ENGINEERING MECHANICS

UNIT-I

Introduction

Science: *It is the knowledge about the structure and behavior of natural and physical world based on facts that one can prove. Science is the study of living and non living things. There are various branches of science such as Natural science, Domestic science, Earth science, Life science, Political science, Social science, etc. Physics, Chemistry, Mathematics, biology, etc., are a few subjects of science.*

Physics: *It is sub-branch of science which studies the property of matter and energy. The field of physics deals with the study of mechanics, thermodynamics, electricity, magnetism, sound, light, nuclear physics, electronics, etc.*

Mechanics: *It is the branch of physics which deals with the study of effect of force system acting on a particular or a rigid body which maybe at rest or in motion. The purpose of mechanics is to explain and predict physical phenomena and thus to lay the foundation for engineering applications.*

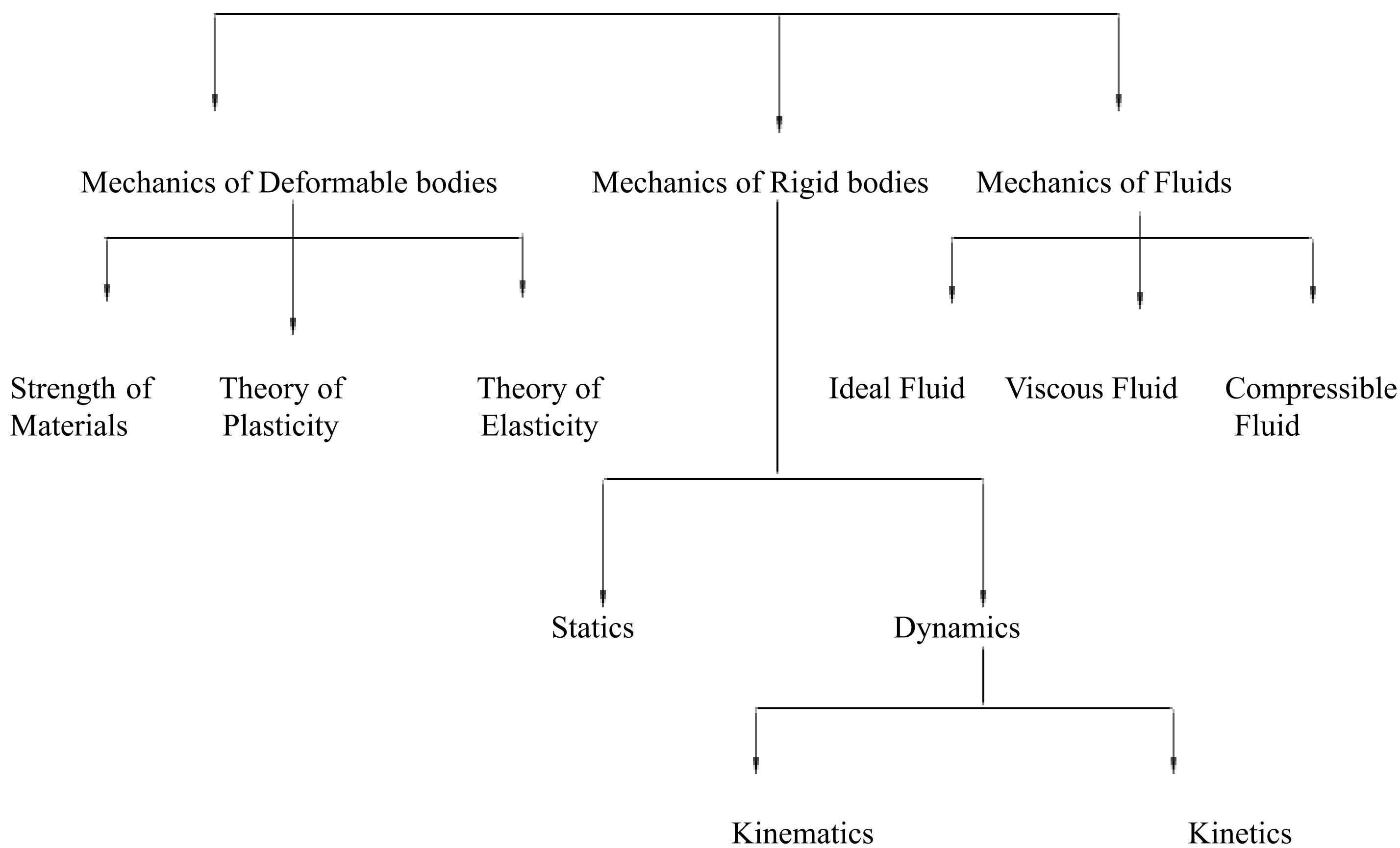
Applications of Engineering Mechanics

Engineering Mechanics is considered as one of the basic subjects for engineering students irrespective of branches as it develops the thinking and imaginative skill of students. It supports many other subjects in manufacturing of various products and projects.

Engineering: *It is the application of scientific knowledge which is used by an engineer to design and manufacture the product that serve the human society.*

Apart from God gifted nature, man has produced many artificial goods which range from a small pin to a huge multistory building. The main role of an engineer is to apply the engineering concepts for manufacturing various products and projects such as automobiles, aircrafts, electric motors, robots, television, mobile, satellite, construction of roadways, railways, bridges, dams, power transmission towers, skyscrapers, projectile of missiles, launching of rockets, radar communication structure, trusses, lifting machines like crane, hoist, screw jack, elevators, conveyor belt, cargo ship, submarine, etc.

Classification of Mechanics



Statics: It is the study of effect of force system acting on a particular or rigid body which is at rest.

Dynamics: It is the study of effect of force system acting on a particular or rigid body which is in motion. It can also be stated as the study of geometry of motion with or without reference to the cause of motion.

It must be noted that:

1. the study of motion means relationship between displacement, velocity, acceleration and time.

2. with reference to cause of motion means mass and the force causing the motion are considered.

The sub-branches of dynamics are Kinematics and Kinetics.

Kinematics: it is the study of geometry of motion without reference to cause of motion (i.e., mass and force causing the motion are not considered).

Kinetics: it is the study of geometry of motion with reference to cause of motion (i.e., mass and force causing the motion are considered).

<u>Basic Concepts</u>

Space: Concept of Space is essential to fix the position of a point. To fully define the position of a point in space we shall need to define some frame of reference and coordinate system.

Time: Concept of time is essential to relate the sequence of events, for example, starting and stopping of the motion of a body.

Mass: Mass is the quantity of matter in a body. Matter refers to the substance of which the body is composed. Concept of mass is essential to distinguish between the behavior of the two bodies under the action of an identical force.

Force: Concept of force is essential as an agency which changes or tends to change the state of rest or of uniform motion of a body.

But in order to describe the state of rest or motion of a body, some datum or reference is required. A body can be said to be at rest or in motion only with respect to some reference frame. This reference, preferably, should be fixed in space. As it is doubtful to locate any fixed reference in the universe, so the earth surface is usually employed as a reference frame. Such a reference serves as inertial frame. A truly inertial frame is one which moves at constant velocity.

Idealization of Bodies:

While studying the effect of force system acting on a particle or a rigid body which may be at rest or in motion, in mechanics some assumptions (idealizations) are made to simplify the problem without affecting the results.

Continuum: Matter is made up of atoms and molecules. But the real picture of matter as atoms and molecules is too complex to deal with. So to study the average measurable behavior of bodies, we assume that the matter is continuously distributed. Such a continuous description of matter is called a continuum.

A continuum can be rigid or deformable depending upon the assumptions we make.

Rigid Body: A body is said to be a rigid if it does not deform or the distance between any two points of the body does not change under the action of an applied force. In practice, however a body gets deformed when acted upon by forces. But in many situations this deformation is negligibly small to affect the results, the body can be assumed rigid.

Particle: It is a matter having considerable mass but negligible dimension. A body whose shape and size is not considered in analysis of problems and all the forces acting on a given body is assumed to act at a single point is considered to be a particle.

Units and Dimensions:

The following units of different systems are mostly used:

i) C.G.S. System of Units: In this system length is expressed in centimeter, mass in gram and time in second. The unit of force in this system is dyne, which is defined as the force acting on a mass of one gram and producing an acceleration of one centimeter per second square.

ii) M.K.S. System of Units: In this system length is expressed in meter, mass in kilogram and time in second. The unit of force in this system is expressed as kilogram force and is represented as kgf.

iii) S.I. System of Units: S.I. is abbreviation for 'The system of International units.' It is also called the International System of units. In this system length is expressed in meter, mass in kilogram and time in second. The unit of force in this system is expressed as Newton and is represented as N. Newton is the force acting on a body of mass one kilogram and producing an acceleration of one meter per Second Square.

The relation between Newton (N) and dyne is obtained as:

$$One\ Newton\ = One\ Kilogram\ Mass \times \frac{One\ meter}{s^2}$$

$$= 1000\ grams \times \frac{100\ cm}{s^2} \qquad (\because\ one\ kg\ =\ 1000\ grams)$$

$$= 1000 \times 100 \times \frac{gm \times cm}{s^2}$$

$$= 10^5\ dynes \qquad \left\{\because\ dyne\ =\ \frac{gm \times cm}{s^2}\right\}$$

S.I. Prefixes

Prefix	**Symbol**	**Multiplying Factor**
Tera	T	10^{12}
Giga	G	10^{9}
Mega	M	10^{6}
Kilo	K	10^{3}
Milli	m	10^{-3}

Micro	µ	10^{-6}
Nano	n	10^{-9}
Pico	p	10^{-12}

Basic Units

Physical Quantity	Notation or Unit	Dimension or Symbol
Length	Meter	m
Mass	Kilogram	kg
Time	Second	s
Electric current	Ampere	A
Temperature	Kelvin	K
Luminous Intensity	Candella	cd

Supplementary Units

Physical Quantity	Notation or Unit	Dimension or Symbol
Plane Angle	Radian	rad
Solid Angle	Steridian	sr

Derived Units

Physical Quantity	Notation or Unit	Dimension or Symbol
Acceleration	meter/second^2	m/s^2
Angular Velocity	radian/second	rad/s
Angular Acceleration	radian/second^2	rad/s^2
Force	Newton	$N = kg\ m/s^2$
Moment of Force	Newton meter	Nm
Work, Energy	Joule	$J = Nm = kg\ m^2/s^2$
Torque	Newton meter	Nm
Power	Watt	W = J/s
Pressure	Pascal	$Pa = N/m^2$
Frequency	Hertz	$Hz = s^{-1}$

Laws of Mechanics:

The study of mechanics depends on few fundamental principles which are based on experimental observations.

i) Newton's First Law of motion: Every body continues in its state of rest or of uniform motion in a straight line unless an external unbalanced force act on it. Newton's first law contains the principles of equilibrium of forces.

ii) Newton's Second Law of motion: The rate of change of momentum of a body is directly proportional to the force acting on it and takes place in the direction of applied force.

Thus, $F = \frac{mv-mu}{t} = m\frac{(v-u)}{t}$

$\therefore F = ma$ where F is the resultant force acting on a body of mass m and moving with an acceleration a.

As per Newton's Second law, the acceleration of a particle is proportional to the resultant force acting on it and is in the direction of this force ($F = ma$).

iii) Newton's Third Law of Motion: To every action there is an equal and opposite reaction. The forces of action and reaction between interacting bodies are equal in magnitude, opposite in direction and collinear. It means that forces always occur in pairs of equal and opposite forces.

iv) Newton's Law of Gravitation: The force of attraction between any two bodies in the universe is directly proportional to the product of their masses and inversely proportional to the square of the distance between them. For example, if m_1 and m_2 are the masses of two bodies and r is the distance between them, then the force of attraction F between them is given by

$$F \propto \frac{m_1 m_2}{r^2} \quad \therefore F = \frac{G m_1 m_2}{r^2}$$

Where G = Universal gravitational constant of proportionality and is given by $G = 6.673 \times 10^{11}$ Nm^2/kg^2 ($m^3/kg\ s^2$)

v) Law of Parallelogram of Force: The law of parallelogram of forces is used to determine the resultant of two forces acting at a point in a plane. It states, "if two forces, acting at a point be represented in magnitude and direction by the two adjacent sides of a parallelogram, then their resultant is represented in magnitude and direction by the diagonal of the parallelogram passing through that point."

Let two forces P and Q act at a point O as shown in the figure. The force P is represented in magnitude and direction by OA whereas the force Q is represented in magnitude and direction by OB. Let the angle between the two forces be 'α.' The resultant of these two forces will be obtained in magnitude and direction by the diagonal passing through O of the parallelogram of which OA and OB are two adjacent sides.

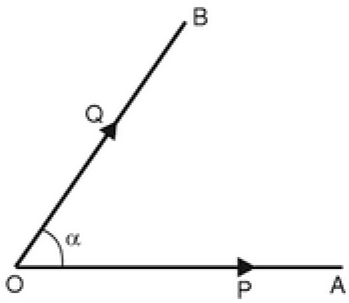


Fig. 1.1

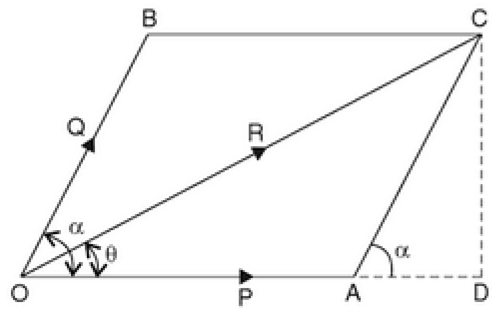


Fig. 1.2

Magnitude of Resultant (R)

From C draw CD perpendicular to OA produced.

Let α = Angle between two forces P and Q = $\angle AOB$

Now $\angle DAC = \angle AOB = \alpha$

In Parallelogram OACB, AC is parallel and equals to OB.

$\therefore$ AC = Q.

In triangle ACD,

$$AD = AC \cos\alpha = Q \cos\alpha$$

And $$CD = AC \sin\alpha = Q \sin\alpha$$

In triangle OCD,

$$OC^2 = OD^2 + DC^2$$

But OC = R, OD = OA+AD = P + Q cos α and DC = Q sin α.

$\therefore R^2 = (P + Q\cos\alpha)^2 + (Q\sin\alpha)^2 = P^2 + Q^2\cos^2\alpha + 2PQ\cos\alpha + Q^2\sin^2\alpha$

$= P^2 + Q^2(\cos^2\alpha + \sin^2\alpha) + 2\,PQ\cos\alpha$

$= P^2 + Q^2 + 2\,PQ\cos\alpha$

$$\therefore R = \sqrt{P^2 + Q^2 + 2\,PQ\cos\alpha}$$

Direction of Resultant

Let θ = Angle made by the resultant with OA.

Then from triangle OCD,

$$\tan\theta = \frac{CD}{OD} = \frac{Q\sin\alpha}{P + Q\cos\alpha}$$

$$\therefore \theta = \tan^{-1}\left(\frac{Q\sin\alpha}{P + Q\cos\alpha}\right)$$

Two case are important.

1st case. If the two forces P and Q act at right angles, then $\alpha = 90^o$

$$R = \sqrt{P^2 + Q^2 + 2PQ\cos\alpha} = \sqrt{P^2 + Q^2 + 2PQ\cos\cos 90^o}$$

$$= \sqrt{P^2 + Q^2} \qquad (\because \cos\cos 90^o = 0)$$

and the direction of resultant is obtained as

$$\theta = \tan^{-1}\left(\frac{Q\sin\alpha}{P + Q\cos\alpha}\right)$$

$$= \tan^{-1}\left(\frac{Q\sin 90^o}{P + Q\cos 90^o}\right) = \tan^{-1}\left(\frac{Q}{P}\right)$$

($\because \cos\cos 90^o = 0 \; and \sin\sin 90^o = 1$)

2nd Case. The two forces P and Q are equal and are acting at an angle α between them.

Then the magnitude and direction of resultant is given as

$$R = \sqrt{P^2 + Q^2 + 2PQ\cos\alpha} = R = \sqrt{P^2 + P^2 + 2P\times P\cos\alpha}$$

$$= \sqrt{2P^2 + 2P^2\cos\alpha} = \sqrt{2P^2(1 + \cos\alpha)}$$

$$= \sqrt{2P^2\times 2\cos^2\frac{\alpha}{2}} = \sqrt{4P^2\cos^2\frac{\alpha}{2}} = 2P\cos\frac{\alpha}{2}$$

and $\theta = \tan^{-1}\left(\frac{Q\sin\alpha}{P + Q\cos\alpha}\right) = \theta = \tan^{-1}\left(\frac{P\sin\alpha}{P + P\cos\alpha}\right)$

$$= \tan^{-1}\left(\frac{P \sin\alpha}{P(1+\cos\alpha)}\right) = \theta = \tan^{-1}\left(\frac{\sin\alpha}{1+\cos\alpha}\right)$$

$$= \tan^{-1}\left(\frac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{2\cos^2\frac{\alpha}{2}}\right) = \tan^{-1}\left(\frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}\right) = \tan^{-1}\left(\tan\frac{\alpha}{2}\right) = \frac{\alpha}{2}$$

vi) Principle of Transmissibility of Forces: It states that if a force, acting on a rigid body, is shifted to any other point which is on the line of action of the force, the external effect of the force on the body remains unchanged.

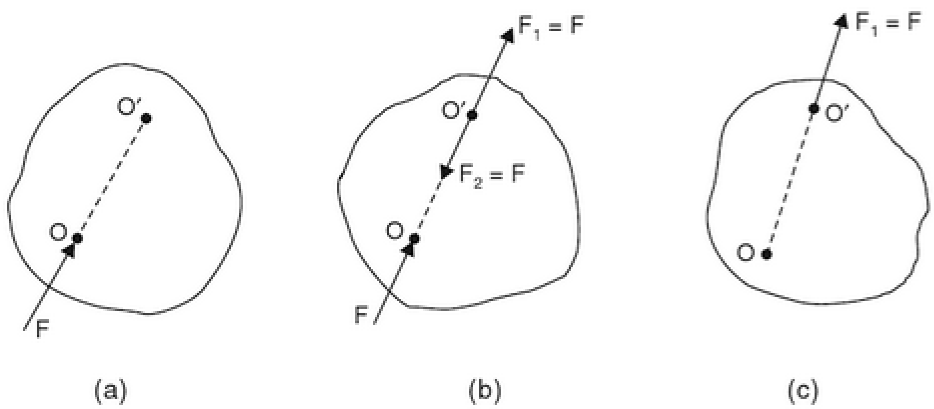


Fig 1.3

For example, consider a force F acting at point O on a rigid body as shown in the Fig. 1.3. On this rigid body, “there is another point O' in the line of action of the force F. Suppose at this point O', two equal and opposite forces F_1 and F_2 (each equal to F and collinear with F) are applied. The force F and F_2, being equal and opposite, will cancel each other, leaving a force F_1 at point O'. But force F_1 is equal to F.

The original force F acting at point O, has been transferred to point O' which is along the line of action of F without changing the effect of the force on the rigid body. Hence any force acting at a point on a rigid body can be transmitted to act at any other point along its line of action without changing its effect on the rigid body. This proves the principle of transmissibility of a force.

vii) Triangle law of Forces: if three forces acting at a point be represented in magnitude and direction by the three sides of a triangle taken in an order then they will be in equilibrium.

(*Or)*

If two forces are represented in magnitude and direction by the adjacent sides of a triangle taken in an order then their resultant is represented in magnitude and direction by the closing side of the triangle taken in the opposite order.

viii) Polygon law of Forces: if a number of coplanar forces are acting at a point such that they can be represented in magnitude and direction by the sides of a polygon taken in the same order, then their resultant is represented in magnitude and direction by the closing side of the polygon taken in the opposite order.

Concept of Force

Everybody has a tendency to remain at rest. Similarly a body in motion has a tendency to remain in motion. This is known as the *property of inertia*. The state of the body changes only if an external agency acts on the body.

Force: An external agency which changes or tends to change the state of rest or of uniform motion of a body upon which it acts is known as *force*.

Characteristics of Force:

i) Magnitude
ii) Direction (Line of action and sense)
iii) Point of application

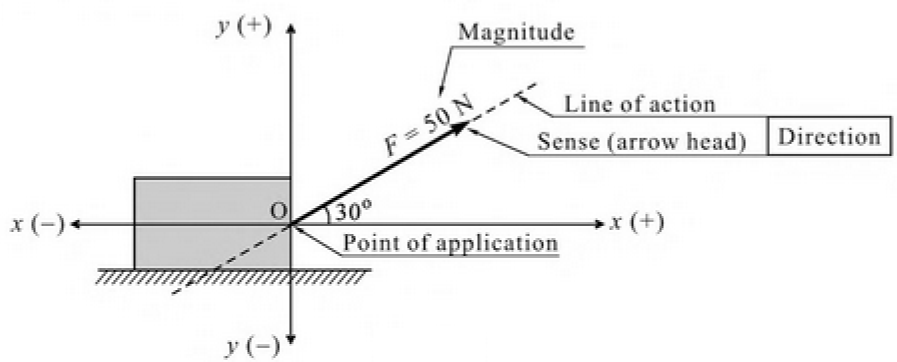


Fig. 1.4

Description of Force:

i) A force is a result of the action of one body on another body. It may be due to direct contact between the bodies example, striking a target or by remote action example gravitational, magnetic. Thus force may be contact force or non-contact force.
ii) Force imparts motion to the body or it can affect the motion of the body on which it acts.

