# Removing magnetic field with a chiral gauge transformation

This paradox is not so knotty, but it can be a bit confusing when you first come across it (I was confused). It is related to a discussion of the theory of the Aharonov–Bohm effect that took place in the late 1970s and early into the 1980s [1–4].

**1Electromagnetic potentials and gauge symmetry:**

We often write Maxwell's equations in terms of electric and magnetic fields, $\boldsymbol{E}$ and $\boldsymbol{B}$,

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B}, \quad \nabla \cdot \mathbf{B} = 0 \tag{1a}$$

$$\nabla \times \mathbf{H} = \frac{\partial}{\partial t}\mathbf{D}+\mathbf{J}, \quad \nabla \cdot \mathbf{D} = \rho \tag{1b}$$

However, the second and third of these equations can be reduced to mathematical identities if we work in terms of the scalar and vector potentials, $\varphi$ and $\boldsymbol{A}$, defined by

with the non locality definitions:

$$\mathbf{B} = \mu_T\left[\mathbf{H} + T(\nabla \times \mathbf{H})\right] \tag{2}$$

$$\mathbf{D} = \epsilon_T\left[\mathbf{E} + T(\nabla \times \mathbf{E})\right] \tag{3}$$

$$\mathbf{B} = \nabla \times \left(\mathbf{A} + T\nabla \times \mathbf{A}\right) = \nabla \times \mathbf{F} \tag{4}$$

$$\left(\mathbf{A} + T\nabla \times \mathbf{A}\right) = \mathbf{F}, \textbf{ with T=c/}\omega \tag{5}$$

$$\nabla \cdot \mathbf{F} = \nabla \cdot \mathbf{A} \tag{6}$$

$$\mathbf{E} = -\frac{\partial}{\partial t}\mathbf{F} - \nabla V \tag{7}$$

In this case we have RCP and LCP in opposite direction having a pure stationary wave.
Also, we have a velocity gauge where Coulomb and Lorentz gauge are particular cases

Local Maxwell's equations (1) can then be reduced to two equations for the potentials,

$$\begin{aligned}\nabla^2\varphi+\frac{\partial\nabla\cdot\boldsymbol{A}}{\partial t} &= -\rho/\varepsilon_0 \\ \nabla\times\nabla\times\boldsymbol{A}+\frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2} &= \mu_0\boldsymbol{j}-\frac{1}{c^2}\frac{\partial\nabla\varphi}{\partial t}\end{aligned} \tag{3}$$

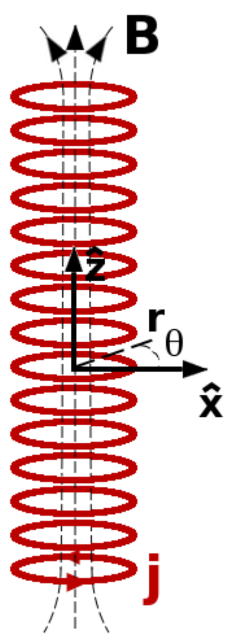
which can often simplify calculations (Landau and Lifshitz [5] were good at this). We might therefore be tempted to immediately abandon the six quantities that define the electromagnetic field in favour of using the four quantities that determine the potentials. However this is not so straightforward as $\varphi$ and $\boldsymbol{A}$ are not uniquely determined by (2). The $\boldsymbol{E}$ and $\boldsymbol{B}$ fields stay the same after the following transformation,

$$\begin{aligned}\varphi &\to \varphi-\frac{\partial\chi}{\partial t} \\ \boldsymbol{A} &\to \boldsymbol{A}+\nabla\chi\end{aligned} \tag{4}$$

where $\chi$ is an arbitrary function of space and time. The transformation, (4) is known as a *gauge transformation*. Classically we don't ascribe any physical meaning to the potentials precisely because of this symmetry - you can't measure something that isn't uniquely defined. Yet when you write the theory of electromagnetism interacting with matter in terms of a Lagrangian or Hamiltonian, then using the potentials rather than the fields becomes unavoidable [5]. As a consequence (4) is a very important symmetry in the quantum theory of charged particles interacting via the electromagnetic field. Generalizations of this symmetry form the basis of our theories of the strong and weak forces.

