)(s+2)}} = \frac{A(s+1)(s+2) + B\,s(s+2) + C[s(s+1)]}{\cancel{s(s+1)(s+2)}}$$

$$1 = A(s^2 + 2s + s + 2) + Bs^2 + 2Bs + Cs^2 + Cs$$

$$1 = As^2 + 3As + 2A + Bs^2 + 2Bs + Cs^2 + Cs$$

$2A = 1$
$A = 1/2$
$A = 0.5$

$\cancel{0A + B + C = 0}$
$\cancel{\frac{1}{2} + B + C}$
$\cancel{B = -C - \frac{1}{2}}$

$0 = A + B + C$
$= \frac{1}{2} + B + C$
$-\frac{1}{2} - C = B$
$\frac{-1-2C}{2} = B$

$3A + 2B + C = 0$

$3\left(\frac{1}{2}\right) + \cancel{2}\left(\frac{-1-2C}{\cancel{2}}\right) + C$

$3\left(\frac{1}{2}\right) - 1 - 2C + C$

$\frac{3}{2} - 1 - C$

$\frac{3-2}{2} - C \Rightarrow \boxed{\frac{1}{2} = C}$

$-\frac{1}{2} - \frac{1}{2} = B$
$\frac{-2-2}{4} = B$
$\frac{-\cancel{4}}{\cancel{4}} = B$
$B = -1$

Scanned with OKEN Scanner

$$ZSR = \mathcal{L}^{-1}\left\{\begin{bmatrix} \frac{0.5}{s} - \frac{1}{s+1} + \frac{0.5}{s+2} \\ \frac{1}{s+1} - \frac{1}{s+2} \end{bmatrix}\right\}$$

$$ZSR = \begin{bmatrix} 0.5(1) - e^{-t} + 0.5e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

Total response = ZIR + ZSR

$$T.R = \begin{bmatrix} 3e^{-t} - 2e^{-2t} \\ -3e^{-t} + 4e^{-2t} \end{bmatrix} + \begin{bmatrix} 0.5 - e^{-t} + 0.5e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

$$= \begin{bmatrix} 3e^{-t} - 2e^{-2t} + 0.5 - e^{-t} + 0.5e^{-2t} \\ -3e^{-t} + 4e^{-2t} + e^{-t} - e^{-2t} \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{-t} - 1.5e^{-2t} + 0.5 \\ -3e^{-t} + 3e^{-2t} + e^{-t} \end{bmatrix}$$

$$\text{Total response} = \begin{bmatrix} 2e^{-t} - 1.5e^{-2t} + 0.5 \\ -2e^{-t} + 3e^{-2t} \end{bmatrix}$$

Controllability and Observability

Controllability :- A linear time invariant System is said to be Completely state Controllable if it is possible to transfer the System state from any initial state $x(0)$ to any other desired state (finite state). $x(t)$ in a Specified time intervals, by a Controll vector that is Output & Input Equation.

The Concept of Controllability and observability is introduced by kalman Hence, kalman test are used to find out whether the System is Controllable and Observable or not.

kalman test for Controllability :-

$$\dot{x} = Ax(t) + Bu(t)$$

$A = n \times n$, $B = n \times 1$

$$\boxed{Q_{c_1} = \left[B : AB : A^2B : A^3B \dots A^{n-1}B \right]}$$

Where $B, AB, A^2B \dots, A^{n-1}B$ are Columns of Matrix Q_c.

$\det |Q_c| \neq 0$.

the sys is Controllable

If

$\det |Q_c| = 0$ the sys is not Controllable.

Observability:-

A linear time invariant System is said to be Completely Observable if Every State Eqn Can be Completely $x(t)$ Can be Completely identified by Measurements of Output $y(t)$. In a finite interval of time. If the system is not Completely observable means that few of state Variables are not practically Measurable.

Kalman's test for Observability

$$\dot{x} = A\,x(t)$$

$$y = C\,x(t)$$

$$A = n \times n, \quad C = 1 \times n$$

$$Q_o = \left(C^T : A^T C^T : (A^T)^2 C^T : \ldots (A^T)^{n-1} C^T \right)$$

$|Q_o| = 0$

the system is not Observable,

$|Q_o| \neq 0$

the system is Observable.

Evaluate the Controllability and observability

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x + [0] u$$

Given $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}_{3\times3}$ $B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$

1) Kalman's test for Controllability.

$n = 3.$

$$Q_C = \begin{bmatrix} B : AB : A^2B \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}_{3\times3} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}_{3\times1} = \begin{bmatrix} 0 \\ 1 \\ -6 \end{bmatrix}$$

$$A^2B = A[AB] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -6 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} 0\times0 + 1\times1 + 0 \\ 0 + 0 - 6 \\ 0 - 11 + 36 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} 1 \\ -6 \\ 25 \end{bmatrix}$$

$$Q_C = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 25 \end{bmatrix} = 0 + 0 + 1(0-1)$$

$$-1 \neq 0.$$

The sys is Controllable.

Q_0

$$C^T = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad A^T = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix}$$

$$Q_0 = \left[C^T \,\vdots\, A^T C^T \,\vdots\, (A^T)^2 C^T \right]$$

$$A^T C^T = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$(A^T)^2 C^T = A^T (A^T C^T) = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$Q_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |Q_0| = 1$$

$\therefore |Q_0| \neq 0$

then the system is completely observable.

$\cancel{4, 8, 12, 14, 21, 34, 35, 56}$