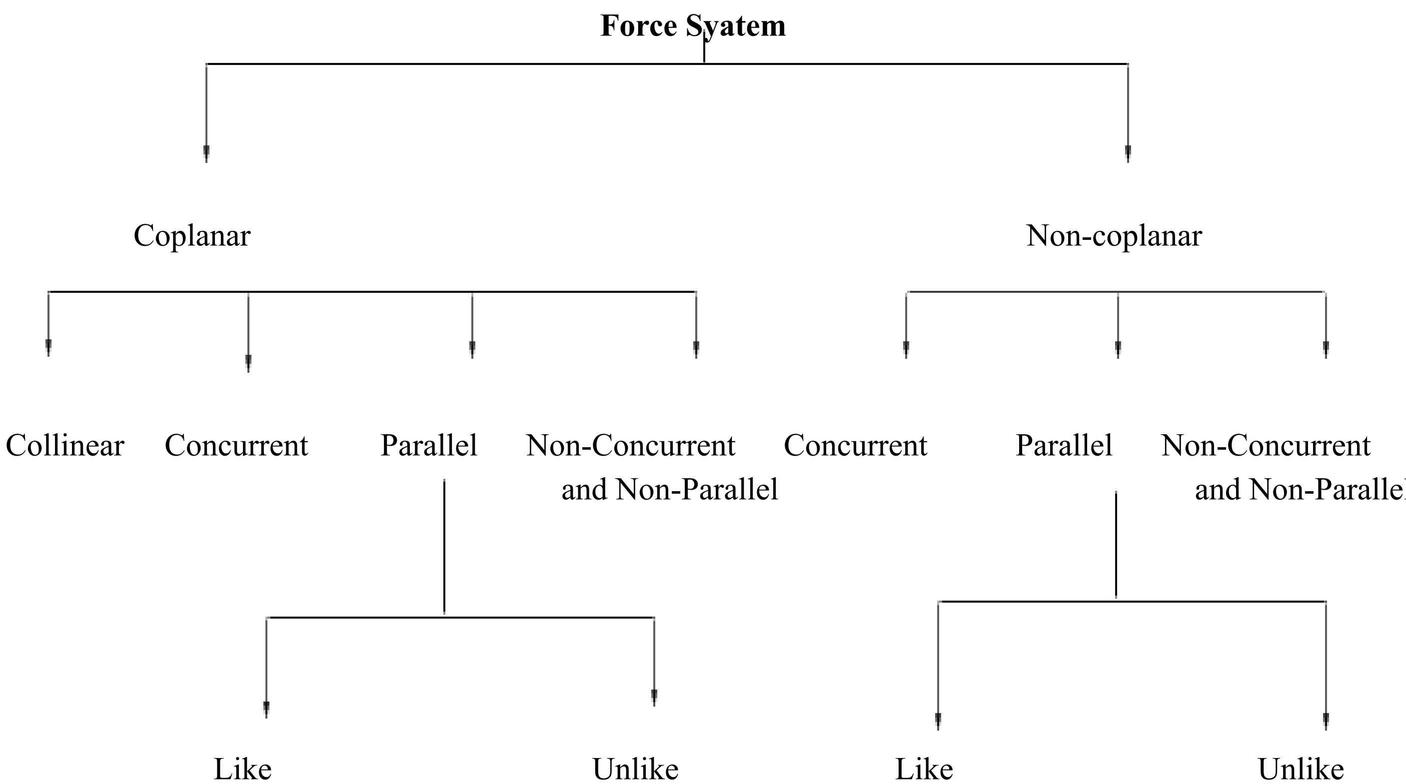
iii) Force can accelerate the motion if applied in same direction of motion or it can decelerate the motion if applied in opposite direction of motion. It can also stop the motion of body.

iv) Force may rotate the body.

v) Force may maintain equilibrium condition of body.

vi) The action of force may be of a push or pull type. It may produce tension or compression in a straight member.

Classification of Force System



Force System: when number of forces act simultaneously on a body then they are said to form a *force system.*

Depending upon whether line of action of all the forces acting on the body lies in the same plane or in different plane, the force system may be classified as follows:

i) **Coplanar force system:** If line of action of all the forces in the system lies on same plane then it is called as *coplanar force system*.

ii) **Non-coplanar force system:** If line of action of all the forces in the system do not lies on same plane then it is called as *non-coplanar force system*.

Collinear Forces: When the lines of action of all the forces act along the same line then it is called as *collinear force system.* Example: Forces acting on a rope in a tug of war.

The above two are sub classified into three groups.

a) **Concurrent force system:** If line of action of all the forces in the system passes through single point then it is called as *concurrent force system.*

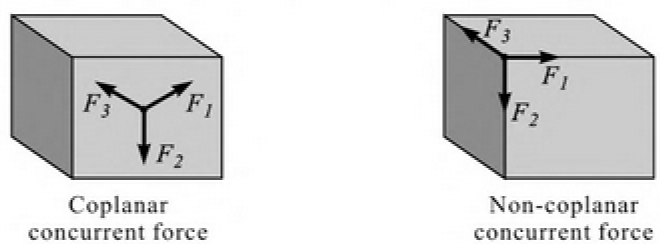


Fig. 1.5

b) **Parallel force system:** If line of action of all the forces in the system are parallel to each other then it is called as *parallel force system.*
It is further sub classified into two groups *like* and *unlike.*

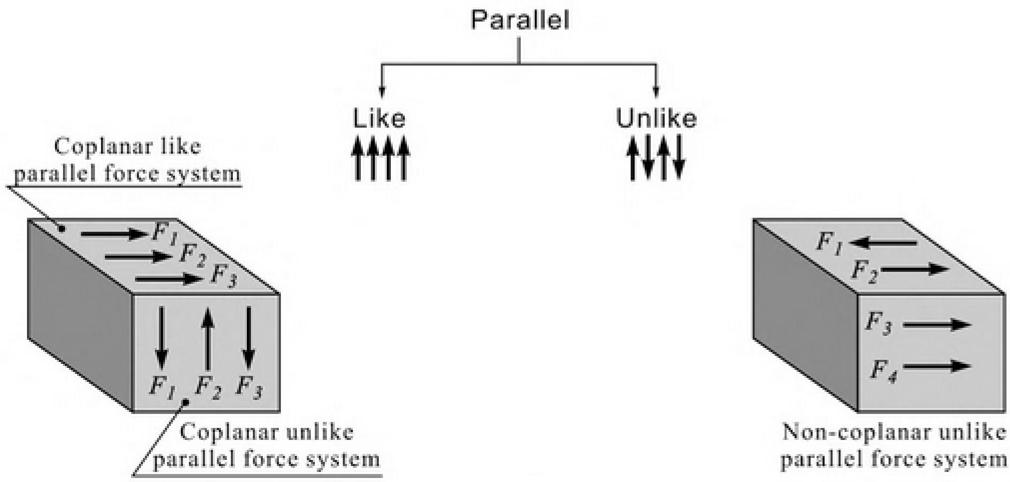


Fig. 1.6

c) **Non-Concurrent and Non-Parallel force system:** If line of action of all the forces in the system are neither concurrent nor parallel then it is called as *non-concurrent and non-parallel force system.*

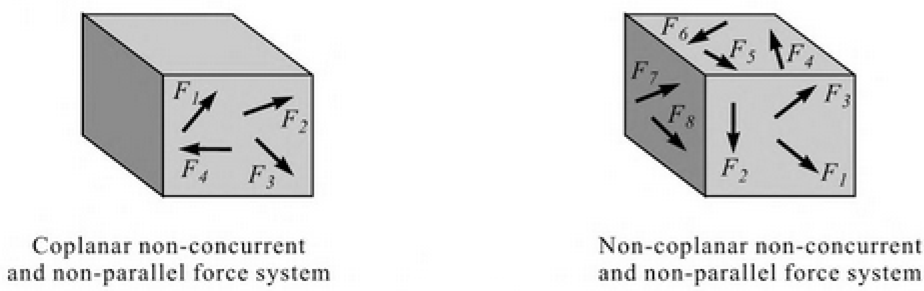


Fig. 1.7

Force System	**Characteristic**	**Examples**
Collinear forces	Line of action of all the forces act along the same line.	Forces on a rope in a tug of war.
Coplanar parallel forces	All forces are parallel to each other and lie in a single plane.	System of forces acting on a beam subjected to vertical loads (including reaction).
Coplanar like parallel forces	All forces are parallel to each other, lie in a single plane and are acting in the same direction.	Weight of a stationary train on a rail when the track is straight.
Coplanar concurrent forces	Line of action of all forces pass through a single point and forces lie in the same plane.	Forces on a rod resting against a wall.

Coplanar non-concurrent forces	All forces do not meet at a point, but lie in a single plane.	Forces on a ladder resting against a wall when a person stands on a rung which is not at its center of gravity.
Non-coplanar parallel forces	All the forces are parallel to each other, but not in same plane.	The weight of benches in a class room.
Non-coplanar concurrent forces	All forces do not lie in the same plane, but their lines of action pass through a single point	A tripod carrying a camera.
Non-coplanar non-concurrent forces	All forces do not lie in the same plane and their lines of action do not pass through a single point.	Forces acting on a moving bus.

Composition of Forces: Forces may be combined (added) to obtain a single force which produces same effect as the original system of forces. This single force is called as *resultant force.* The process of finding the resultant of forces is called as *composition of forces.*

Resolution of Force: The process of breaking the force into number of components which are equivalent to the given force is called *resolution of force.*

Resolution of force into rectangular components of force: The process of breaking the force into mutually perpendicular components which are equivalent to the given force is called *rectangular components of a force.*

Consider a force of magnitude F acting at an angle θ with horizontal, taking O as origin. Let the force F be represented by line OA drawn to the scale. Draw perpendicular from point A on the *x*-axis to mark B. OB(F_x) and AB(F_y) are the mutually perpendicular components of force F.

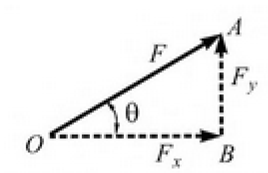


Fig. 1.8

By trigonometry we have the relation of components F_x and F_y with F and θ.

$$\sin sin\,\theta = \frac{F_y}{F} \text{ and } \cos cos\,\theta = \frac{F_x}{F}$$

$$\therefore F_y = F\,sin\theta \text{ and } F_x = F\,cos\theta$$

$$F = \sqrt{F_x^2 + F_y^2} \text{ and } \theta = tan^{-1}\left(\frac{F_y}{F_x}\right)$$

Sign Conventions

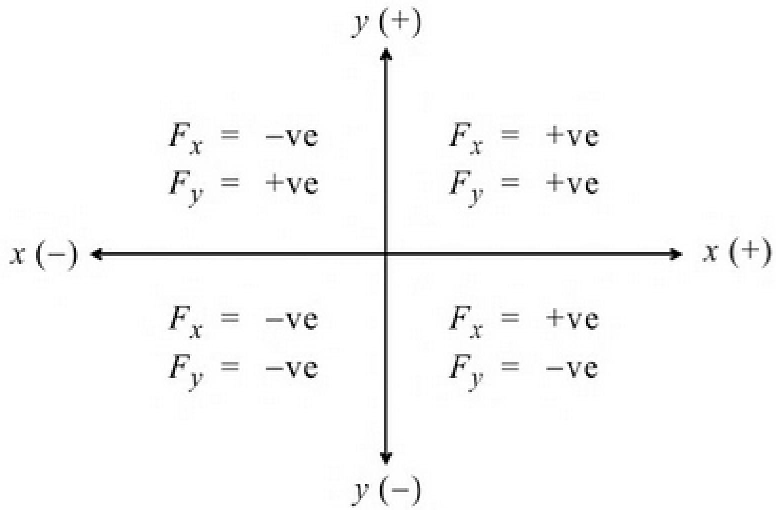
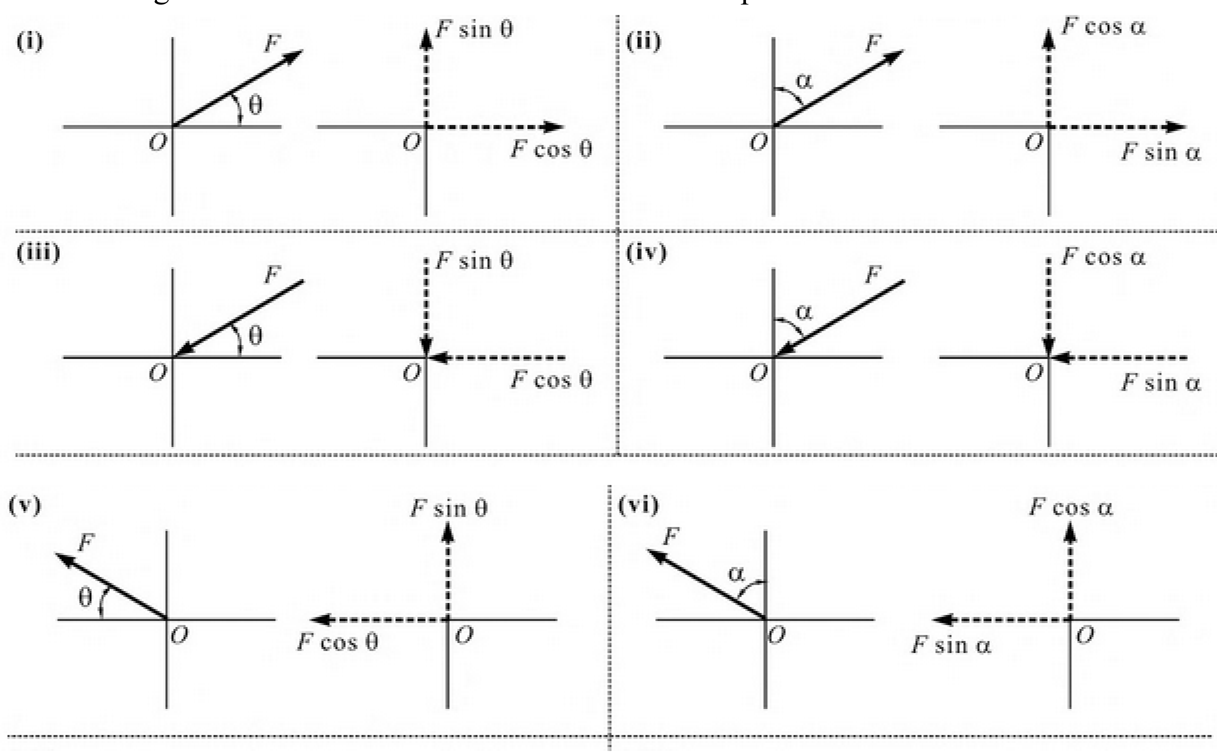
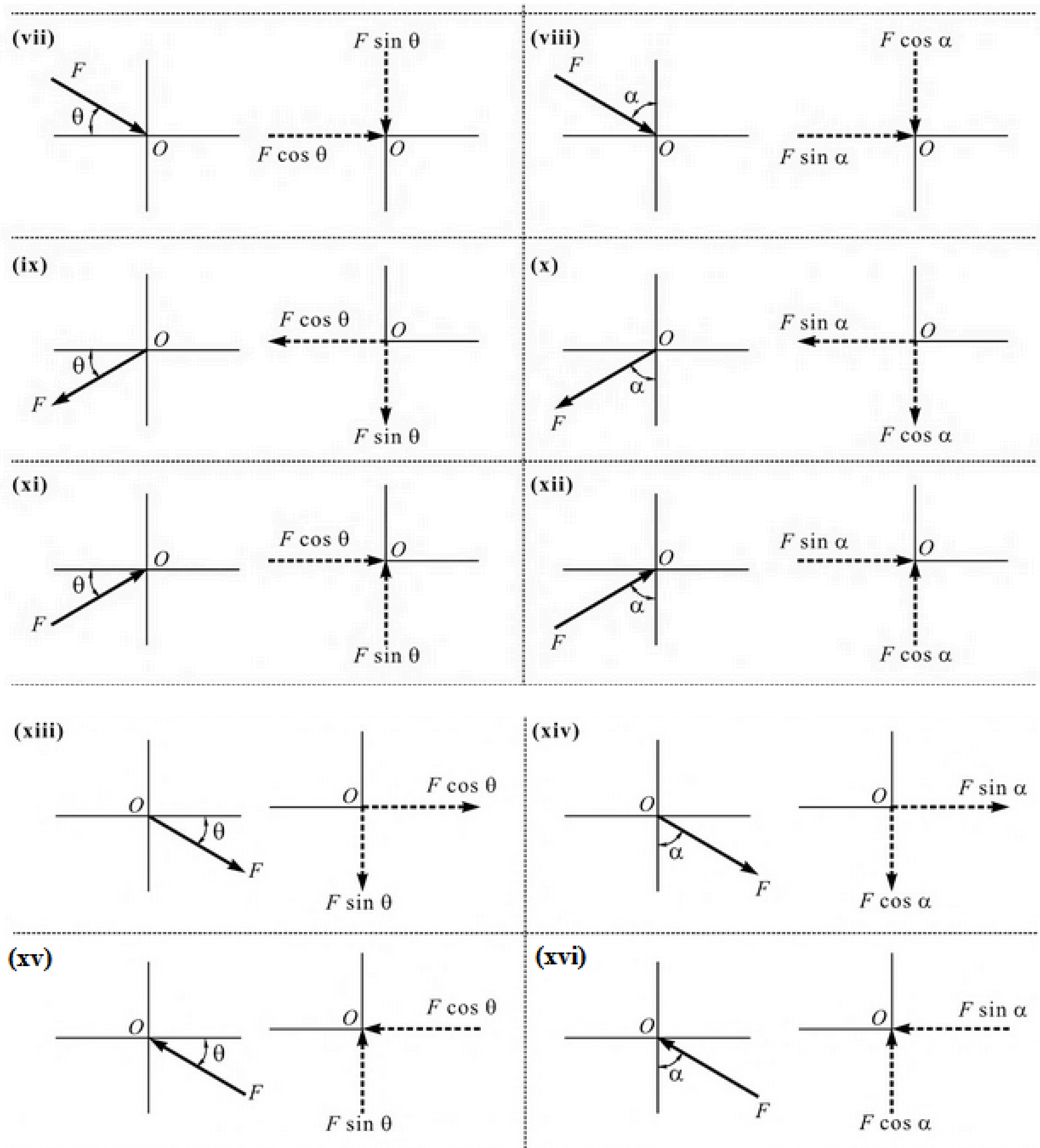


Fig. 1.9

i. Forces acting horizontally towards right are + ve and left are – ve.
ii. Forces acting vertically upward are + ve and downward are – ve.

Resolve the given force F into horizontal and vertical components.





Resultant of Concurrent coplanar force:

Concurrent coplanar forces are those forces which act in the same plane and they intersect or meet at a common point.

1. **When two forces act at a point:**
   a) Analytical Method: When two forces act at a point, their resultant is found by the law of parallelogram of forces.

b) Graphical Method:

i) Choose a convenient scale to represent the forces P and Q.

ii) From point O, draw a vector Oa=P.

iii) Now from point o, draw another vector Ob=Q and at an angle of α.

iv) Complete the parallelogram by drawing a line ac|| Ob and bc || Oa.

v) Measure the length OC. Then resultant R will be equal to length OC× chosen scale.

vi) Also measure the angle θ, which will give the direction of resultant.

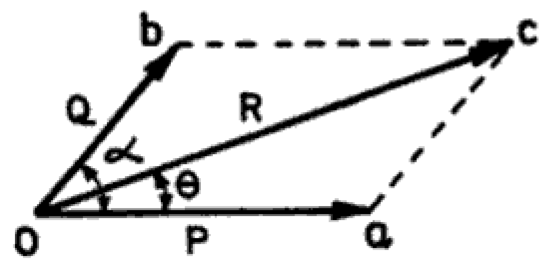


Fig.1.10

The resultant can also be determined graphically by drawing a triangle oac as explained:

i) Draw a line oa parallel to P and equal to P.

ii) From a, draw a vector ac at an angle α with the horizontal and cut ac equal to Q.

iii) Join oc. Then oc represents the magnitude and direction of resultant R. Magnitude of resultant R = Length oc × chosen scale.

iv) The direction of resultant is given by angle θ. Hence measure the angle θ.

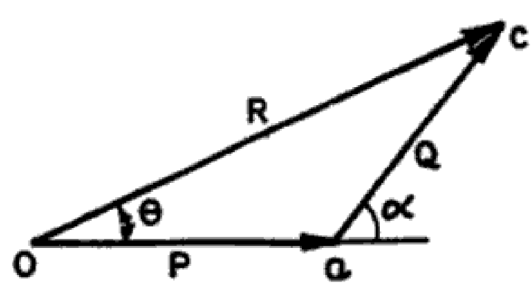


Fig.1.11

Example 1.1 *Two forces of 100 N and 150 N are acting simultaneously at a point. What is the resultant of these two forces, if the angle between them is 45°?*

Solution. Given : First force (*P*) = 100 N; Second force (*Q*) = 150 N and angle between *P* and *Q* (α) = 45°.

We know that the resultant force,

$$R = \sqrt{P^2 + Q^2 + 2\,PQ\,cos\alpha}$$
$$= \sqrt{100^2 + 150^2 + 2\times100\times150\ cos45°}\ \text{N}$$
$$= \sqrt{10\,000 + 22\,500 + (30\,000\times0.707)}\ \text{N}$$
= 232 N **Ans.**

Example 1.2. *Two forces act at an angle of 120°. The bigger force is of 40 N and the resultant is perpendicular to the smaller one. Find the smaller force.*

Solution. Given: Angle between the forces ∠*AOB* = 120°, Bigger force (*P*) = 40 N and angle between the resultant and *Q* (∠*BOC)* = 90°;

Let *Q* = Smaller force in N

From the geometry of the figure, we find that ∠*AOB*,

$$\theta = 120° - 90° = 30°$$

We know that $\tan\theta = \frac{Q\ sin\ \alpha}{P + Q\ cos\ \alpha}$

$$\tan tan\ 30^o = \left(\frac{Q sinsin\ 120^o}{40 + Q\ cos120^o}\right)$$

$$\therefore 0.577 = \left(\frac{Q\times0.866}{40 - Q\times0.5}\right) = \frac{0.866Q}{40 - 0.5Q}$$

$$40 - 0.5Q = \frac{0.866Q}{0.577}\ 1.5Q$$

$$\therefore 2Q = 40\ \ Or\ Q = 20N.$$

Example 1.3. *Find the magnitude of the two forces, such that if they act at right angles, their resultant is* $\sqrt{10}$ *N. But if they Act at 60°, their resultant is* $\sqrt{13}$ *N.*

Solution. Given: Two forces =P and Q.

First of all, consider the two forces acting at right angles. We know that when the angle between the two given forces is 90°, then the resultant force (*R*)

$$\sqrt{10} = \sqrt{P^2 + Q^2}$$

$$10 = P^2 + Q^2$$

Similarly, when the angle between the two forces is 60°, then the resultant force (*R*)

$$\sqrt{13} = \sqrt{P^2 + Q^2 + 2PQ\ Cos\ 60^O}$$

$$\therefore 13 = P^2 + Q^2 + 2PQ\times0.5 \quad PQ = 13 - 10 = 3.$$

We know that $(P + Q)^2 = P^2 + Q^2 + 2PQ = 10 + 6 = 16$.