**2The paradox:**

Let's now consider a specific situation. A line of coils carrying electric current is placed along the $z$ axis, as shown in the figure below,



Now we'll determine the vector potential that describes this situation. In the Coulomb gauge, which is where $\chi$ is chosen such that $\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$, the second line of equation (3) reduces to

$$\boldsymbol{\nabla}^2 \boldsymbol{A} = -\mu_0 \boldsymbol{j}, \tag{5}$$

which has the general solution (equation (5) is just a vector version of Poisson's equation),

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int d^3\boldsymbol{r}' \frac{\boldsymbol{j}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \ . \tag{6}$$

We now apply (6) to the situation shown in the figure, where the current density is given by

$$\boldsymbol{j}(\boldsymbol{r}) = NI\delta(r-a)[\cos(\theta)\hat{\boldsymbol{y}} - \sin(\theta)\hat{\boldsymbol{x}}] \tag{7}$$

where there are $N$ turns of the coil (radius $a$) per unit length along the $z$ axis, with a current, $I$ flowing through each. Inserting (7) into (6) and expanding the resulting expression to leading order in $a/r$ (the first order term integrates to zero), one obtains,

$$\begin{aligned} \boldsymbol{A}(\boldsymbol{r}) &= \frac{\mu_0 NIa^2}{4\pi r^2} \int_{-\infty}^{\infty} dz' \int_0^{2\pi} d\theta' \frac{\cos^2(\theta')\hat{\boldsymbol{y}}}{(1+z'^2/r^2)^{3/2}} \\ &= \frac{\mu_0 NIa^2 \hat{\boldsymbol{y}}}{4r} \int_{-\infty}^{\infty} d\zeta \frac{1}{(1+\zeta^2)^{3/2}} \end{aligned} \tag{8}$$

where we have assumed a point of observation at $\theta = 0$, $z = 0$, and assumed that the line of coils has an infinite length. The final integral can be performed, with the result,

$$\int_{-\infty}^{\infty} d\zeta \frac{1}{(1+\zeta^2)^{3/2}} = \int_{-\infty}^{\infty} d\zeta \frac{d}{d\zeta}\left(\frac{\zeta}{\sqrt{1+\zeta^2}}\right) = 2 \tag{9}$$

So that,

$$\boldsymbol{A}(r,0,0) = \frac{\mu_0 NIa^2 \hat{\boldsymbol{y}}}{2r} \rightarrow \boldsymbol{A}(\boldsymbol{r}) = \frac{\Phi \hat{\boldsymbol{\theta}}}{2\pi r} \tag{10}$$

where the symmetry of the situation determines the direction of the vector in the second step, and, $\Phi = \mu_0 NI\pi a^2$. Equation (10) is the vector potential that describes the field of a long, infinitesimally thin line of coils. However, (10) can also be re-written as the gradient of a scalar function,

$$\boldsymbol{A}(\boldsymbol{r}) = \boldsymbol{\nabla}\left(\frac{\Phi\theta}{2\pi}\right) \tag{11}$$

which suggests that it can be set to zero by a gauge transformation, (4). Initially this seems fine, because the magnetic field everywhere outside of such a long thin line of

coils (a solenoid) ought to vanish. However, there is nevertheless a magnetic flux in the system that is infinitely concentrated along the $z$ axis. Performing such a gauge transformation would remove this flux from the system. Is this really allowed? Is it that an infinitely concentrated line of magnetic flux is somehow inconsistent with Maxwell's equations?
The solution of F with the non local approach can be useful to analyze the phantasmagoric action of distance indicated by Einstein (EPR paradox).

**References:**


Aharonov, Y. and Bohm, D. *"Significance of electromagnetic potentials in quantum theory"*, *Phys. Rev.* **115** 485 (1959)

Bocchieri, P. and Loinger, A. *"Nonexistence of the Aharonov–Bohm Effect"*, *Il Nuovo Cimento* **47** 475 (1978)

Berry, M. V. *"Exact Aharonov–Bohm wavefunction obtained by applying Dirac's magnetic phase factor"*, *Eur. J. Phys.* **1** 240 (1980)

Tonomura, A., Osakabe, N., Matsuda, T., Kawasaki, T., Endo, J., Yano, S. and Yamada, H. *"Evidence for Aharonov–Bohm effect with magnetic field completely shielded from electron wave"*, *Phys. Rev. Lett.* **56** 792 (1986)

Landau, L. D. and Lifshitz, E. M. *"The Classical Theory of Fields"*, Butterworth–Heinemann (2003

Einstein A, Rosen N and Podolsky B, *"Can quantum mechanical description of physical reality be considered complete"*, Phys Rev 47, 777, (1935)