$\therefore P + Q = \sqrt{16} = 4.$ ……… (i)

Similarly, $(P - Q)^2 = P^2 + Q^2 - 2PQ = 10 - 6 = 4$

$\therefore P - Q = \sqrt{4} = 2.$ ……… (ii)

Solving Equations (i) and (ii), P = 3 N, and Q = 1N.

2. **When more than two forces act at a point:**
   a) Analytical method: The resultant of three or more forces acting at a point is found analytically by rectangular components method.
   b) Graphical method: The resultant of several forces acting at a point is found graphically with the help of polygon law of forces.

   Let the four forces F_1, F_2, F3 and F_4 act at a point O as shown in the figure. The resultant is obtained graphically by drawing polygon of forces as explained below:

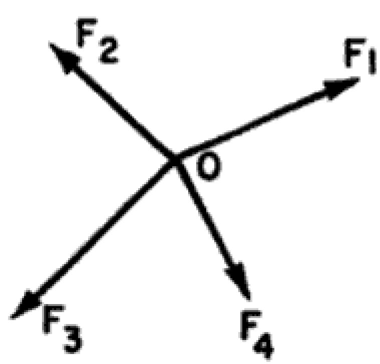


Fig. 1.12

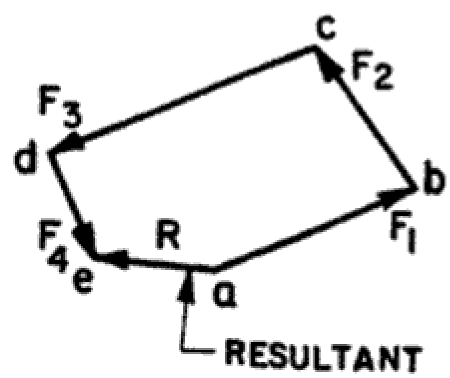


Fig.1.13

i) Choose a suitable scale to represent the given forces.

ii) Take any point a. from a, draw a vector ab parallel to OF_1. Cut ab = force F_1 to the scale.

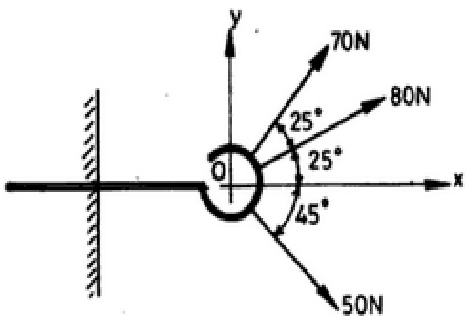
iii) From point b, draw bc parallel to OF_2. Cut bc = force F_2.

iv) From point c, draw cd parallel to OF_3. Cut cd = force F_3.

v) From point d, draw de parallel to OF_4. Cut de = force F_4.

vi) Join point a to e. this is the closing side of the polygon. Hence ae represents the resultant in magnitude and direction.
Magnitude of resultant R = Length ae × scale.
The resultant is acting from a to e.

Example 1.4. *Determine the resultant of three forces acting on a hook as shown in the Figure below.*



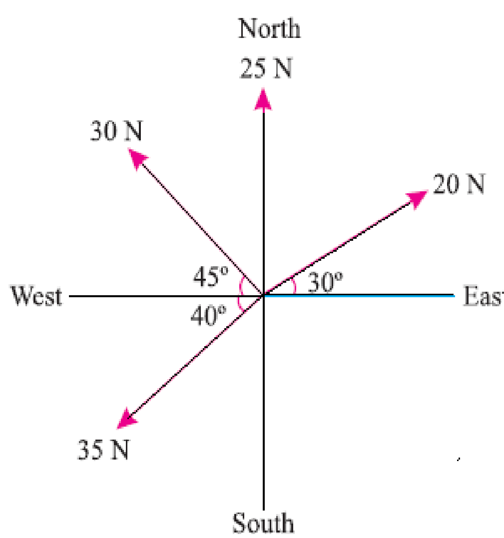
Solution.

Force	x-component	y- component
70 N	70 cos 50° = 45 N	70 sin 50° = 53.62 N
80 N	80 cos 25° = 72.50 N	80 sin 25° = 33.81 N
50 N	50 cos 45° = 35.36 N	-50 sin 45° = -35.36 N
	$\Sigma F_x = 152.86$ N	$\Sigma F_y = 52.07$ N

$$\therefore R = \sqrt{F_x^2 + F_y^2} = \sqrt{152.86^2 + 52.07^2} = 161.48\text{N}.$$

$$\theta = tan^{-1}\left(\frac{F_y}{F_x}\right) = tan^{-1}\left(\frac{52.07}{152.86}\right) = 18.81°.$$

Example 1.5. *The following forces act at a point:*
(*i*) *20 N inclined at 30° towards North of East,*
(*ii*) *25 N towards North,*
(*iii*) *30 N towards North West, and*
(*iv*) *35 N inclined at 40° towards South of West.*
Find the magnitude and direction of the resultant force.
Solution. The system of given forces is shown in Fig.



Force	x-component	y- c
20 N	20 cos 30° = 17.32N	20 s
25 N	25 cos 90° = 0 N	25 s
30 N	-30 cos 45° = -21.21N	30 s

35 N	-35 cos 40º = -26.81N	-35 sin 40º = -22.50 N

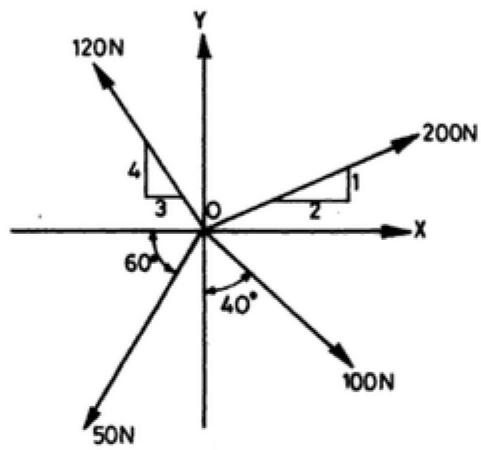
ΣF_x = -30.7 N ΣF_y = 33.7 N

$\therefore R = \sqrt{F_x^2 + F_y^2} = \sqrt{(-30.7)^2 + 33.7^2}$ = 45.6 N.

$\theta = tan^{-1}\left(\frac{F_y}{F_x}\right) = tan^{-1}\left(\frac{33.7}{-30.7}\right)$ = 47.7º.

Since ΣH is negative and ΣV is positive, therefore resultant lies between 90° and 180°. Thus actual angle of the resultant = 180° – 47.7° = 132.3°

Example 1.6. *A system of four forces are acting as shown in the figure below. Determine the resultant.*



Let θ_1 is the inclination of 200 N force with horizontal

$\therefore = \frac{1}{2}$; $sin\theta_1 = \frac{1}{\sqrt{5}}$; cos $cos\ \theta_1 = \frac{2}{\sqrt{5}}$

Let θ_2 is the inclination of 120 N force with horizontal

$\therefore = \frac{4}{3}$; $sin\theta_2 = \frac{4}{5}$; cos $cos\ \theta_2 = \frac{3}{5}$

Force	x-component	y- component
200 N	$200 \times \frac{2}{\sqrt{5}}$ = 178.57N	$200 \times \frac{1}{\sqrt{5}}$ = 89.29 N
120 N	$-120 \times \frac{3}{5}$ = -72 N	$120 \times \frac{4}{5}$ = 96 N
50 N	-50 cos 60° = - 25 N	-50 sin 60° = -43.30 N
100 N	100 sin 40° = 64.28 N	-100 cos 40° = -76.6 N
	ΣF_x = 145.85 N	ΣF_y = 65.39 N

$$\therefore R = \sqrt{F_x^2 + F_y^2} = \sqrt{(145.85)^2 + 65.39^2} = 159.84 \text{ N.}$$

$$\theta = tan^{-1}\left(\frac{F_y}{F_x}\right) = tan^{-1}\left(\frac{65.39}{145.85}\right) = 24.15^o.$$

Example 1.7. *A force of 100 units acts along the line OP, terminating at P. If the coordinates of point O and P are (3,-1, 2) and (10,5,8), respectively, specify the force in terms of the unit vectors.*

Solution. Unit vector along $OP\rightarrow = \frac{(10-3)i+(5-(-1))j+(8-2)k}{\sqrt{(10-3)^2+(5+1)^2+(8-2)^2}}$

$$= \frac{7i+6j+6k}{11}$$

So, the force vector $F = 100\times \frac{7i+6j+6k}{11} = 63.64i + 54.54j + 54.54k$

Example 1.8. *A point P is located as (-5, 2, 14) with respect to an origin O (0,0,0). Specify its position vector (i) in terms of the rectangular components, (ii) in terms of its unit vector, and (iii) in terms of its direction cosines.*

Solution. Position vector of P with respect to origin O

$$= (-5 - 0)i + (2 - 0)j + (14 - 0)k = -5i + 2j + 14k$$

Therefore,

Unit vector along $OP\rightarrow = \frac{-5i+2j+14k}{\sqrt{-5^2+2^2+14^2}} = \frac{-5i+2j+14k}{15}$

$$= -0.33i + 0.13j + 0.93k$$

So, the position vector of P in terms of unit vector

$$= \sqrt{-5^2 + 2^2 + 14^2}\,(-0.33i + 0.13j + 0.93k)$$

$$= -5i + 2j + 14k$$

Direction cosines, $l = cos\alpha = \frac{-5}{\sqrt{-5^2+2^2+14^2}} = -0.33$

$$m = cos\,\beta = \frac{2}{\sqrt{-5^2+2^2+14^2}} = 0.13$$

$$n = cos\gamma = \frac{14}{\sqrt{-5^2+2^2+14^2}} = 0.93$$

So, the position vector of P in terms of direction cosine

$$= \sqrt{-5^2 + 2^2 + 14^2}\,(li + mj + nk)$$

$$= 15li + 15mj + 15nk$$

Example 1.9. *A force of 210N forms angles 53°, 77°, 142° with x-, y-, and z-axes, respectively. Express the force as a vector.*

Solution. Here, direction cosines are

$$l = cos\alpha = cos53^o = 0.6018$$

$$m = cos\beta = \cos cos\,77^o = 0.2249$$

$$n = cos\gamma = \cos cos\,142^o =- 0.788$$

Thus, the vector will be $= 210(li + mj + nk)$

$$= (210\times0.6018)i + (210\times0.2249)j - (210\times0.788)k$$

$$= 126.378i + 47.229j - 165.48k$$

Example 1.10. *Compute the magnitude of the force F, whose components along x-,y- and z-directions are 15kN, -26kN, and -33kN, respectively. Also compute the inclinations with all axes.*

Solution. The force vector is $F = 15i - 26j - 33k$

Therefore, magnitude of $F = \sqrt{15^2 + -26^2 + -33^2} = 44.609$kN.

The inclinations are

$$\alpha = cos^{-1}\left(\frac{15}{44.609}\right) = 70.351^o$$

$$\beta = cos^{-1}\left(\frac{-26}{44.609}\right) = 125.65^o$$

$$\gamma = cos^{-1}\left(\frac{-33}{44.609}\right) = 137.711^o$$

Example 1.11. *Forces 30kN, 40kN, 50kN, and 60kN are concurrent at O(1.2.3) and are directed through M(6,3,-2), N(-4,-2,5), P(-3,2,4), and Q(4,-3,6), respectively. Determine the resultant of the system.*

Solution. Unit vector along $OM\rightarrow = \frac{(6-1)i+(3-2)j+(-2-3)k}{\sqrt{5^2+1^2+-5^2}} = \frac{1}{\sqrt{51}}(5i + j - 5k)$

Force vector along $OM\rightarrow$ is $F_1 = \frac{30}{\sqrt{51}}(5i + j - 5k)$

Unit vector along $ON\rightarrow = \frac{(-4-1)i+(-2-2)j+(5-3)k}{\sqrt{-5^2+-4^2+2^2}} = \frac{1}{\sqrt{45}}(-5i - 4j + 2k)$

Force vector along $ON\rightarrow$ is $F_2 = \frac{40}{\sqrt{45}}(-5i - 4j + 2k)$

Unit vector along $OP\rightarrow = \frac{(-3-1)i+(2-2)j+(4-3)k}{\sqrt{-4^2+1^2}} = \frac{1}{\sqrt{17}}(-4i + k)$

Force vector along $OP\rightarrow$ is $F_3 = \frac{50}{\sqrt{17}}(-4i + k)$

Unit vector along $OQ\rightarrow = \frac{(4-1)i+(-3-2)j+(6-3)k}{\sqrt{3^2+-5^2+3^2}} = \frac{1}{\sqrt{43}}(3i - 5j + 3k)$

Force vector along $OQ\rightarrow$ is $F_4 = \frac{60}{\sqrt{43}}(3i - 5j + 3k)$

Therefore, resultant of the force system

$$R = F_1 + F_2 + F_3 + F_4$$

$$\left(\frac{30\times5}{\sqrt{51}} - \frac{40\times5}{\sqrt{45}} - \frac{50\times4}{\sqrt{17}} + \frac{60\times3}{\sqrt{43}}\right)i$$

$$+\left(\frac{30}{\sqrt{51}}-\frac{40\times4}{\sqrt{45}}-\frac{60\times5}{\sqrt{43}}\right)j$$

$$+\left(-\frac{30\times5}{\sqrt{51}}+\frac{40\times2}{\sqrt{45}}+\frac{50}{\sqrt{17}}+\frac{60\times3}{\sqrt{43}}\right)k$$

$$= -29.86i - 65.4j + 30.49k$$

Magnitude of the resultant = $\sqrt{-29.86^2 + -65.4^2 + 30.49^2}$ = 78.09kN

Inclination with *x*-.*y*-.and *z*-axes, respectively,

$$\alpha = cos^{-1}\left(\frac{-29.86}{78.09}\right) = 112.48^o$$

$$\beta = cos^{-1}\left(\frac{-65.4}{78.09}\right) = 146.87^o$$

$$\gamma = cos^{-1}\left(\frac{30.49}{78.09}\right) = 67.01^o$$

Resultant of coplanar non-concurrent forces:

If all the forces lie in the same plane and lines of action of all the forces do not intersect or meet at a single point the system is said to be coplanar non-concurrent force system.

Parallel forces are the special cases of non-concurrent force system. The resultant of parallel forces cannot be determined by the law of parallelogram of forces. Hence, the resultant of such forces can be determined with the help of *principle of moments (Varignon's Principle).*

Moment of a Force: The product of a force and the perpendicular distance of the line of action of the force from a point is known as moment of the force about that point.

Let F = A force acting on a body.

r = Perpendicular distance from the point O on the line of action of force F.

Then moment (M) of the force F about O is given by, M =F × r.

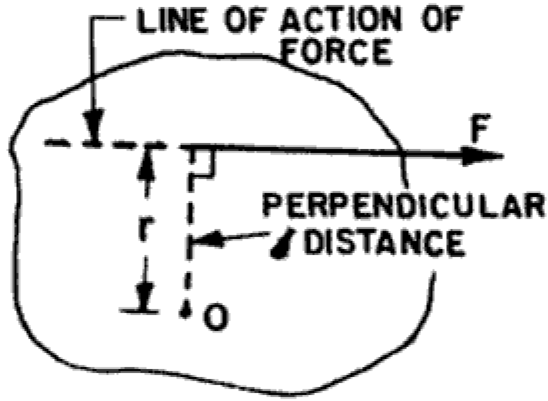


Fig.1.14

The tendency of this moment is to rotate the body in the clockwise direction about O. Hence this moment is called *clockwise moment.* If the tendency of a moment is to rotate the body in anti-clockwise direction, then that moment is known as *anti-clockwise moment*. If clockwise moment is taken – ve then anti-clockwise moment will be + ve.

In S.I. system, moment is expressed in N m (Newton meter).

Principle of Moments (Varignon's principle):

Principle of moments states that the moment of the resultant of number of forces about any point is equal to the algebraic sum of the moments of all the forces of the system about the same point.

According to *Varignon's principle* the moment of a force about any point is equal to the algebraic sum of the moments of its components about the same point.

Proof of Varignon's Principle:

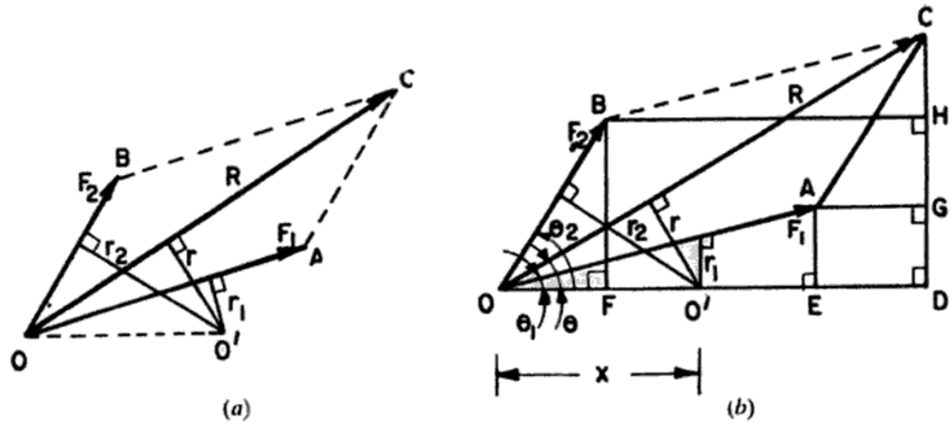


Fig. 1.15

Let us consider two forces F_1 and F_2 are acting at point O. These forces are represented in magnitude and direction by OA and OB. Their resultant R is represented in magnitude and direction by OC which is the diagonal of parallelogram OACB. Let O' is the point in the plane about which moments of F_1, F_2 and R are to be determined. From Point O' draw perpendiculars on OA, OC and OB.

Let r_1 = Perpendicular distance between F_1 and O'.

r = Perpendicular distance between R and O'.

r_2 = Perpendicular distance between F_2 and O'.

Then according to Varignon's principle;

Moment of R about O' must be equal to algebraic sum of moments of F_1 and F_2 about O'.

Or $$R \times r = F_1 \times r_1 + F_2 \times r_2.$$

Join OO' and produce it to D. From points C.A and B draw perpendiculars on OD meeting at D,E and F respectively. From A and B also draw perpendiculars on CD meeting the line CD at G and H respectively.

Let θ_1 = Angle made by F_1 with OD,

θ = Angle made by R with OD, and

θ_2 = Angle made by F_2 with OD.

From the Fig.1.15 (b) OA = BC, hence the projection of OA and BC on the same vertical line CD will be equal i.e. GD = CH as GD is the projection of OA on CD and CH is the projection of BC on CD.

Then from the Fig.1.15 (b) , we have

$F_1 \sin \theta_1 = AE = GD = CH$

$F_1 \cos \theta_1 = OE$

$F_2 \sin \theta_2 = BF = HD$

$F_2 \cos \theta_2 = OF = ED$

From triangle OCD

$R \sin \theta = CD$ and

$R \cos \theta = OD$.

Let the length OO' = x

Then $x \sin \theta_1 = r_1$, $x \sin \theta = r$ and $x \sin \theta_2 = r_2$

Now moment of R about O'

$= R \times (\perp$ distance between O' and R$) = R \times r$

$= R \times x \sin \theta$

$= (R \sin\theta) \times x$

$= CD \times x$

$= (CH+HD) \times x$

$= (F_1 \sin \theta_1 + F2 \sin \theta_2) \times x$

$= F_1 \times x \sin \theta_1 + F_2 \times x \sin \theta_2$

$= F_1 \times r_1 + F_2 \times r_2$

= Moment of F_1 about O' + Moment of F_2 about O'

Hence moment of R about any point is the algebraic sum of moments of its components about the same point. Hence Varignon's principle is proved.

The principle of moments (or Varignon's principle) is not restricted only to concurrent forces but is also applicable to any coplanar force system, i.e. concurrent or non-concurrent or parallel force system.

Resultant of Parallel force system:

Procedure

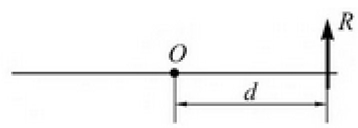
Step1: Find resultant $R = \Sigma F$. Take the algebraic sum of all the parallel forces considering proper sign convention. (+ ve ↑ / - ve ↓)

Step 2: Find ΣM_o. Take the algebraic sum of moments of forces about a point (say O) considering proper sign conventions. (+ ve ↺ / - ve ↻)

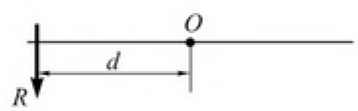
Step 3: Apply Varignon's theorem. $\Sigma M_o = R \times d$ (where d is the perpendicular distance between line of action of R and reference point O)

Step 4: Position of resultant w.r.t. point O. Resultant may lie to the right or left of the reference point O at a distance d, depending on sign of ΣF and ΣM_o.

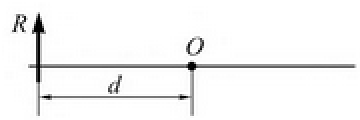
i) ΣF ↑ and ΣM_o (+ ve ↺)



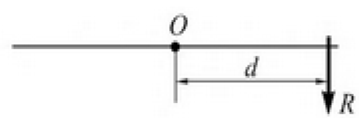
ii) ΣF ↓ and ΣM_o (- ve ↻)



iii) ΣF ↑ and ΣM_o (- ve ↻)



iv) ΣF ↓ and ΣM_o (+ ve ↺)



Couple

Two non collinear parallel forces of equal magnitude and in opposite direction forms a *couple*. It is a special case of parallel forces which produces the *rotary effect* on a rigid body.

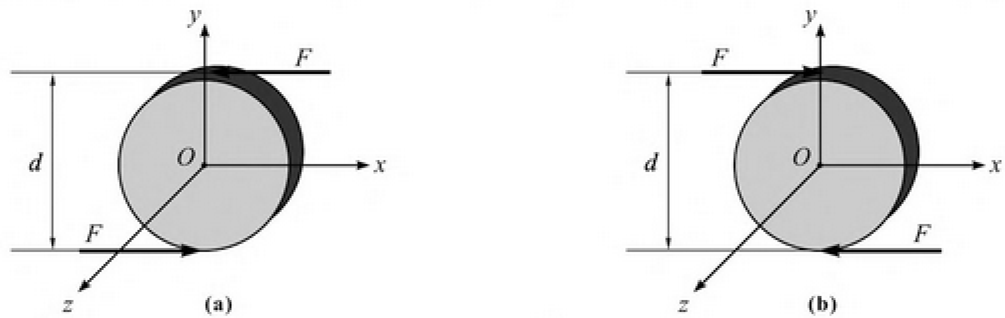


Fig. 1.16

Moment of Couple

The magnitude of rotation known as the *moment of couple* which is the product of common magnitude of the two forces F and of the perpendicular distance d (arm of the couple) between the lines of action.

From the Fig. 1.16 (a) M = F× d (↺) and from the Fig. 1.16 (b) M = F× d ↻)

Sign Convention

The couple has not only a magnitude but also a direction. This direction is perpendicular to the plane of the paper. The couple will try to rotate the body in clockwise or anticlockwise. *Anticlockwise* is taken as + ve. And *clockwise* is taken as – ve.

Properties of Couple

i) Moment of couple is equal to the product of one of the force and the arm of couple.
ii) The tendency of couple is to rotate the body about an axis perpendicular to the plane containing the two parallel forces.
iii) The resultant force of a couple system is zero.
iv) Couple can only rotate the body but cannot translate the body.
v) Moment of couple can be added algebraically as scalar quantity with proper sign convention.
vi) A couple can be replaced by a couple only and it cannot be replaced by a single force.
vii) Couple is a pure turning moment which is always constant. Couple may be moved anywhere in its own plane on a body without any change of its effect on the body. Thus a couple acting on a rigid body is known as a free vector.
viii) A system of parallel forces whose resultant is a couple can attain equilibrium only by another couple of same magnitude and opposite direction.
ix) Couple does not have moment center, like moment of force.

Why Couple is a free vector?

Case (i): Magnitude of moment of couple M is equal to the product of the common magnitude of the two force F and of the perpendicular distance between the lines of action. The sense of couple M is anticlockwise.

Moment of couple M = F× d (↺) ……. (i)

Case (ii): Take moment of forces about a point O. As per sign convention anticlockwise direction of rotation is positive.

From Fig. 1.17 (b), we have

$$M = F\times \frac{d}{2} + F\times \frac{d}{2} = F\left(\frac{d}{2} + \frac{d}{2}\right)$$

M = F× d (↺) …….. (ii)

Case (iii): Take moment of two forces about a point A.

From Fig. 1.17 (c), we have

$M = F\times d_2 - F\times d_1$

$M = F\,(d_2 - d_1)$

M = F× d (↺) ……… (iii)

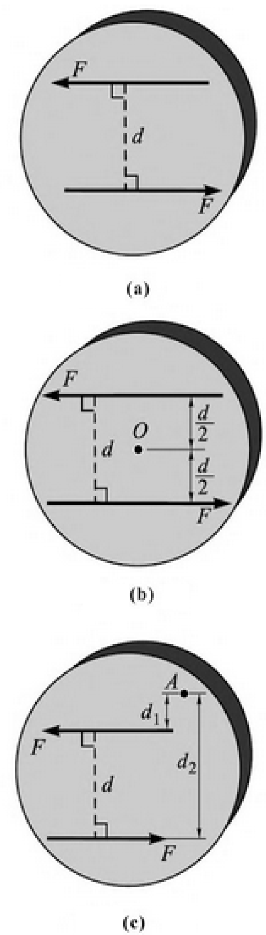


Fig. 1.17

It may be seen that equation (i), (ii) and (iii) has same result which shows that moment of couple is constant and independent of any point. So '*couple is a free vector.*'

Resolution of a Force into a Force and Couple:

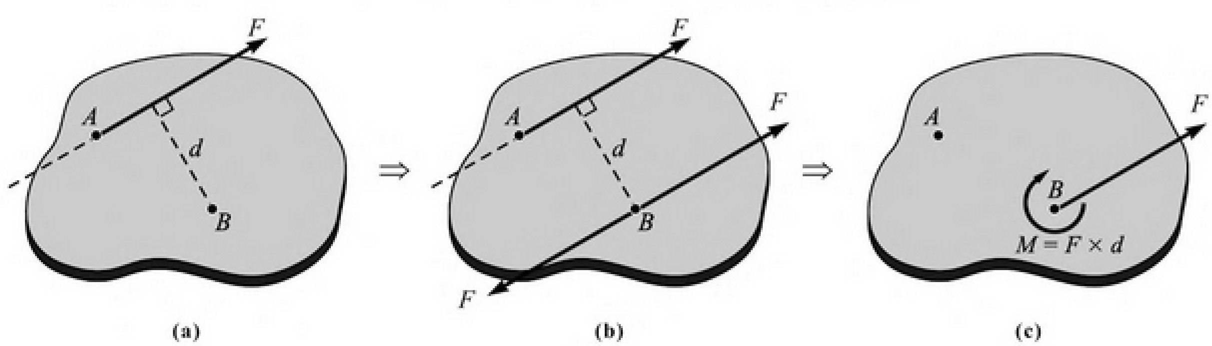


Fig. 1.18

Consider a force acting at point A on a rigid body. This force is to be replaced by a force and a couple at point B.

Now apply two forces equal in magnitude and opposite in direction parallel to force F at point B. This addition of forces does not change the effect on rigid body. Out of these three forces two forces are acting in opposite direction at A and B form a couple.

Moment of couple M = F× d (↻).

Thus, to shift a force to new parallel position, a couple is required to be added to the system. Here we can see that moment of couple (M) is equal to the moment of force about point B (M_B = F× d (↻)).

Example 1.12. *Three like parallel forces 100N, 200N and 300N are acting at points A,B and C respectively on a straight line ABC as shown in the Fig. The distances are AB = 30cm and BC = 40cm. Find the resultant and also the distance of the resultant from point A on line ABC.*

Solution. Given:

Force at A = 100 N

Force at B = 200 N

Force at C = 300 N

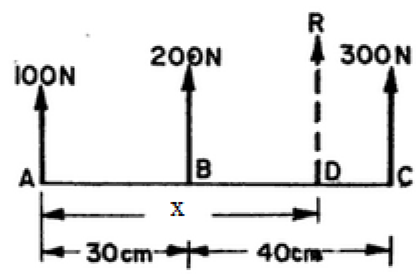


Fig. 1.19

Distance AB = 30cm, BC =40cm. As all the forces are parallel and acting in the same direction, their resultant R is given by R = 100+ 200+ 300 = 600 N

Let the resultant is acting at a distance of *x* cm from the point A.

Now take the moments of all forces about point A. The force 100N is passing through A, hence its moment about A will be zero.

∴ Moment of 100 N force about A = 0

Moment of 200 N force about A = 200 × 30 = 6000 N cm (anti-clockwise)

Moment of 300 N force about A = 300 × 70 = 21000 N cm (anti-clockwise)

Algebraic sum of moments of all forces about A = 0 +6000 + 21000 = 270000 N cm

Moment of resultant R about A = R × *x* = 600 × *x* N cm (anti-clockwise)

But algebraic sum of moments of all forces about A = Moment of resultant about A

$27000 = 600 \times x \therefore x = \frac{27000}{600} 45\ cm$

Example 1.13. *The three like parallel forces of magnitude 50 N, F and 100 N are as shown in the Fig. If the resultant R = 250N and is acting at a distance of 4m from A, then find i)Magnitude of force F and ii) Distance of F from A.*

Solution. Given:

Forces at A = 50 N, at B = F and at D = 100 N, at R = 250 N, Distance AC = 4m , CD = 3m.

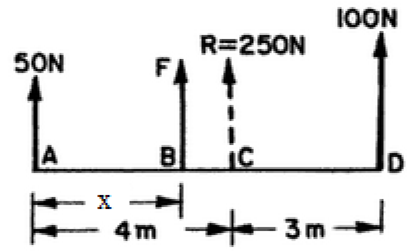


Fig. 1.20

i) Magnitude of force F

The resultant R of three like forces is given by,

$$R = 50+F+100$$

$$250 = 50 + F + 100$$

$$\therefore F = 250 - 50 - 100 = 100\ N.$$

ii) Distance of F from A

Take the moments of all forces about point A.

Moment of force %0 N about A =0

Moment of force about A = F × x (anti-clockwise)

Moment of force 100 N about A = 100 × 7 = 700 N m (anti-clockwise)

Algebraic sum of moments of all forces about A

= 0 + F × x + 700

Moment of resultant R about A = R × 4 = 250 × 4 =1000 N m (anti-clockwise)

But algebraic sum of moments of all forces about A = Moment of resultant about A

$$\therefore F \times x + 700 = 1000$$

$$F \times x = 300$$

$$x = \frac{300}{F} = \frac{300}{100}$$

x = 3 m.

Example 1.14. *Four parallel forces of magnitudes 100 N, 150 N, 25 N and 200 N are acting as shown in the Fig. Determine the magnitude of the resultant and also the distance of the resultant from point A.*

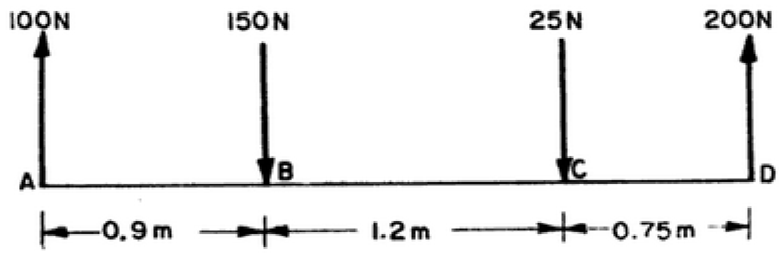


Fig. 1.21

Solution. Given:

Forces are 100 N, 150 N, 25 N and 200 N.

Distances AB = 0.9m, BC = 1.2m, CD = 0.75m.

As all the forces are acting vertically, hence their resultant R is given by

$$R = 100 - 150 - 25 + 200 = 125 \text{ N.}$$

+ ve sign shows that R is acting vertically upwards. To find the distance of R from point A, take the moments of all forces about point A.

Let x = Distance of R from A in meter.

As the force 100 N is passing through A, its moment about A will be zero.

Moment of 150 N force about A = 150 × AB = 150 × 0.9 (clockwise) = -135 N m

Moment of 25 N force about A = 25 × AC = 25 × 2.1 (clockwise) = -52.5 N m

Moment of 200 N force about A = 200 × AD = 200 × 2.85 (anti-clockwise) = 570 N m

Algebraic sum of moments of all forces about A = -135- 52.5 + 570 = 382.5 N m

+ve sign shows that this moment is anti-clockwise. Hence the moment of resultant R about A must be 382.5 N m. The moment of R about A will be anti-clockwise if R is acting upwards and towards the right of A.

Now moment of R about A = R × x = 125 × x

But algebraic sum of moments of all forces about A = Moment of resultant about A

$$\therefore 382.5 = 125 \times x$$

$$x = \frac{382.5}{125} = 3.06 \text{ m}$$

∴ Resultant will be 125 N upwards and is acting at a distance of 3.06 m to the right of point A.

Example 1.15. *A system of parallel forces are acting on a rigid bar as shown in the Fig. Reduce this system to :*

i) *A single force*
ii) *A single force and a couple at A*
iii) *A single force and a couple at B.*

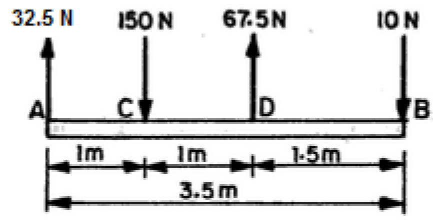


Fig. 1.22

Solution. Given:

Forces at A,C,D and B are 32.5 N, 150 N, 67.5 N and 10 N respectively.

Distances AC = 1 m, CD = 1 m and BD =1.5 m.

i) Single force system

The single force system will consist only resultant force in magnitude and location. All the forces are acting in the vertical direction and hence their resultant in magnitude is given by

$$R = 32.5 - 150 + 67.5 - 10 = -60 \text{ N}.$$

Negative sign shows that resultant is acting vertically downwards.

Let x = Distance of resultant from A towards right. To find the location of the resultant take the moments of all forces about A, we get moment of resultant about A.

= Algebraic sum of moments of all forces about A.

$R \times x = -150 \times AC + 67.5 \times AD - 10 \times AB$

$-60 \times x = -150 \times 1 + 67.5 \times 2 - 10 \times 3.5$

$-60x = -150 + 135 - 35 = -50$

$$\therefore x = \frac{-50}{-60} = 0.833 \text{ m}$$

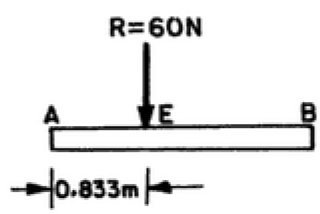


Fig. 1.23

Hence the given system of parallel forces is equivalent to a single force 60 N acting vertically downwards at point E at a distance of 0.833m from A.

ii) A single force and a couple at A

The resultant force R acting at point E can be replaced by an equal force applied at point A in the same direction together with a couple.

The moment of the couple = 60 × 0.833 =-49.98 Nm ($\because clockwise - ve$)

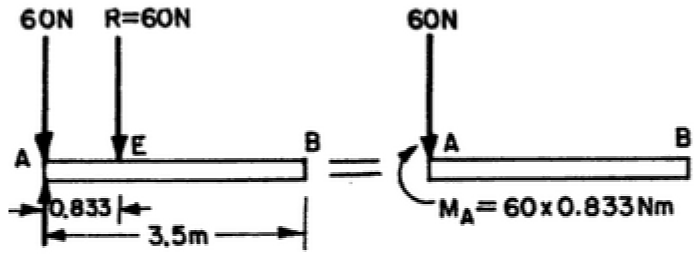


Fig. 1.24

iii) A single force and a couple at B

First find the distance BE.

BE = AB – AE = 3.5 – 0.833 = 2.667 m.

Now if the force R = 60 Nis moved to the point B, it will be accompanied by a couple of moment 60 × BE = 60 × 2.667 N m.

= 160 N m.

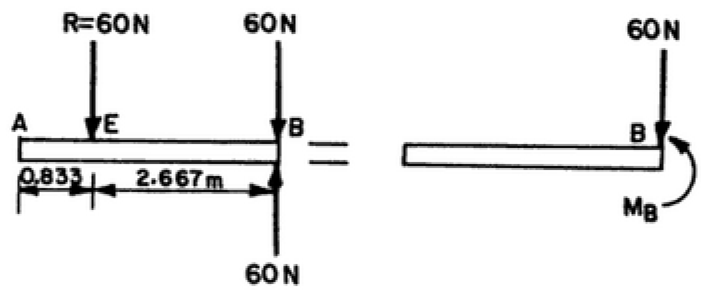


Fig. 1.25

Example 1.16. *The collar A as shown in the Fig. can slide on a frictionless vertical rod and is attached with a spring. The spring constant is 600 N/m and the spring is unstretched when H =300mm. Knowing that the system is in equilibrium when H = 400 mm, determine the weight of the collar.*

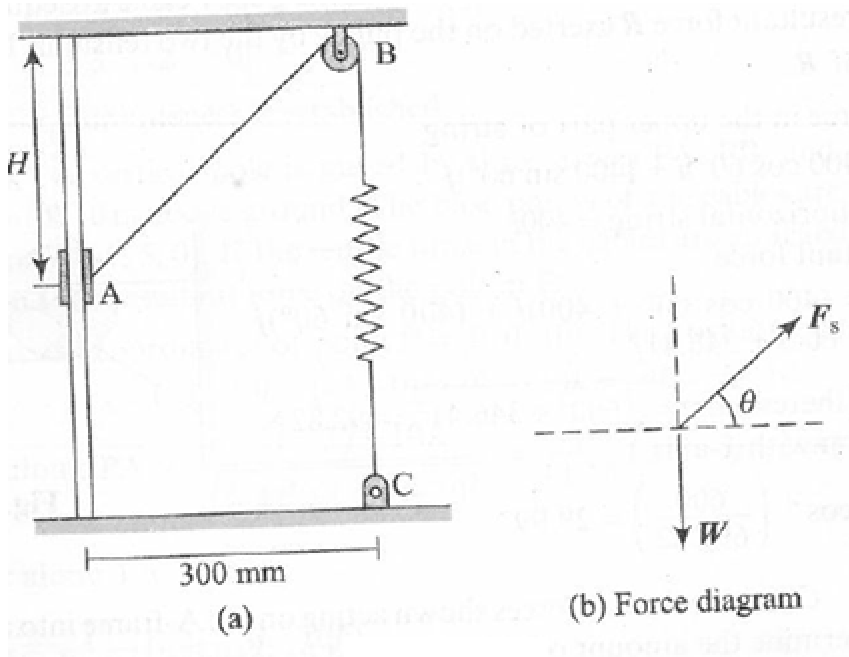


Fig. 1.26

Solution. When H = 300mm, the length of AB = $300\sqrt{2}$ mm and when H = 400 mm, the length of AB = $\sqrt{300^2 + 400^2}$ = 500mm.

So, stretching of the spring during equilibrium = (500 - $300\sqrt{2}$) mm = 75.736 mm

The force induced in the spring

$F_s = \frac{600}{1000} \times 75.736$ = 45.44 N

From force diagram, $\theta = tan^{-1}\left(\frac{400}{300}\right)$ = 53.13°.

Now, $F_s = (45.44\, cos\theta)i + (45.44\, sin\theta)j = 27.26\, i + 36.35\, j$

And W = -Wj

So from the equations of equilibrium, 36.35-W = 0. Hence, the weight of the collar is 36.35N.

Example 1.17. *A block of weight W is suspended by a 600 mm long cord PR and by two springs PQ and PS, each having an unstretched length of 450 mm. Determine the i) tension in the cord and ii) weight of the block. Take* k_{PQ} *=1500 N/m.* k_{PS}*= 500 N/m.*

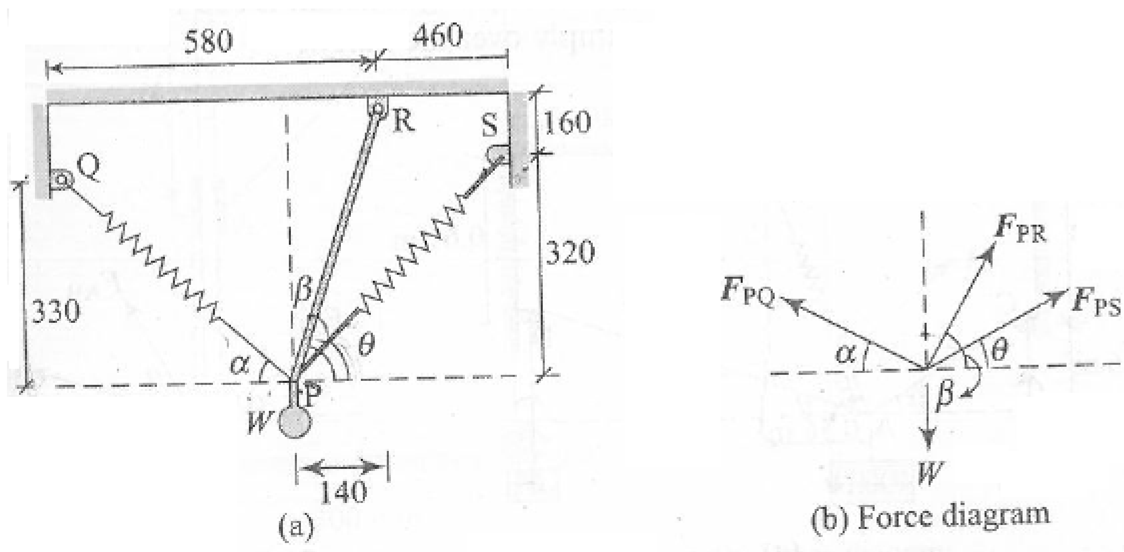


Fig. 1.27

Solution. Length of PS in stretched condition:

$$= \sqrt{320^2 + (460 + 140)^2} = 680 \text{ mm}$$

Length of PQ in stretched condition:

$$= \sqrt{330^2 + (580 - 140)^2} = 550 \text{ mm}$$

Stretched length of PQ = 550 – 450 = 100 mm

Stretched length of PS = 680 – 450 = 230 mm

Therefore, due to stretching, force in spring PQ,

$$F_{PQ} = \frac{1500}{1000} \times 100 = 150 \text{ N}$$

Force in spring PS,

$$F_{PS} = \frac{500}{1000} \times 230 = 115 \text{ N}$$

From the Fig.,

$$\alpha = tan^{-1}\left(\frac{330}{440}\right) = 36.87^{\circ}$$

$$\beta = tan^{-1}\left(\frac{160+320}{140}\right) = 73.74^{\circ}$$

$$\theta = tan^{-1}\left(\frac{320}{600}\right) = 28.07^o$$

Now, $F_{PQ} = (-\ 150\ cos\alpha)i\ +\ (150\ sin\alpha)j$ = -120i + 90j

$F_{PS} = \ (115\ cos\theta)i\ +\ (115\ sin\theta)j$ = 101.47i + 54.11j

$F_{PR} = \left(F_{PR}\ cos\beta\right)i\ +\ \left(F_{PR}\ sin\beta\right)j$ = (0.22799F_{PR}) i +(0.96F_{PR}) j

W = -$W\,j$

The equations of equilibrium are

$$-120 + 101.47 + 0.2799F_{PR} = 0$$

$$90+54.11+0.96F_{PR}-W = 0$$

Solving the above equations, F_{PR} = 66.2 N and W = 207.66 N.

Example 1.18. *From the arrangement shown in the figure, determine the value of W and unstretched length of the spring if the spring constant is 800N/m. Assume the pulleys as frictionless and strings pass simply over the pulleys.*

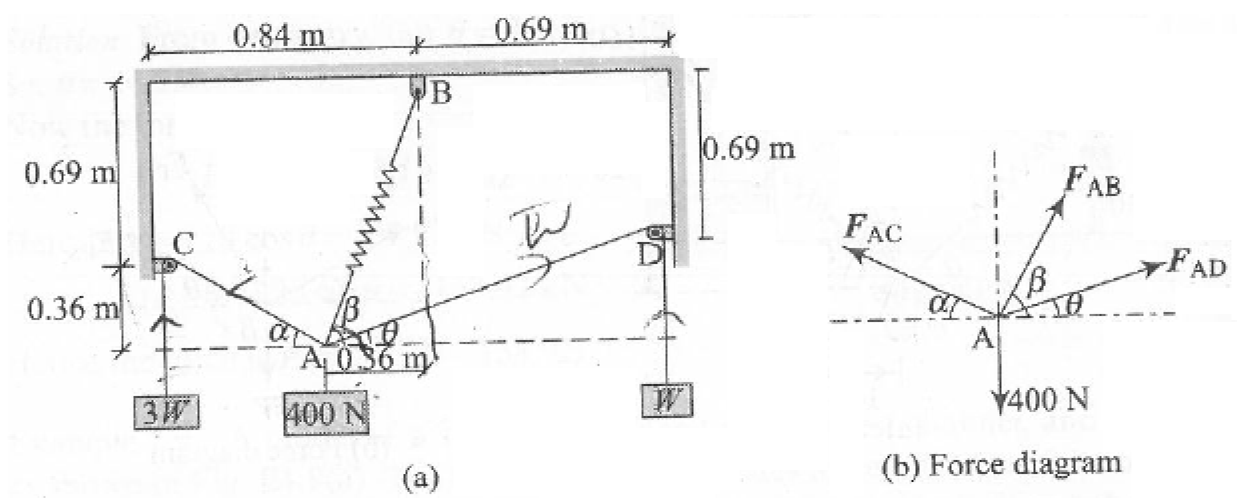


Fig. 1.28

Solution. From the Fig.

$$\alpha = tan^{-1}\left(\frac{0.360}{0.84-0.36}\right) = 36.87^o$$

$$\beta = tan^{-1}\left(\frac{0.69+0.36}{0.36}\right) = 71.07^o$$

$$\theta = tan^{-1}\left(\frac{0.36}{0.69+0.36}\right) = 18.92^o$$

Length of AB in stretched condition

$$= \sqrt{0.36^2 + (0.69 + 0.36)^2} = 1.11 \text{ m}$$

As the strings are simply passing over the pulleys at C and D, the force in string AC is $3W$and that in AD is W. So, $F_{AC} = 3W$ AND $F_{AD} = W$. Let us consider the unstretched length of spring = L m.

So, $F_{AB} = (1.11\text{-L}) \times 800$ N

Now, $F_{AB} = \{(1.11\text{-L}) \times 800 \cos \beta\}i + \{(1.11\text{-L}) \times 800 \sin \beta\}j$

$= 259.53\ (1.11\text{- L})\ i + 756.73\ (1.11\text{-L})\ j$

$F_{AD} = (F_{AD} \cos\theta)\ i + (F_{AD} \sin\theta)\ j$

$= (0.946\ F_{AD})\ i + (0.324\ F_{AD})\ j$

$F_{AC} = -\ (F_{AC} \cos \alpha\ )\ i + (F_{AC} \sin \alpha\ )\ j$

$= (-0.79\ F_{AC}\ )\ i + (0.6\ F_{AC}\ )\ j$

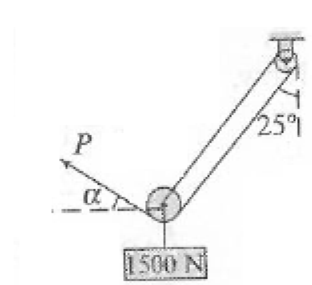
The resultant of the force system is

$259.53\ (1.11\text{- L}) + (0.946 \times W) - 0.79 \times 3W = 0$

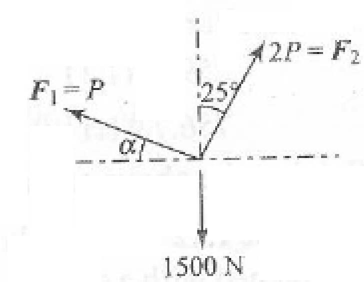
$756.73\ (1.11\text{-L}) + 0.324 \times W + 0.6 \times 3\text{W} = 400$

Solving the above equations of equilibrium, we obtain, W = 63.78 N, L = 0.76 m.

Example 1.19. *A 1500 N load is supported by the rope and smooth pulley arrangement as shown in Fig. Determine the magnitude and the direction of force P which should be exerted at the free end of the rope to maintain the equilibrium.*



(a)



(b) Force diagram

Solution. Here everywhere in the rope, same amount of force P will be effective. From the force diagram, we can write

$F_2 = (2\text{P} \sin 25^o)\ i + (2\text{P} \cos 25^o)\ j$

$F_1 = (-P \cos \alpha)\, i + (P \sin \alpha)\, j$

So, the equations of equilibrium are

$2P \sin 25^o - P \cos \alpha = 0$

$2P \cos 25^o + P \sin \alpha - 1500 = 0$

Solving the above equations, we get

$\alpha = 32.3^o$ and P = 639.12 N.

Example 1.20. *In the Fig. 1.30, if the tensions in the pulley cable are equal, i.e., 400 N, express the resultant force R exerted on the pulley by the two tensions. Determine the magnitude of R.*

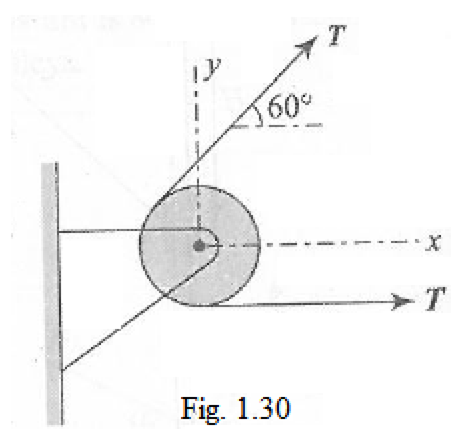


Fig. 1.30

Solution. Force in the upper part of the string = $(400 \cos 60^o)\, i + (400 \sin 60^o)\, j$

Force in the horizontal string = 400 i

So, the resultant force

$R = (400 \cos 60^o + 400)\, i + (400 \sin 60^o)\, j = 600\, i + 346.41\, j$

Magnitude of the resultant = $\sqrt{600^2 + 346.41^2}$ = 692.82 N

And inclination with x- axis, $\alpha = cos^{-1}\left(\frac{600}{692.82}\right) = 29.99^o$

Moment of a force about a point other than Origin:

Suppose the moment of a force about a point other than the origin is required then we need to consider the vector from the point about which the moment is required to the given point of application of the force. For instance, if the moment of the force acting at A is required about a point B then the position of point A with respect to point B is

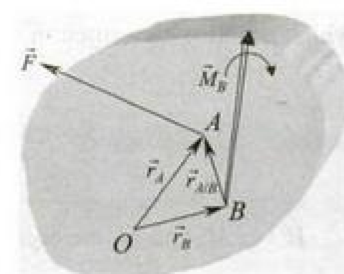


Fig. 1.31

$$\vec{r}_{A/B} = \vec{BA} = \vec{OA} - \vec{OB} = \vec{r}_A - \vec{r}_B$$

Therefore, the moment of the force about the point B is

$$\vec{M}_B = \vec{r}_{A/B} \times \vec{F}$$

This moment is perpendicular to the plane formed by $\vec{BA}$ and $\vec{F}$.

Example 1.21. *A force F = (6i+8j-13k) N passes through point P (1.2.3). Compute the moments of the force about point Q (4,5,6). The coordinate distances are measured in meters.*

Solution. Position vector r_{QP} = (1-4)i + (2-5)j + (3-6) k = -3i -3j-3k

Therefore,

Moment of the force $\vec{M}_Q = r_{QP} \times F = \begin{vmatrix} i & j & k \\ -3 & -3 & -3 \\ 6 & 8 & -13 \end{vmatrix}$ = 63i -57j-6k

And Magnitude of the moment = $\sqrt{63^2 + (-57)^2 + (-6)^2}$ = 85.17 Nm

Example 1.22. *A force F = (32i – 15j +50k) N acts at point P (4,-6,3) m. Determine the moment of this force about y-axis.*

Solution. Considering origin O = (0, 0, 0), the position vector of point P with respect to origin, $\vec{OP} = (4 - 0)i + (-6 - 0)j + (3 - 0)k$ = 4i -6j +3k

Therefore,

Moment of the force about the origin = $\begin{vmatrix} i & j & k \\ 4 & -6 & 3 \\ 32 & -15 & 50 \end{vmatrix}$ = (-255i- 104j +132k) Nm

Moment about y- axis = -104 N m.

Moment of a force about an Axis:

Further, if the moment of a force about a point (say B) is known then the component of this moment about an axis passing through that point, say BC, is given as a dot product of the unit vector along BC and $\vec{M}_B$, i.e.,

$$\left|\vec{M}_{BC}\right| = \vec{n}_{BC} \cdot \vec{M}_B$$

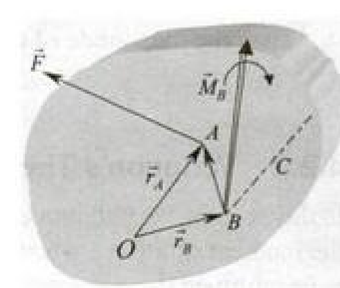


Fig. 1.32

It should be noted that moment of a force about an axis is a scalar quantity.

Example 1.23. *Moment of a certain force about a point P(3,7,-2) is* $\vec{M}_P = (10i - 8j + 40k)$ *kN m. Determine the moment of the same force about the line PQ, where the coordinates of Q are (1,8,5).*

Solution. Position vector $\vec{PQ} = (1 - 3)i + (8 - 7)j + (5 - (-2))k = -2i + j + 7k$

Unit vector, $\vec{n}_{PQ} = \frac{-2i+j+7k}{\sqrt{(-2)^2+1^2+7^2}} = -0.272i + 0.136j + 0.953k$

Therefore, Moment about $\vec{PQ}$ $\left|\vec{M}_{PQ}\right| = \vec{n}_{PQ} \cdot \vec{M}_P$

$= (-0.272i + 0.136j + 0.953k) \cdot (-2i + j + 7k)$

= 34.312 kN m.

Example 1.24. *A force of 8 kN acts along OC. Compute the moment of the force about the line BQ.*

Solution. The points are O (0,0,0), C (4,3,3), B (4,0,3), and Q (4,3,0).

Position vector

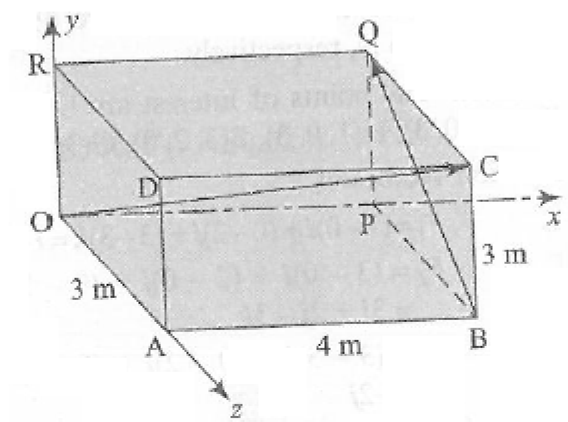


Fig. 1.33

$\vec{OC} = (4 - 0)i + (3 - 0)j + (3 - 0)k = 4i + 3j + 3k$

Unit vector along $\vec{OC}$, $\vec{n}_{OC} = \frac{4i+3j+3k}{\sqrt{4^2+3^2+3^2}} = 0.69i + 0.51j + 0.51k$

Force vector $\vec{F}_{OC} = 8(0.69i + 0.51j + 0.51k) = 5.52i + 4.08j + 4.08k$

Also, position vector $\vec{BC} = (4 - 4)i + (3 - 0)j + (3 - 3)k = 3j$

Therefore, moment of force about point B, $\vec{M}_B = \vec{r}_{BC} \times \vec{F}_{OC} = |i\ j\ k\ 0\ 3\ 0\ 5.52\ 4.08\ 4.08\ |$

$= 12.24i - 16.56k$

Position vector $\vec{BQ} = (4 - 4)i + (3 - 0)j + (0 - 3k) = 3j - 3k$

Unit vector along $\vec{BQ}$, $\vec{n}_{BQ} = \frac{3j-3k}{\sqrt{3^2+(-3)^2}} = 0.707\,j - 0.707k$

Therefore, Moment about $\vec{BQ}$ $\left|\vec{M}_{BQ}\right| = \vec{n}_{BQ}.\ \vec{M}_B$

$= (0.707\,j - 0.707k)\,.\,(12.24i - 16.56k)$

$= 11.708\ kNm$

Example 1.25. *A vertical pole is guyed by three cables PA, PB, and PC tied at a common point P 10 m above the ground. The base points of the cables are A(-4,-3,0), B(5,1,-1), and C(-1,5,0). If the tensile force in the cables are adjusted to be 15,18, and 20 kN, find the resultant force on the pole at P.*

Solution. Here the coordinates of point P = (0,0,10). The position vector of A

$$\vec{PA} = (-4 - 0)i + (-3 - 0)j + (0 - 10)k = -4i - 3j - 10k$$

Unit vector along $\vec{PA}$, $\vec{n}_{PA} = \frac{-4i-3j-10k}{\sqrt{(-4)^2+(-3)^2+10^2}} = \frac{1}{11.18}(-4i - 3j - 10k)$

Force vector along $\vec{PA}$ is $F_{PA} = \frac{15}{11.18}(-4i - 3j - 10k)$

In the similar fashion, we can write the force vectors along $\vec{PB}$ and $\vec{PC}$ as

$$F_{PB} = \frac{18}{\sqrt{5^2+1^2+(-11)^2}}\,[(5 - 0)i + (1 - 0)j + (-1 - 10)k$$

$$= \frac{18}{12.124}(5i + j - 11k)$$

And

$$F_{PC} = \frac{20}{\sqrt{(-1)^2+5^2+(-10)^2}}\,[(-1 - 0)i + (5 - 0)j + (0 - 10)k$$

$$= \frac{20}{11.224}(-i + 5j - 10k)$$

Now, the resultant of F_{PA}, F_{PB} and F_{PC} is

$$R = \left(-\frac{15\times4}{11.18} + \frac{18\times5}{12.124} - \frac{20}{11.224}\right)i + \left(-\frac{15\times3}{11.18} + \frac{18}{12.124} + \frac{20\times5}{11.224}\right)j + \left(-\frac{15\times10}{11.18} - \frac{18\times11}{12.124} - \frac{20\times10}{11.224}\right)k$$

$$= 0.247i + 6.369j - 47.567k$$

Magnitude of the resultant = $\sqrt{0.247^2 + 6.369^2 + (-47.567)^2}$ = 47.99 kN.

Example 1.26. *A transmission tower is held by three guy wires AB, AC, and AD anchored by bolts at B, C, and D, respectively. If the tension in AB is 2100 N, determine the components of the force exerted by the wire on bolt B.*

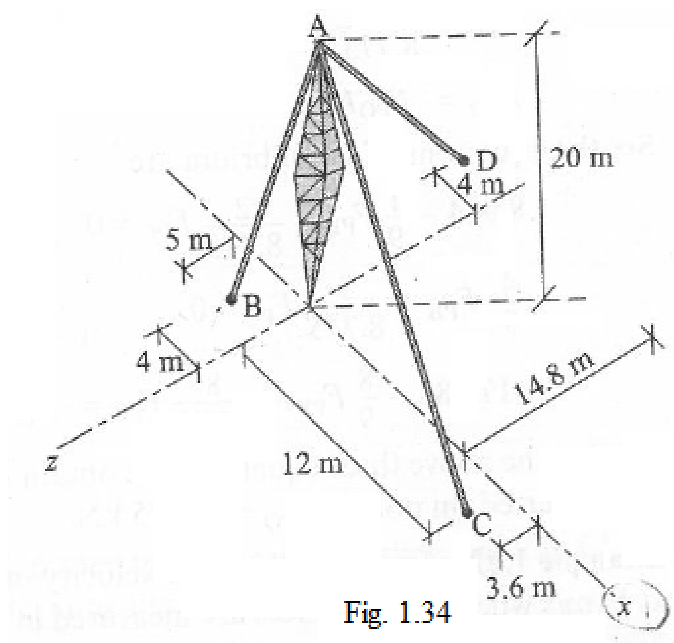


Fig. 1.34

Solution. The coordinates of A = (0, 20, 0) and B = (-4, 0, 5)

Position vector

$$\vec{BA} = (0 - (-4))i + (20 - 0)j + (0 - 5)k = 4i + 20j - 5k$$

Unit vector along $\vec{BA}$ $\vec{n}_{BA} = \frac{4i+20j-5k}{\sqrt{4^2+20^2+(-5)^2}} = \frac{1}{21}(4i + 20j - 5k)$

So, the force vector

$$F_{BA} = \frac{2100}{21}(4i + 20j - 5k) = 400i + 2000j - 500k$$

Hence, the components along *x*-, *y*- and *z*- directions are 400 N, 2000 N, and -500 N, respectively.

Example 1.27. *A vertical pole is guyed by three cables PA, PB, and PC tied at a common point P, 8 m above the ground. The base points of the cables are A(4, 0, 0), B(-1, 4, 0), and C(-2, -3, 0) m. If the tension in PA is 20 kN, calculate the tensions to be provided in PB and PC so that the resultant force exerted on the pole is vertical. Find the force exerted on the pole.*

Solution. As per the given conditions, the coordinates of P is (0, 0, 8) considering ground as datum plane and origin O (0, 0, 0). So the position vectors

$$\vec{PA} = (4 - 0)i + (0 - 0)j + (0 - 8)k = 4i - 8k$$

$$\vec{PB} = (-1 - 0)i + (4 - 0)j + (0 - 8)k =- i + 4j - 8k$$

$$\vec{PC} = (-2 - 0)i + (0 - 0)j + (0 - 8)k = -2i - 3j - 8k$$

$$\vec{PO} = (0 - 0)i + (0 - 0)j + (0 - 8)k = -8k$$

Unit vectors

$$\vec{n}_{PA} = \frac{4i-8k}{\sqrt{(4)^2+(-8)^2}} = \frac{1}{8.9443}(4i - 8k)$$

$$\vec{n}_{PB} = \frac{-i+4j-8k}{\sqrt{(-1)^2++4^2+(-8)^2}} = \frac{1}{9}(- i + 4j - 8k)$$

$$\vec{n}_{PC} = \frac{-2i-3j-8k}{\sqrt{(-2)^2++(-3)^2+(-8)^2}} = \frac{1}{8.775}(- 2i - 3j - 8k)$$

$$\vec{n}_{PO} = \frac{-8k}{\sqrt{(-8)^2}} = -k$$

Therefore, force vectors,

$$F_{PA} = \frac{20}{8.9443}(4i - 8k) = 8.944i - 17.888k$$

$$F_{PB} = \frac{F_{PB}}{9}(- i + 4j - 8k)$$

$$F_{PC} = \frac{F_{PC}}{8.775}(- 2i - 3j - 8k)$$

$$F_{PO} =- F_{PO}k$$

So, the equations of equilibrium are

$$8.994 - \frac{1}{9}F_{PB} - \frac{2}{8.775}F_{PC} = 0$$

$$\frac{4}{9}F_{PB} - \frac{3}{8.775}F_{PC} = 0$$

$$-17.888 - \frac{8}{9}F_{PB} - \frac{8}{8.775}F_{PC} = -F_{PO}$$

Solving the above three equations, we obtain F_{PB} = 21.9525 kN, F_{PC} = 28.534 kN, and force exerted on pole P, $F_{PO} = 63.475\ kN$.

Example 1.28. *Three cables DA, DC, and DB are used to tether a balloon as shown in Fig. 1.35. Determine the vertical force P exerted by the balloon at D when tension in cable DC is 270N.*

Solution. Here the points of interest are A(-4, 0, 0), B(0, 0, -3), C(2, 0, 4), D(0, 5, 0), and O(0, 0, 0). So, the unit vector along the lines of action of forces are

$$\vec{n}_{DA} = \frac{(-4-0)i+(0-5)j+(0-0)k}{\sqrt{(4)^2+(-5)^2}} = \frac{1}{6.403}(-4i - 5j)$$

$$\vec{n}_{DB} = \frac{(0-0)i+(0-5)j+(-3-0)k}{\sqrt{(-5)^2+(-3)^2}} = \frac{1}{5.831}(-5j - 3k)$$

$$\vec{n}_{DC} = \frac{(2-0)i+(0-5)j+(4-0)k}{\sqrt{(2)^2(-5)^2+(3)^2}} = \frac{1}{6.708}(2i - 5j + 4k)$$

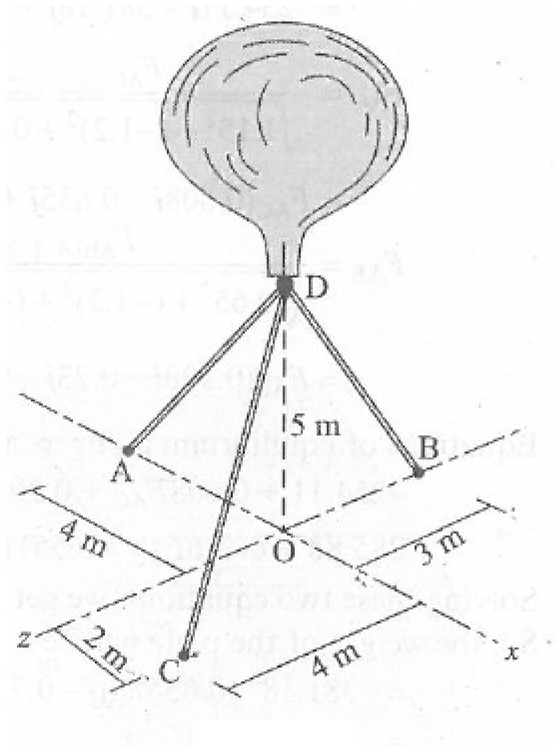


Fig. 1.35

Now, the force vectors are

$$F_{DA} = \frac{F_{DA}}{6.403}(-4i - 5j)$$

$$F_{DB} = \frac{F_{DB}}{5.831}(-5j - 3k)$$

$$F_{Dc} = \frac{270}{6.708}(2i - 5j + 4k) = 80.5i - 201.25j + 161k$$

Equations of equilibrium are

$$-0.6247F_{DA} + 80.5 = 0$$

$$-0.78F_{DA} - 0.8575\,F_{DB} - 201.25 = -P$$

$- 0.514\, F_{DB} + 161 = 0$

Solving the above equations, we obtain,

$P = 570.355$ N (upwards)

Example 1.29. *A rectangular plate is supported by three cables as shown in Fig. 1.36. If tension in the cable AD is 540 N, determine the weight of the plate.*

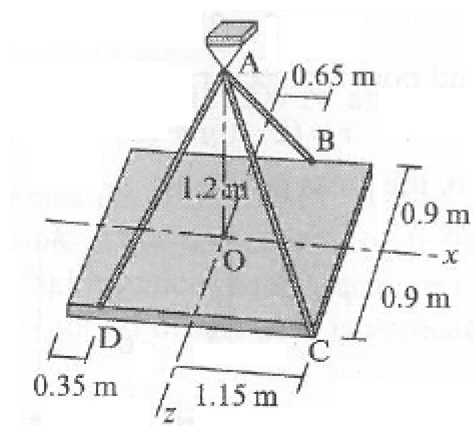


Fig. 1.36

Solution. The coordinates of the points are A(0, 1.2, 0), B(0.65, 0, -0.9), C(1.15, 0, 0.9), and D(-0.8, 0, 0.9). So, the unit vectors are

$$\vec{n}_{AD} = \frac{(-0.8-0)i+(0-1.2)j+(0.9-0)k}{\sqrt{(-0.8)^2+(-1.2)^2+0.9^2}}$$

$$= \frac{1}{1.7}(-\ 0.8i - 1.2j + 0.9k)$$

$$\vec{n}_{AC} = \frac{(1.15-0)i+(0-1.2)j+(0.9-0)k}{\sqrt{(1.15)^2+(-1.2)^2+0.9^2}}$$

$$= \frac{1}{1.89}(1.15i - 1.2j + 0.9k)$$

$$\vec{n}_{AB} = \frac{(0.65-0)i+(0-1.2)j+(-0.9-0)k}{\sqrt{(0.65)^2+(-1.2)^2+(-0.9)^2}}$$

$$= \frac{1}{1.63}(0.65i - 1.2j - 0.9k)$$

Now, the force vectors are

$$F_{AD} = \frac{540}{1.7}(-\ 0.8i - 1.2j + 0.9k)$$

$$= -\ 254.11\, i - 381.18\, j + 285.88\, k$$

$$F_{AC} = \frac{F_{AC}}{1.89}(1.15i - 1.2j + 0.9k)$$

$$= F_{AC}(0.608\, i - 0.635\, j + 0.476\, k)$$

$$F_{AB} = \frac{F_{AB}}{1.63}(0.65i - 1.2j - 0.9k)$$

$= F_{AB}(0.398\, i - 0.736\, j - 0.552\, k)$

Equations of equilibrium along *x*- and *z*- directions

$$-254.11 + 0.608\, F_{AC} + 0.398 F_{AB} = 0$$

$$285.88 + 0.476\, F_{AC} - 0.552 F_{AB} = 0$$

Solving these two equations, we get $F_{AB} = 561.4\, N$ $F_{AC} = 50.4\, N$

So, the weight of the plate will be

$= -381.18 - 0.635 F_{AC} - 0.736\, F_{AB}$

= -826.37 N = 826.37 N (downwards)

Example 1.30. *To stabilize a tree against storm, two cables AB and AC are attached to the trunk and fastened on the ground at B and C, respectively. If the tensions in AB and AC are 4.2 kN and 3.6 kN, compute in each case i) components of force exerted by cables, and ii) α, β, γ angles the forces form with axes at A which are parallel to the coordinate axes.*

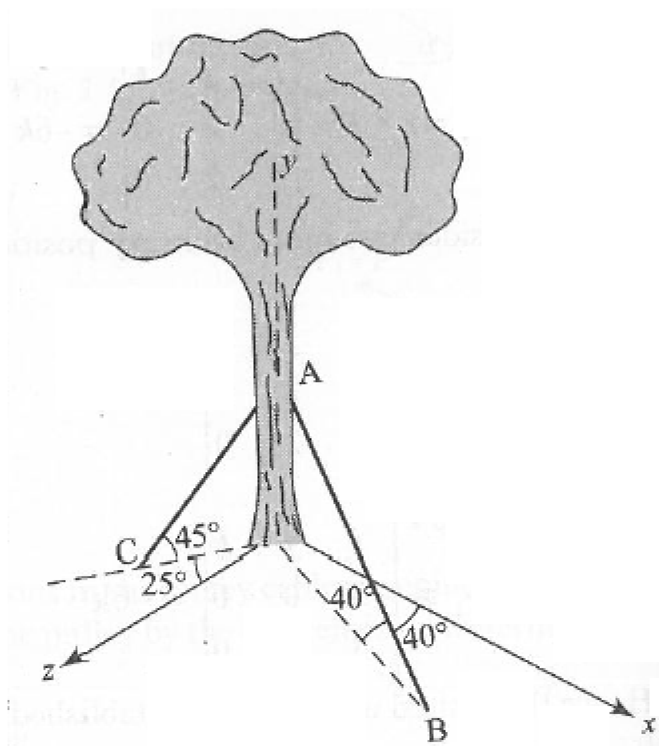


Fig. 1.37

Solution. For cable AB:

Component of 4.2 kN force on *x*-*z* plane = 4.2 cos 40° =3.2174 kN

And along *y*- axis = -4.2 sin 40° = -2.6997 kN

The component 3.2174 kN on *x*-*z* plane will have to be resolved to obtain components along *x*- and *z*- axis.

Therefore, component along *x*- axis = 3.2174 cos 40° = 2.465 kN

Component along *z*- axis = 3.2174 sin 40° = 2.068 kN

So, the inclinations of components with the three axes are

$$\alpha = cos^{-1}\left(\frac{2.465}{4.2}\right) = 54.06^{o}$$

$$\beta = \cos^{-1}\left(\frac{-2.6997}{4.2}\right) = 129.99^o$$

$$\gamma = \cos^{-1}\left(\frac{2.068}{4.2}\right) = 60.5^o$$

For cable AC:

Component of 3.6 kN force along *y*- axis = -3.6 sin 45° = -2.546 kN

Component of 3.6 kN force along *x-z* plane = 3.6 cos 45° = 2.546 kN

Component of 2.546 kN along *z*- axis = 2.546 cos 25° = 2.307 kN

Component of 2.546 kN along *x*- axis = -2.546 sin 25° = -1.076 kN

So, the inclinations are

$$\alpha = \cos^{-1}\left(\frac{-1.076}{3.6}\right) = 107.39^o$$

$$\beta = \cos^{-1}\left(\frac{-2.546}{3.6}\right) = 135^o$$

$\gamma = \cos^{-1}\left(\frac{2.307}{3.6}\right) = 50.15^o$

Example 1.31. *A horizontal circular plate having a mass of 28 kg is suspended with the help of three wires as shown in Fig. 1.38. Each wire forms 30° with the vertical. Determine the tension in each wire.*

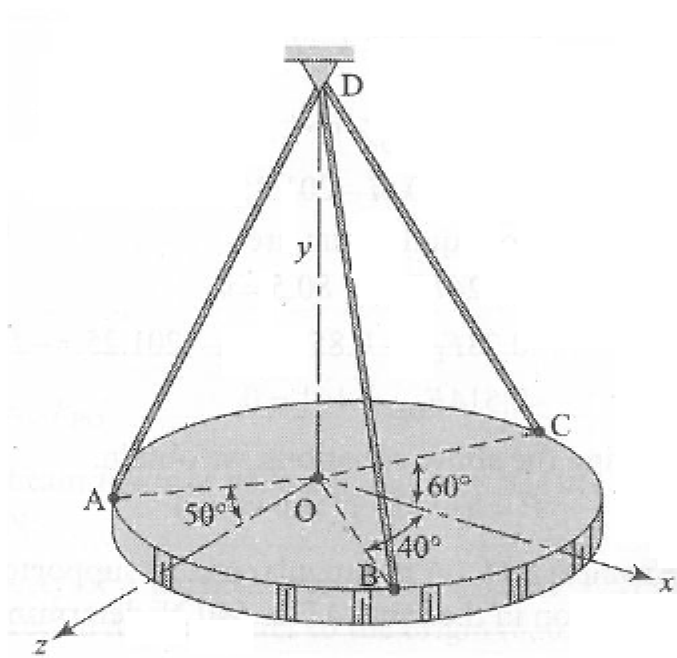


Fig. 1.38

Solution. Here the vertical components, i.e., the components along *y*- axis, of F_{DA}, F_{DB}, and F_{DC} are $-F_{DA}\cos 30^o$, $-F_{DB}\cos 30^o$, and $-F_{DC}\cos 30^o$ respectively. Components of F_{DA}, F_{DB}, and F_{DC} on *x-z* plane are $F_{DA}\sin 30^o$, $F_{DB}\sin 30^o$, and $F_{DC}\sin 30^o$ respectively.

Now, *x*- component of $F_{DA} = -(F_{DA}\sin 30^o)\sin 50^o = -0.383F_{DA}$

z-component of $F_{DA} = (F_{DA}\sin 30^o)\cos 50^o = 0.321F_{DA}$

x- component of $F_{DB} = (F_{DB}\sin 30^o)\cos 40^o = 0.383\,F_{DB}$

z- component of $F_{DB} = (F_{DB} \sin 30^o)\sin 40^o = 0.321F_{DB}$

x- component of $F_{DC} = (F_{DC} \sin 30^o)\cos 60^o = 0.25\, F_{DC}$

z- component of $F_{DC} = -(F_{DC} \sin 30^o)\sin 60^o = -0.433\, F_{DC}$

So, the vectors can be written as

$F_{DA} = F_{DA}\,(-0.383i - 0.866j + 0.321k)$

$F_{DB} = F_{DB}\,(0.383i - 0.866j + 0.321k)$

$F_{DC} = F_{DC}\,(0.25i - 0.866j - 0.433k)$

And the weight vector = $-28g\, j = -274.596\, j$

So, the equations of equilibrium are

$-0.383\, F_{DA} + 0.383\, F_{DB} + 0.25\, F_{DC} = 0$

$-0.866\, F_{DA} - 0.866\, F_{DB} - 0.866\, F_{DC} = -274.596$

$0.321\, F_{DA} + 0.321\, F_{DB} - 0.433\, F_{DC} = 0$

Solving the above three equations, we obtain.

$F_{DA} = 135.1\ N$, $F_{DB} = 46.9\ N$, $F_{DC} = 135.1$ N.

Friction:

Introduction:

When a body moves or tends to move over another body, a force opposing the motion develops at the contact surfaces. This force which opposes the movement or tendency of movement is called a frictional force or simply friction.

Definition: The property of the bodies by virtue of which a force is exerted by a body over the surface of another body to resist the motion is called *friction* and is always acting in the direction opposite to the direction of motion. Friction act parallel to the surface of contact and depends upon the nature of surface of contact.

Limiting Friction:

Frictional force has the remarkable property of adjusting itself in magnitude to the force producing or tending to produce the motion so that the motion is prevented. However, there is a limit beyond which the magnitude of this force cannot increase. If the applied force is more than this maximum frictional force, there will be movement of one body over the other. This maximum value of frictional force which comes into play, when the motion is impending is known as *Limiting Friction.*

Static Friction: When the applied force is less than the limiting friction, the body remains at rest and such frictional force is called *Static Friction*, which may have any value between zero and limiting friction.

Dynamic Friction: If the value of applied force exceeds the limiting friction, the body starts moving over other body and the frictional resistance experienced by the body while moving is known as *Dynamic Friction*. Dynamic friction is found to be less than limiting friction. Dynamic friction may be grouped into the following two:

i) **Sliding friction:** It is the friction experienced by a body when it slides over the other body.

ii) **Rolling friction:** It is the friction experienced by a body when it rolls over the other body.

<u>Impending Motion:</u>

Let us consider a body of weight W resting on a horizontal surface as shown in the Fig. 1.39. Let P be the horizontal force applied, R be the normal reaction and F be the limiting friction.

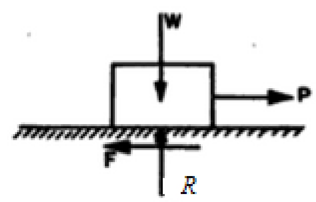


Fig. 1.39

If P is small, the body will not move as the force of friction acting on the body in the direction opposite to P will be more than P. But if magnitude of P goes on increasing, a stage comes when the solid body is on the point of motion. This stage is called *impending motion* or *verge of motion.*

Resolving the forces horizontally and vertically, we get F = P and W = R.

<u>Coefficient of friction (μ):</u>

It is defined as the ratio of liming force of friction (F) to the normal reaction (R) between the two bodies. It is denoted by the symbol μ.

$$\text{Thus } \mu = \frac{Limiting\ force\ of\ friction}{Normal\ reaction} = \frac{F}{R}$$

<u>Coulomb's laws of dry friction:</u>

The friction that exists between the two surfaces which are not lubricated is known as *solid* or *dry friction*. The following are the laws of dry friction:

i) The force of friction always acts in a direction opposite to that in which the body tends to move.

ii) Till the limiting value is reached, the magnitude of friction is exactly equal to the force which tends to move the body.

iii) The magnitude of limiting friction bears a constant ratio to the normal reaction between the two surfaces.

iv) The force of friction depends upon the roughness/smoothness of the surfaces.

v) The force of friction is independent of the area of contact between the two surfaces.

vi) After the body starts moving, the dynamic friction comes into play, the magnitude of which is less than that of limiting friction and it bears a constant ratio with the normal reaction. This ratio is called coefficient of dynamic friction.

vii) The force of friction is independent of velocity of sliding.

Angle of Friction:

Consider a block of weight W subjected to a pull of P. Let F be the frictional force developed and R be the normal reaction. Thus, at the contact surface, the reactions are F and R. They can be combined to get resultant reaction R_s which acts at an angle ϕ with normal reaction. This angle ϕ, called the angle of friction, is given by: $\tan tan\,\phi = \frac{F}{R}$

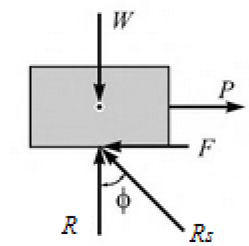


Fig. 1.40

Thus it is defined as the angle made by the resultant of normal reaction (N) and limiting force of friction (F) with the normal reaction.

$$\tan tan\,\phi = \frac{F}{R} = \frac{\mu R}{R} = \mu$$

Angle of Repose:

Consider a block of weight W resting on an inclined plane, which makes an angle of θ with the horizontal as shown in the Fig. 1.41. When θ is small the block will rest on the plane. If θ is increased gradually a slope is reached at which the block is about to start sliding. The angle at which the body is about to start sliding down is called as *angle of repose.*

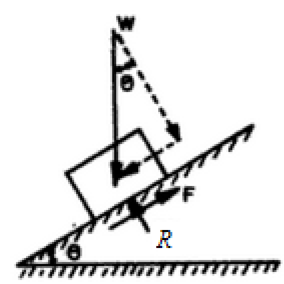


Fig. 1.41

Angle of Friction equals to Angle of Repose:

Consider the equilibrium of the block. Since the surface of contact is not smooth, not only normal reaction but frictional force also develops. As the body tends to slide down, the frictional resistance will be up the plane.

Σ Forces parallel to the plane = 0, gives F = W sin θ ---------- (i)

Σ Forces normal to the plane = 0, gives N = W cos θ ---------- (ii)

Dividing equation (i) by equation (ii), we get

$$\frac{F}{R} = \tan tan\,\theta$$

But we have,

$$\tan tan\,\phi = \frac{F}{R} = \mu$$

$$\tan tan\,\phi = \tan tan\,\theta$$

$$\phi = \theta$$

Thus angle of friction = angle of repose.

Cone of Friction:

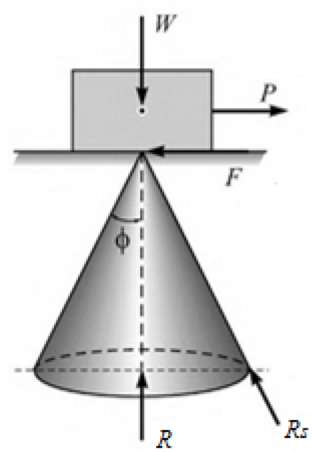


Fig. 1.42

It is defined as the right circular cone with vertex at the point of contact of the two bodies (or surfaces), axis in the direction of normal reaction (R) and semi vertical angle equal to angle of friction (ϕ).

When the applied force P is just sufficient to produce the impending motion of given body, angle of friction ϕ is obtained which is the angle made by resultant of limiting friction force and normal reaction with the normal reaction as shown in Fig. 1.42. If the direction of applied force P is gradually changed through 360°, the resultant R_s generates a right circular cone with semi vertex angle equal to ϕ. This is called the *Cone of friction.*

Example 1.32. *A body of weight 100 N is placed on a rough horizontal plane. Determine the co-efficient of friction if a horizontal force of 60 N just causes the body to slide over the horizontal plane.*

Solution. Given:

Weight of body W = 100 N

Horizontal force applied, P = 60 N

Therefore, limiting force of friction, $F = P$ = 60 N

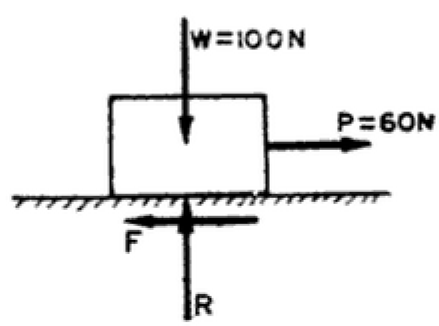


Fig. 1.43

Let μ = Co-efficient of friction.

The normal reaction of the body is given as $R = W = 100$ N

We know that $F = \mu R$

Or $\mu = \frac{F}{R} = \frac{60}{100} = 0.6$

Example 1.33. *The force required to pull a body of weight 50 N on a rough horizontal plane is 15N. Determine the co-efficient of friction if the force is applied at an angle of 15° with the horizontal.*

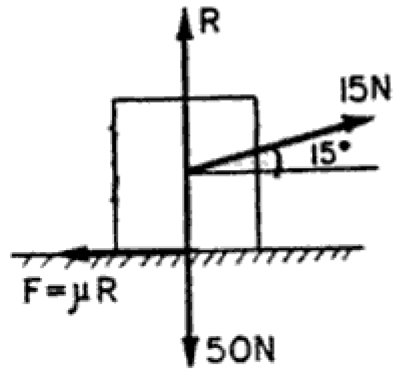


Fig. 1.44

Solution. Given:

Weight of the body $W = 50$ N

Force applied, $P = 15$ N

Angle made by the force P, with horizontal $\theta = 15^o$

Let the co-efficient of friction = μ

Normal reaction = R

When a force equal to 15 N is applied to the body at an angle 15° to the horizontal, the body is on the point of motion in the forward direction. Hence a force of friction equal to μR will be acting in the backward direction. The body is in equilibrium under the action of Forces as shown in Fig.1.44

Resolving the forces along the plane, $\mu R = 15 \cos 15^o$ ……….. (i)

Resolving the forces normal to the plane $R + 15 \sin 15^o = 50$

$\therefore R = 50 - 15 \sin 15^o = 46.12$ N

By substituting the value of R in equation (i), we get

$\mu \times 46.12 = 15 \cos 15^o$

$\therefore \mu = \frac{15 \cos 15^o}{46.12}$

$= 0.314$

Example 1.34. *A pull of 20 N, inclined at 25º to the horizontal plane, is required just to move a body placed on a rough horizontal plane. But the push required to move the body is 25 N. If the push is inclined at 25º to the horizontal, find the weight of the body and co-efficient of friction.*

Solution. Given:

Pull required, P = 20 N

Inclination of pull, $\theta = 25^o$

Push required $P' = 25$ N

Inclination of push, $\theta' = 25^o$

Let W = weight of the body

μ = co-efficient of friction

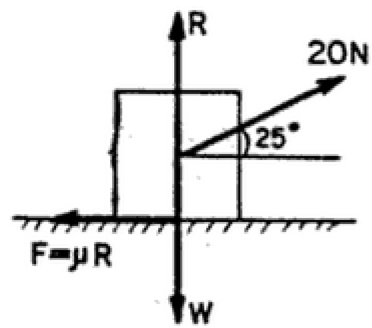
R = Normal reaction when body is pulled

R' = Normal reaction when body is pushed

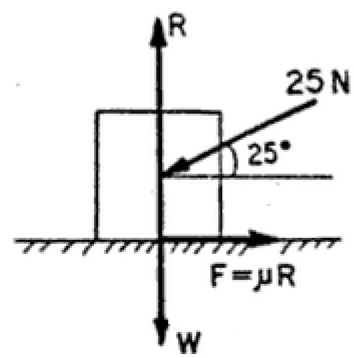
i) When body is pulled: The body is in equilibrium under the action of forces as shown in the Fig. 1.45 (a)
Resolving the forces along the plane,
$\mu R = 20 \cos 25^o = 18.126$ ………….. (i)
Resolving the forces normal to the plane,
$R + 20 \sin 25^o = W$
$\therefore R = W - 20 \sin 25^o = W - 8.452$
Substituting the value of R in equation (i),
$\mu (W - 8.452) = 18.126$ ………….. (ii)

ii) When body is pushed: The body is in equilibrium under the action of forces as shown in Fig. 1.45 (b).
Resolving the forces along the plane,
$\mu R' = 25 \cos 25^o = 22.657$ …………… (iii)
Resolving the forces normal to the plane,
$R' = W + 25 \sin 25^o = W + 10.565$
Substituting the value of R' in equation (iii),
$\mu (W + 10.565) = 22.567$ …………… (iv)
Dividing equation (ii) by equation (iv),

$$\frac{\mu (W - 8.452)}{\mu (W + 10.565)} = \frac{18.126}{22.567}$$



(a) Body is Pulled



(b) Body is Pushed

Fig. 1.45

$$22.567\,(W - 8.452) = 18.126\;(W + 10.565)$$

$$4.53\,W = 383$$

$$\therefore W = 84.547\,N$$

Substituting the value of *W* in equation (ii),

$$\mu\,(84.547 - 8.452) = 18.126$$

$$\mu = \frac{18.126}{76.095} = 0.238$$

Example 1.35. *Find the least force required to drag a body of weight W, placed on a rough inclined plane having inclination α to the horizontal. The force is applied to the body in such a way that it makes an angle θ to the inclined plane and the body is a) on the point of motion up the plane and b) on the point of motion down the plane.*

Solution. Given:

Weight of body = *W*

Inclination of plane = α

Force applied = *P*

Angle made by force *P* with inclined plane = θ

a) Least force when the body is on the point of motion up the plane:

When the body is on the point of motion up the plane, the force of friction ($F=\mu R$) is acting down the plane. The body is in equilibrium under the action of following forces as shown in Fig. 1.46.

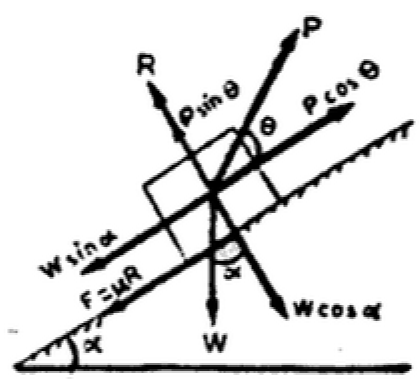


Fig. 1.46

Weight (W) of the body acting vertically downwards,

Normal reaction (R), perpendicular to the inclined plane,

The force of friction, $F = \mu R$ acting down the plane, and

Force P, inclined at an angle θ to the plane.

Resolving the forces along the plane, we get

$W \sin \alpha + \mu R = P \cos \theta$ ………. *(i)*

Resolving the forces perpendicular to the inclined plane,

$W \cos \alpha = R + P \sin \theta$

$R = W \cos \alpha - P \sin \theta$

Substituting the value of R in equation *(i),* we get

$W \sin \alpha + \mu (W \cos \alpha - P \sin \theta) = P \cos \theta$

$W \sin \alpha + \mu W \cos \alpha = P \cos \theta + \mu P \sin \theta$

$$\therefore P = \frac{W(\sin \sin \alpha + \mu \cos \cos \alpha)}{\cos \cos \theta + \mu \sin \sin \theta}$$

$$\therefore P = \frac{W(\sin \sin \alpha + \frac{\sin \sin \emptyset}{\cos \cos \emptyset} \cos \cos \alpha)}{\cos \cos \theta + \frac{\sin \sin \emptyset}{\cos \cos \emptyset} \sin \sin \theta} = \frac{W(\sin \sin \alpha \cos \cos \emptyset + \sin \sin \emptyset \cos \cos \alpha)}{\cos \cos \theta \cos \cos \emptyset + \sin \sin \emptyset \sin \sin \theta}$$

$$= \frac{W \sin(\alpha + \emptyset)}{\cos(\theta - \emptyset)}$$

The force P will be least if the denominator cos $(\theta - \emptyset)$ is maximum. But the maximum value of cos $(\theta - \emptyset)$ will be equal to one.

$$\therefore \cos \cos (\theta - \emptyset) = 1 = \cos \cos 0$$

$$\theta - \emptyset = 0 \; or \; \theta = \emptyset$$

Substituting this value of $\theta = \emptyset$ in P, we get

$P_{min} = W \sin (\alpha + \theta)$

Thus the force P will be minimum if the angle of inclination of the force with the inclined plane is equal to the angle of friction.

b) Least force when the body is on the point of motion down the plane:

When the body is on the point of motion down the plane, the force of friction $F = \mu R$ is acting up the plane as shown in Fig.1.47.

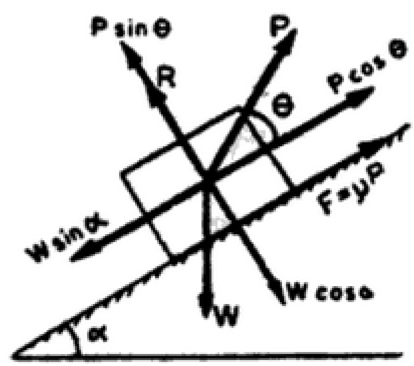


Fig. 1.47

Resolving the forces along the plane,

$W \sin \alpha = P \cos \theta + \mu R$

………… *(ii)*

Resolving forces perpendicular to the inclined plane

$R + P \sin \theta = W \cos \alpha$

$R = W \cos \alpha - P \sin \theta$

Substituting the value of R in equation *(ii)*, we get

$W \sin \alpha = P \cos \theta + \mu(W \cos \alpha - P \sin \theta)$

$\qquad = P \cos \theta + \mu W \cos \alpha - \mu P \sin \theta$

$W[\sin \alpha - \mu \cos \alpha] = P [\cos \theta - \mu \sin \theta]$

$$\therefore P = \frac{W(\sin \alpha - \mu \cos \alpha)}{(\cos \theta - \mu \sin \theta)}$$

$$= \frac{W(\sin\alpha - \frac{\sin\sin\emptyset}{\cos\cos\emptyset}\cos\alpha)}{\left(\cos\theta - \frac{\sin\sin\emptyset}{\cos\cos\emptyset}\sin\theta\right)}$$

$$= \frac{W(\sin\alpha\cos\cos\emptyset - \sin\sin\emptyset\ \cos\alpha)}{(\cos\theta\cos\emptyset - \sin\sin\emptyset\ \sin\theta)}$$

$$P = \frac{W\sin(\alpha-\emptyset))}{\cos(\theta+\emptyset)}$$

Example 1.36. *A body of weight 500 N is pulled up an inclined plane, by a force of 350N. The inclination of the plane is 30°to the horizontal and the force is applied parallel to the plane. Determine the co-efficient of friction.*

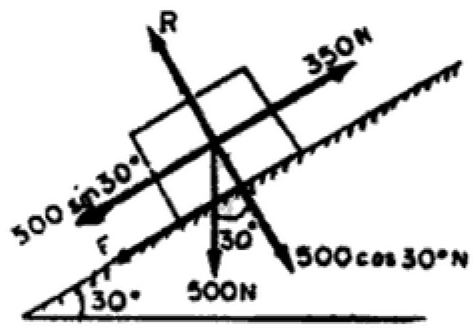


Fig. 1.48

Solution. Given

Weight of the body, W = 500 N

Force applied, P = 350 N

Inclination α = 30°

Let μ = Co-efficient of friction

R = Normal reaction

F = Force of friction = μR

Resolving the forces along the plane,

500 sin 30° + μR = 350 ……….. (*i*)

Resolving the forces normal to the plane,

R = 500 cos 30° = 500 × 0.866 = 433 N

Substituting the value of R in equation (*i*), we get

500 sin 30° + μ × 433 = 350

$$\therefore \mu = 0.23$$

Example 1.37. *An effort of 200 N is required to move a certain body up an inclined plane of angle 15°, the force acting parallel to the plane. If the angle of inclination of the plane is made*

20^o, the effort, again applied parallel to the plane, is found to be 230 N. Find the weight of the body and the co-efficient of friction.

Solution. Given

Effort required, $P_1 = 200$ N; when inclination, $\theta_1 = 15^o$

Effort required, $P_2 = 230$ N; when inclination, $\theta_2 = 20^o$

In both the cases, the effort is applied parallel to the inclined plane and body is just to move up. Hence the force of friction (*$F=\mu R$*) will be acting downwards.

1st case

$P_1 = 200$ N, $\theta_1 = 15^o$

Let W = weight of the body,

μ = Co-efficient of friction,

R_1 = Normal reaction, and

F_1 = Force of friction = μR_1

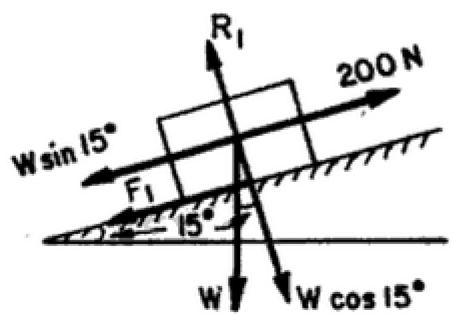


Fig. 1.49

Resolving the forces along the plane,

$$W \sin sin\ 15^o + \mu R_1 = 200 \qquad \text{.......... } (i)$$

Resolving the forces normal to the plane,

$$R_1 = W \cos cos\ 15^o$$

Substituting the value of $R1$ in equation (i),

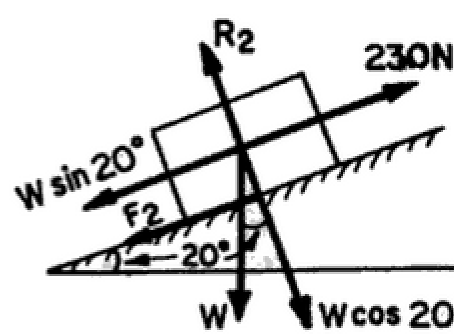
$$W \sin sin\ 15^o + \mu (W \cos cos\ 15^o) = 200$$

$$W\left(\sin sin\ 15^o + \mu \cos cos\ 15^o\right) = 200 \qquad \text{............ } (ii)$$

2nd case

$P_2 = 230$ N, $\theta_2 = 20^o$

Let R_2 = Normal reaction, and



F_2 = Force of friction = μR_2

Resolving the forces along the plane,

$W \sin sin\, 20^o + \mu R_2 = 230$ .... (*iii*)

Resolving the forces normal to the plane,

$$R_2 = W \cos cos\, 20^o$$

Substituting the value of R_2 in equation (*iii*),

$W \sin sin\, 20^o + \mu \left(W \cos cos\, 20^o\right) = 230$

$W\left(\sin sin\, 20^o + \mu \cos cos\, 20^o\right) = 230$ .... (*iv*)

Dividing equation (*iv*) by equation (*ii*),

$$\frac{W\left(\sin sin\, 20^o + \mu \cos cos\, 20^o\right)}{W\left(\sin sin\, 15^o + \mu \cos cos\, 15^o\right)} = \frac{230}{200} = 1.15$$

$$\sin sin\, 20^o + \mu \cos cos\, 20^o = 1.15\left(\sin sin\, 15^o + \mu \cos cos\, 15^o\right)$$

$$\therefore \mu = 0.26$$

The weight of the body is obtained by substituting the value of μ in equation (*iv*),

$$W\left(\sin sin\, 20^o + 0.26 \times \cos cos\, 20^o\right) = 230$$

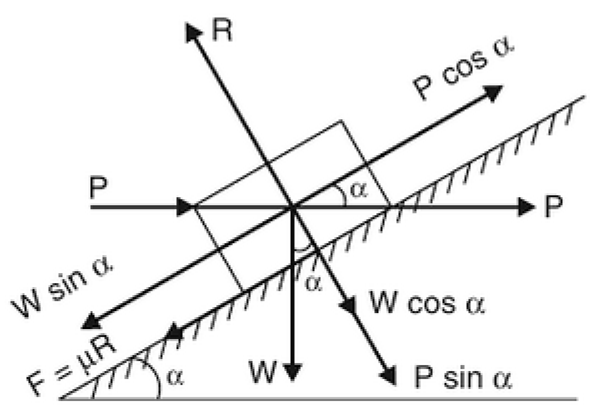
$$\therefore W = 392.3\ N$$

Example 1.38. *Find the force required to drag a body of weight W, placed on a rough inclined plane having inclination α to the horizontal. The force P is applied to the body horizontally and the body is (a) on the point of motion up the plane, and (b) on the point of motion down the plane.*

Solution. Given

Weight of the body = W

Inclination of the plane = α



Force applied = P

a) Force required to drag the body when it is on the point of motion up the plane:
When the body is on the point of motion up the plane, the force of friction $F = \mu R$ is acting down the plane.
Resolving the forces along the plane,

$$W \sin sin\, \alpha \ + \mu R = P \cos cos\, \alpha \qquad \ldots\ldots\ldots\ldots (i)$$

Resolving the forces normal to the inclined plane,

$$W \cos cos\, \alpha \ + P \sin sin\, \alpha \ = R$$

Substituting the value of R in equation (i),

$$W \sin sin\, \alpha \ + \mu(W \cos cos\, \alpha \ + P \sin sin\, \alpha\,) = P \cos cos\, \alpha$$

$$P \cos cos\, \alpha \ - \mu P \sin sin\, \alpha \ = w \sin sin\, \alpha \ + \mu\, W \cos cos\, \alpha$$

$$P\left(\cos cos\, \alpha \ - \frac{\sin sin\, \emptyset}{\cos cos\, \emptyset} \sin sin\, \alpha\right) = W\left(\sin sin\, \alpha + \frac{\sin sin\, \emptyset}{\cos cos\, \emptyset} \cos cos\, \alpha\ \right)$$

$$P(\cos cos\, \alpha\ \ \cos cos\, \emptyset \ - \sin sin\, \alpha\ \ \sin sin\, \emptyset \ = W(\sin sin\, \alpha\ \ \cos cos\, \emptyset \ + \cos cos\, \alpha\ \ \sin sin\, \emptyset\,)$$

$$\therefore P = W \tan tan\, (\alpha + \ \emptyset))$$

b) Force required to drag the body when it is on the point of motion down the plane:
In this case the force of friction $F = \mu R$ will be acting up the plane,

Resolving the forces along the plane,

$$W \sin sin\, \alpha \ = P \cos cos\, \alpha \ + \mu R \qquad \ldots\ldots\ldots\ldots (ii)$$

Resolving the forces normal to the inclined plane,

$$W \cos cos\, \alpha \ + P \sin sin\, \alpha \ = R$$

Substituting t

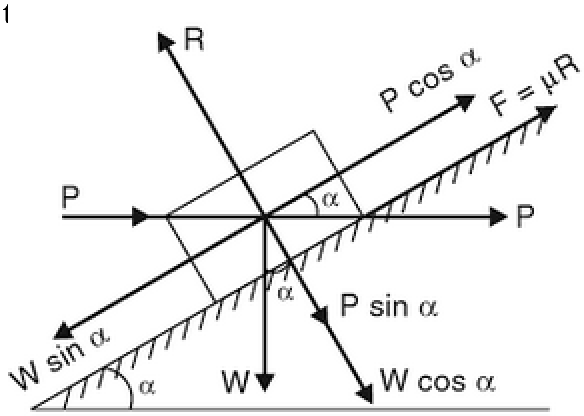


Fig. 1.52

$$W \sin sin\, \alpha \;=\; P \cos cos\, \alpha \;+\; \mu(W \cos cos\, \alpha \;+\; P \sin sin\, \alpha)$$

$$P \cos cos\, \alpha \;+\; \mu\; P \sin sin\, \alpha \;=\; W \sin sin\, \alpha \;-\; \mu\, W \cos cos\, \alpha$$

$$P\left(\cos cos\, \alpha \;+\; \frac{\sin sin\, \emptyset}{\cos cos\, \emptyset} \sin sin\, \alpha\right) = \; W(\sin sin\, \alpha \;-\; \frac{\sin sin\, \emptyset}{\cos cos\, \emptyset} \cos cos\, \alpha)$$

$$P(\cos cos\, \alpha \cos cos\, \emptyset \;+\; \sin sin\, \alpha\; \sin sin\, \emptyset) \;=\; W(\sin sin\, \alpha \cos cos\, \emptyset \;-\; \cos cos\, \alpha \sin sin\, \emptyset\;)$$

$$\therefore P \;=\; W \tan tan\, (\alpha \;-\; \emptyset)$$

Example 1.39. *Two blocks A and B of weights 1 kN and 2 kN respectively are in equilibrium position as shown in Fig.1.53. If the coefficient of friction between the two blocks as well as the block B and the floor is 0.3, find the force (P) required to move the block B.*

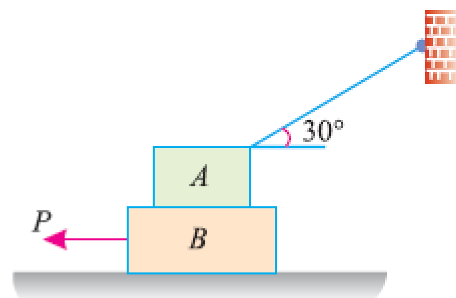
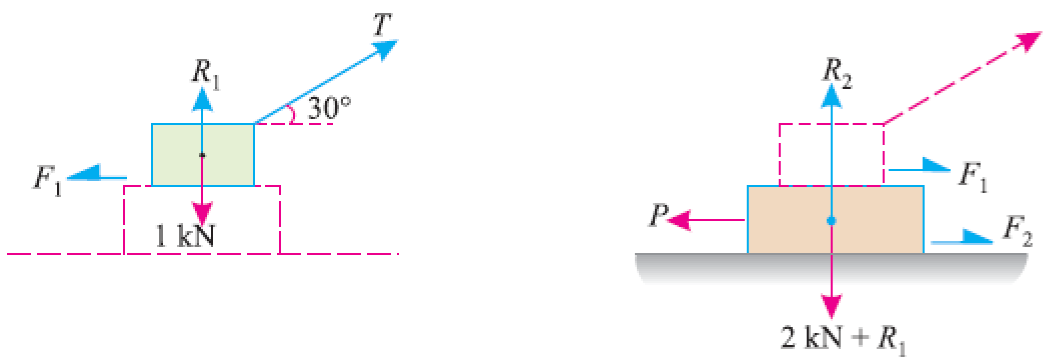


Fig. 1.53

Solution. Given: Weight of block *A* (W_A) = 1 kN;
Weight of block *B* (W_B) = 2 kN and coefficient of friction (μ) = 0.3.



(*a*) Block *A* Fig. 1.54 (*b*) Block *B*

The forces acting on the two blocks *A* and *B* are shown in Fig. 1.54 (*a*) and (*b*) respectively.
First of all, consider the forces acting in the block *A*.
Resolving the forces vertically,

$R_1 + T \sin 30° = 1$ kN

or $T \sin 30° = 1 - R_1$ ... (*i*)

and now resolving the forces horizontally,

$T \cos 30° = F_1 = \mu R_1 = 0.3\, R_1$ ... (*ii*)

Dividing equation (*i*) by (*ii*)

$$\frac{T\sin\sin 30^{o}}{T\cos\cos 30^{o}} = \frac{1-R_1}{0.3\,R_1}$$

$$Tan\ 30^{o} = \frac{1-R_1}{0.3\,R_1}$$

$$\therefore R_1 = 0.85\ kN$$

and $F1 = \mu R_1 = 0.3 \times 0.85 = 0.255$ kN ... (*iii*)

Now consider the block *B*. A little consideration will show that the downward force of the block *A* (equal to R_1) will also act along with the weight of the block *B*.

Resolving the forces vertically,

$R_2 = 2 + R_1 = 2 + 0.85 = 2.85$ kN

$F_2 = \mu R_2 = 0.3 \times 2.85 = 0.855$ kN ...(*iv*)

and now resolving the forces horizontally,

$P = F_1 + F_2 = 0.255 + 0.855 = 1.11$ kN

Example 1.40. *Block A weighing 1000N rests over block B which weighs 2000N as shown in Fig. 1.55. Block A is tied to a wall with a horizontal string. If the coefficient of friction between A and B is 1/4 and that between B and the floor is 1/3, what value of force P is required to create impending motion if a) P is horizontal, b) P acts 30°upwards to horizontal?*

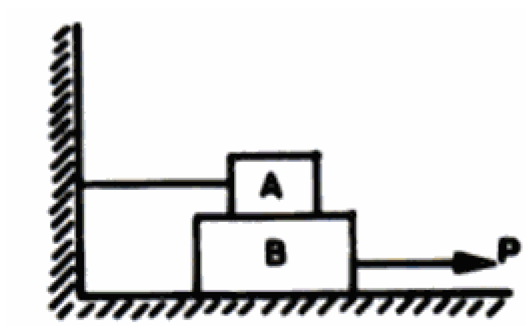


Fig. 1.55

Solution.

a) When P is horizontal:

Now consider the equilibrium of block A

Resolving forces vertically,

$$R_1 - 1000 = 0$$

$$\therefore R_1 = 1000\ N$$

Since F_1 is limiting friction,

$$\frac{F_1}{R_1} = \frac{1}{4}$$

$$\therefore F_1 = 250\ N$$

And now, resolving the forces horizontally,

$$F_1 - T = 0$$

$$\therefore T = F_1 = 250\ N$$

Now consider the equilibrium of block B

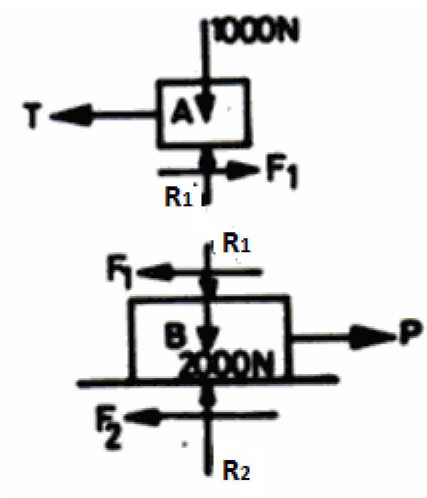


Fig. 1.56 (a)

Resolving the forces vertically,

$$R_2 - R_1 - 2000 = 0$$

$$\therefore R_2 = 2000 + 1000 = 3000\,N$$

Since F_2 is limiting friction,

$$\frac{F_2}{R_2} = \frac{1}{3}$$

$$\therefore F_2 = 1000\,N$$

And now resolving the forces horizontally,

$$P - F_1 - F_2 = 0$$

$$\therefore P = F_1 + F_2 = 250 + 1000 = 1250N$$

b) When P is inclined:
Now consider the equilibrium of block A
Resolving forces vertically,

$$R_1 - 1000 = 0$$

$$\therefore R_1 = 1000\,N$$

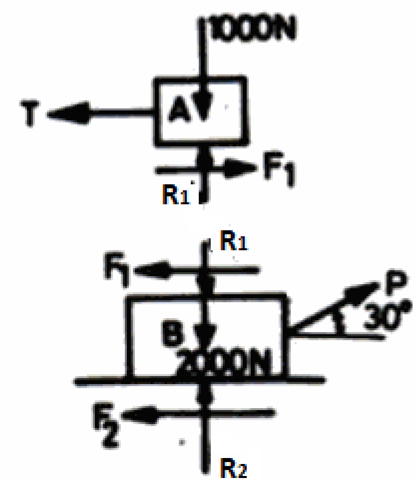


Fig. 1.56 (b)

Since F_1 is limiting friction,

$$\frac{F_1}{R_1} = \frac{1}{4}$$

$$\therefore F_1 = 250\,N$$

And now, resolving the forces horizontally,

$$F_1 - T = 0$$

$$\therefore T = F_1 = 250\,N$$

Now consider the equilibrium of block B
Resolving the forces vertically,

$$R_2 - R_1 - 2000 + P \sin sin\, 30^o = 0$$

$$R_2 + 0.5P = 3000$$

From law of friction

$$F_2 = \mu R_2 = \frac{1}{3} \times (3000 - 05P)$$

$$\therefore F_2 = 1000 - \frac{0.5}{3} P$$

And now resolving the forces horizontally,

$$P \cos \cos 30^o - F_1 - F_2 = 0$$

$$P \cos \cos 30^o - 250 - \left(1000 - \frac{0.5}{3} P\right)$$

$$\therefore P = 1210.4\ N$$

Example 1.41. *What should be the value of angle θ so that the motion of the 90 N block impends down the plane? The co-efficient of friction μ for all the surfaces is 1/3.*

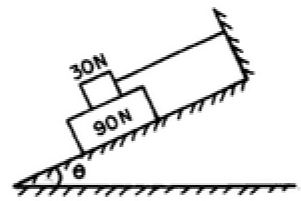


Fig. 1.57

Solution.

Consider the equilibrium of weight 30 N

Resolving the forces along the plane,

$$T = 30 \sin sin\, \theta + \mu R_1$$

$$T = 30 \sin sin\, \theta + \frac{1}{3} R_1 \quad \ldots\ldots\ldots\ldots (i)$$

Resolving the forces normal to the plane,

$$R_1 = 30 \cos cos\, \theta$$

Substituting the value of R_1 in equation (*i*),

$$T = 30 \sin sin\, \theta + \frac{1}{3} \times 30 \cos cos\, \theta$$

$T = 30 \sin sin\, \theta + 10 \cos cos\, \theta$

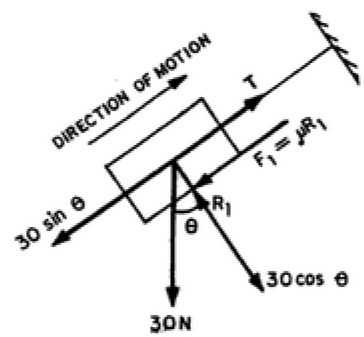


Fig. 1.58

Now consider the equilibrium of weight 90 N

Resolving the forces along the plane,

$$90 \sin sin\, \theta = \mu R_1 + \mu R_2$$

$$= \frac{1}{3} R_1 + \frac{1}{3} R_2$$

$$90 \sin sin\, \theta = 10 \cos cos\, \theta + \frac{1}{3} R_2$$

………….. (*ii*)

Resolving the forces normal to the plane,

$R_2 = R_1 + 90 \cos cos\, \theta$

$$= 30 \cos cos\, \theta + 90 \cos cos\, \theta$$

$$R_2 = 120 \cos cos\, \theta$$

Substituting the value of R_2 in equation (*ii*),

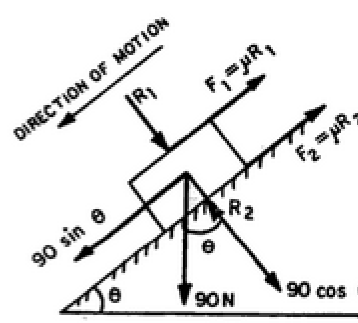


Fig. 1.59

$$90 \sin sin\, \theta \; = \; 10 \cos cos\, \theta \; + \frac{1}{3} \times 120 \cos cos\, \theta$$

$$90 \sin sin\, \theta \; = \; 50 \cos cos\, \theta$$

$$\frac{\sin sin\, \theta}{\cos cos\, \theta} = \frac{50}{90}$$

$$\tan tan\, \theta \; = \; 0.5555$$

$$\therefore \theta \; = \; tan^{-1}(0.5555) = \; 29.05^{o}$$

Example 1.42. *A cord connects two bodies of weights 400 N and 800 N. The two bodies are placed on an inclined plane and cord is parallel to inclined plane. The co-efficient of friction for weight 400 N is 0.15 and that for 800 N is 0.4. Determine the inclination of the plane to the horizontal and the tension in the cord when the motion is about to take place, down the inclined plane. The body weighing 400 N is below the body weighing 800 N.*

Solution.

Consider the equilibrium of weight 400 N

Resolving the forces along the plane,

$400 \sin sin\, \alpha \; = T \; + F_1 \; = T \; + \; 0.15\, R_1$

…….. (*i*)

Resolving the forces normal to the plane,

$$400 \cos cos\, \alpha \; = \; R_1$$

Substituting the value of R_1 in equation (*i*),

$$400 \sin sin\, \alpha \; = \; T \; + \; 0.15 \times 400 \cos cos\, \alpha$$

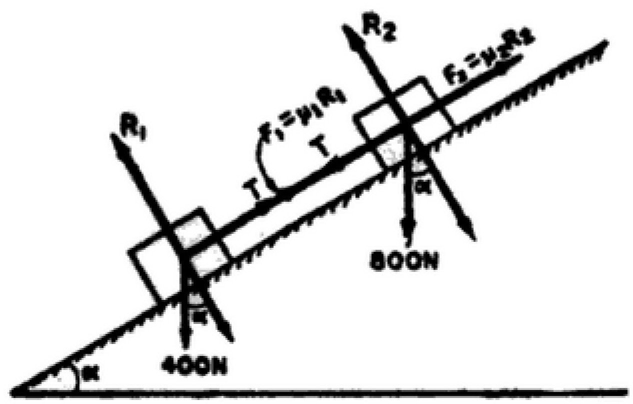


Fig. 1.60

$\therefore T \; = \; 400 \sin sin\, \alpha \; - \; 60 \cos cos\, \alpha$ ……. (*ii*)

Consider the equilibrium of weight 800 N

Resolving the forces along the plane,

$800 \sin sin\, \alpha \; + \; T \; = F_2 = \mu_2 R_2 \; = \; 0.4 R_2$ ……. (*iii*)

Resolving the forces normal to the plane,

$$R_2 \; = \; 800 \cos cos\, \alpha$$

Substituting the value of R_2 in equation (*iii*),

$$800 \sin sin\, \alpha \; + \; T \; = \; 0.4 \times 800 \cos cos\, \alpha$$

$\therefore T \; = \; 320 \cos cos\, \alpha \; - \; 800 \sin sin\, \alpha$ ……… (*iv*)

Equating the values of *T* given by the equations (*ii*) and(*iv*)

$$400 \sin sin\, \alpha \; - \; 60 \cos cos\, \alpha \; = \; 320 \cos cos\, \alpha \; - \; 800 \sin sin\, \alpha$$

$$1200 \sin sin\, \alpha \; = \; 380 \cos cos\, \alpha$$

$$\frac{\sin sin\, \alpha}{\cos cos\, \alpha} = \; \frac{380}{1200} \; = \; 0.3166$$

$$\therefore \alpha = tan^{-1}(0.3166) = 17.56^o$$

Substituting the value of α in equation (*ii*),

$$T = 400 \sin sin\, 17.56^o - 60 \cos cos\, 17.56^0$$

$$\therefore T = 120.68 - 57.20 = 63.48\, N$$

Example 1.43. *What is the value of Pin the system shown in Fig. 1.61 to cause the motion to impend? Assume the pulley is smooth and coefficient of friction between the other contact surfaces is 0.2.*

Solution.

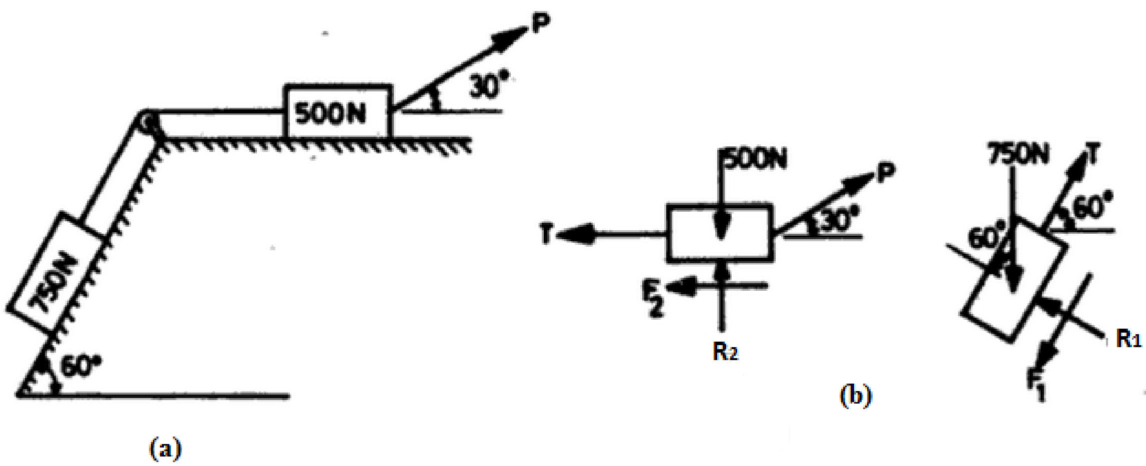


Fig. 1.61

Consider the equilibrium of 750 N block,
Resolving the forces normal to the plane,

$$R_1 - 750 \cos cos\, 60 = 0$$

$$\therefore R_1 = 375\, N$$

From law of friction,

$$F_1 = \mu R_1 = 0.2 \times 375 = 75\, N$$

Now, resolving the forces parallel to the plane,

$$T - F_1 - 750 \sin sin\, 60 = 0$$

$$\therefore T = 75 + 750 \sin sin\, 60 = 724.5\, N$$

Now, consider the equilibrium of 500 N block,
Resolving the forces normal to the plane,

$$R_2 - 500 + P \sin sin\, 30 = 0$$

$$\therefore R_2 + 0.5\,P = 500$$

From law of friction,

$$F_2 = \mu R_2 = 0.2(500 - 0.5P) = 100 - 0.1P$$

Resolving the forces along the plane,

$$P\cos cos\,30 - T - F_2 = 0$$

$$P\cos cos\,30 - 724.5 - 100 + 0.1\,P = 0$$

$$\therefore P = 853.5\,N$$

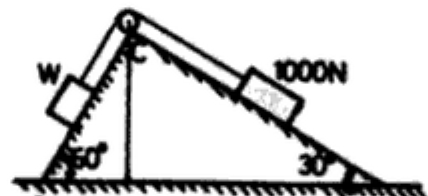


Fig. 1.62

Example 1.44. *Two identical planes AC and BC, inclined at 60º and 30º to the horizontal meet at C as shown in the Fig. 1.62. A load of 1000 N rests on the inclined plane BC and is tied by a rope passing over a pulley to a block weighing W newtons and resting on the plane AC. If the coefficient of friction between the load and the plane BC is 0.28 and that between the block and the plane AC is 0.2, find the least and greatest values of W for the equilibrium of the system.*

Solution.

Least value of *W*:

In this case motion of 1000 N block is impending down the plane and block *W* has impending motion up the plane.

Considering the equilibrium of 1000 N block,

Resolving the forces normal to the plane,

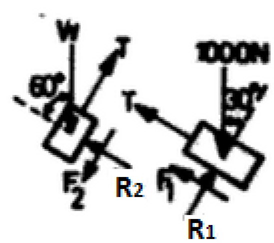


Fig. 1.63

$$R_1 - 1000\cos cos\,30 = 0$$

$$\therefore R_1 = 866\,N$$

From the law of friction

$$F_1 = \mu_1 R_1 = 0.28\times866 = 242.5\,N$$

Now, resolving the forces along the plane,

$$T - 1000\sin sin\,30 + F_1 = 0$$

$$\therefore T = 500 - 242.5 = 257.5\,N$$

Consider the equilibrium of block weighing *W*

Resolving the forces normal to the plane,

$$R_2 - W\cos cos\,60 = 0$$

$$\therefore R_2 = 0.5\,W$$

From the law of friction

$$F_2 = \mu_2 R_2 = 0.2\times0.5W = 0.1\,W$$

Resolving the forces along the plane,

$$T - F_2 - W\sin sin\,60 = 0$$

Substituting the values of T and F_2 , we get

$$257.5 - 0.1W - W\sin sin\,60 = 0$$

$$\therefore W = 266.6\,N$$

Greatest Value of *W*:

In such case 1000 N block is on the verge of moving up the plane and *W* is on the verge of moving down the plane.

Consider the equilibrium of 1000 N block,

$$R_1 - 1000\cos cos\,30 = 0$$

$$\therefore R_1 = 866\,N$$

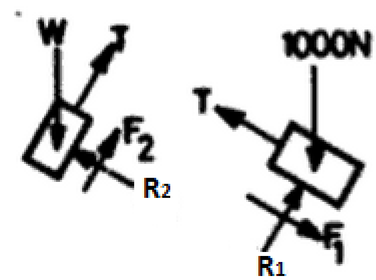


Fig. 1.64

From the law of friction

$$F_1 = \mu_1 R_1 = 0.28\times866 = 242.5\,N$$

Now, resolving the forces along the plane,

$$T - 1000\sin sin\,30 - F_1 = 0$$

$$\therefore T = 500 + 242.5 = 742.5\,N$$

Consider the equilibrium of block weighing *W*

Resolving the forces normal to the plane,

$$R_2 - W\cos cos\,60 = 0$$

$$\therefore R_2 = 0.5\,W$$

From the law of friction

$$F_2 = \mu_2 R_2 = 0.2\times0.5W = 0.1\,W$$

Resolving the forces along the plane,

$$T + F_2 - W\sin sin\,60 = 0$$

Substituting the values of T and F_2, we get

$$742.5 + 0.1W - W\sin sin\,60 = 0$$

$$\therefore W = 969.3\,N$$

Problems on Ladder Friction:

Example 1.45. *A uniform ladder of length 10 m and weighing 20 N is placed against a smooth vertical wall with its lower end 8m from the wall. In this position the ladder is just to slip. Determine:*

i) *The co-efficient of friction between the ladder and the floor, and*

ii) *Frictional force acting on the ladder at the point of contact between ladder and floor.*

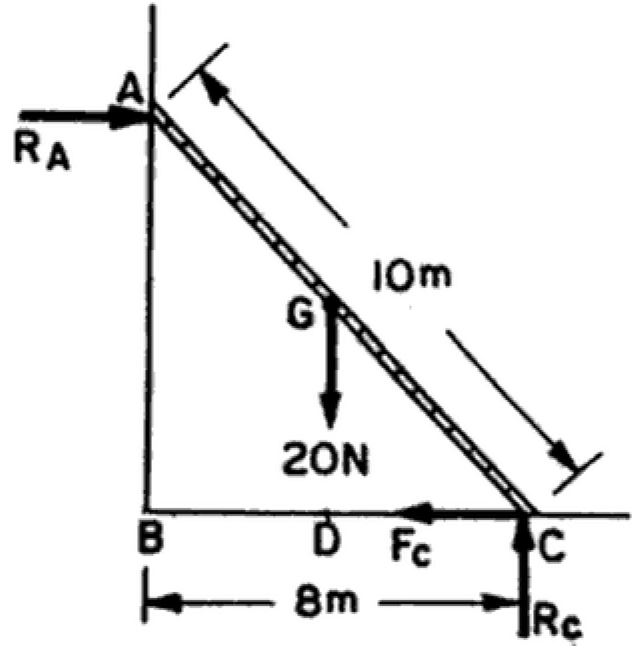


Fig. 1.65

Solution. Given:

Weight of ladder, $W = 20$ N

Length of ladder, AC = 10 m

Distance of lower end of ladder from wall, BC = 8 m

In right-angled triangle ABC,

$$AB = \sqrt{AC^2 - BC^2} = \sqrt{10^2 - 8^2} = 6\,m$$

As vertical wall is smooth, hence there will be no force of friction between ladder and wall.

At the lower end C, the ladder will tend to move towards right and hence a force of friction between ladder and floor will be acting towards left.

Let R_A = Reaction at A

R_C = Reaction at C

μ = co-efficient of friction between ladder and floor at C

F_C = Force of friction at C = μR_C

Ladder is uniform and the weight of ladder is acting at the middle point of AC to G. the line of action of W will pass through the middle point of BC. Hence distance

$CD = \frac{1}{2} \times BC = 4\,m$

Resolving the forces vertically,

$R_C = 20$ N

Resolving the forces horizontally,

$R_A = F_C = \mu R_C = 20\,\mu$

Taking the moment of all forces about point C,

$$R_A \times AB = 20 \times CD$$

$$20\,\mu \times 6 = 20 \times 4$$

$$\therefore \mu = 0.67$$

Frictional force acting at C is given as

$$F_C = \mu \times R_C = 0.67 \times 20 = 13.4\,N$$

Example 1.46. *A uniform ladder of length 13 m and weight 25 N is placed against a smooth vertical wall with its lower end 5 m from the wall. The co-efficient of friction between the ladder* *and the floor is 0.3. Show that the ladder will remain in equilibrium in this position. What is the frictional force acting on the ladder at the point of contact between the ladder and floor?*

Solution. Given:

Length of ladder, $L = 13\ m$

Weight of ladder, $W = 25$ N

Distance of lower end of ladder from wall, AC = 5 m

Co-efficient of friction between ladder and floor, $\mu = 0.3$

Vertical wall is smooth and hence there will be no force of friction between ladder and wall.

Let F_A = Limiting frictional force acting at A = 0.3 R_A

R_A = Normal reaction at A

R_B = Normal reaction at B

The weight of 25 N is acting at the middle point of AB vertically downwards. If the ladder is not in equilibrium, it will start moving at A towards right and force of friction (F_A) will act towards left.

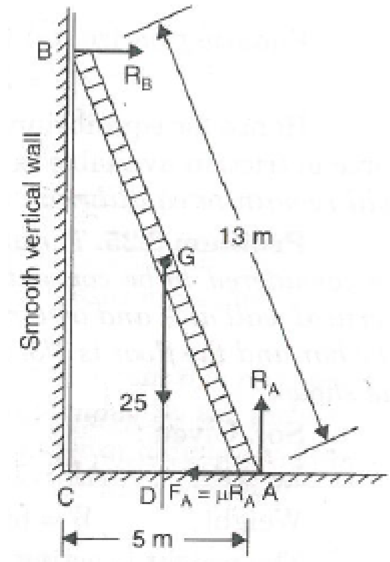


Fig. 1.66

Resolving the forces vertically,

$R_A = 25$ N

Resolving the forces horizontally,

$R_B = F_A = 0.3\ R_A = 0.3 \times 25 = 7.5$ N

Therefore, maximum amount of frictional force available at A is

$F_A = 7.5$ N

To prove that ladder is in equilibrium for the given position:

Let $F_A{}'$ = Limiting frictional force acting at A = 0.3 R_A

$R_A{}'$ = Normal reaction at A

$R_B{}'$ = Normal reaction at B

The ladder will be in equilibrium if $F_A{}'$ is less than F_A

From the Fig. 1.66, ABC is a right-angled triangle. G is the middle point of AB and GD is normal to AC. Hence D is the middle point of AC.

$$\therefore AD = CD = 2.5\ m$$

Also $AB^2 = AC^2 + BC^2$

$$\therefore BC = \sqrt{AB^2 - AC^2} = \sqrt{13^2 - 5^2} = 12\ m$$

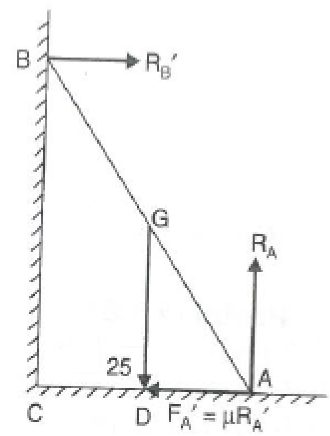


Fig. 1.67

The ladder will be in equilibrium, if the moments of all forces acting on the ladder about any point is zero.

Taking moments of all forces about A,

$$25 \times AD = R_B{}' \times BC$$
$$25 \times 2.5 = R_B{}' \times 12$$
$$\therefore R_B{}' = 5.21\ N$$

Hence for equilibrium, the force of friction required is 5.21 N. But maximum amount of force of friction available is 7.5 N which is more than the required amount. Hence the ladder will remain in equilibrium in the given position.

Example 1.47. *A uniform ladder of weight 850N and of length 6 m rests on a horizontal ground and leans against a smooth vertical wall. The angle made by the ladder with the horizontal is*

65°. When a man of weight 750 N stands on the ladder at a distance 4 m from the top of the ladder, the ladder is at the point of sliding. Determine the co-efficient of friction between the ladder and the floor.

Solution. Given:

Length of ladder, $L = 6\ m$

Weight of ladder, $W = 850$ N

Angle made by ladder with horizontal, $\alpha = 65°$

Weight of man, $W_1 = 750$ N

Distance of man from top of the ladder, $L_1 = 4\ m$

∴ Distance of man from the foot of ladder, $L_2 = L - L_1 = 6 - 4 = 2\ m$

Let μ = Co-efficient of friction between the ladder and floor. Vertical wall is smooth and hence there will be no force of friction between the ladder and vertical wall.

Let AB is the ladder and G is the middle point of the ladder at which the weight 850 N is acting. The man of weight 750 N is standing at E. At this position, the ladder is at the point of sliding. This means that ladder at A will be start moving towards right. Hence a force of friction $F_A = \mu R_A$ will be acting towards left.

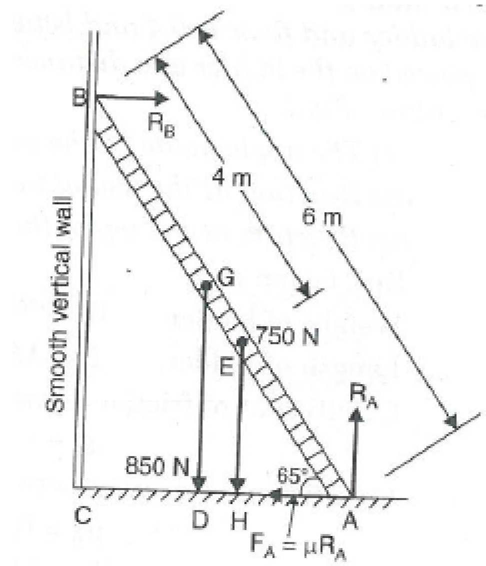


Fig. 1.68

R_A = Normal reaction at A

R_B = Normal reaction at B

Resolving the forces vertically,

$$R_A = 850 + 750 = 1600 \text{ N}$$

Resolving the forces horizontally,

$$R_B = F_A = \mu R_A = \mu \times 1600 = 1600\ \mu\ N \ldots\ldots\ldots (i)$$

From triangle ABC, $BC = AB \sin 65° = 6 \times 0.9063 = 5.437\ m$

$$AC = AB \cos 65° = 6 \times 0.4226 = 2.5357\ m$$

As G is the middle point of AB and GD is normal to AC.

∴ D is the middle point of AC

$$\therefore AD = \frac{AC}{2} = \frac{2.5357}{2} = 1.267\ m$$

$$AH = AE \cos 65° = (AB - BE) \cos 65° = 2 \times 0.4226 = 0.8452\ m.$$

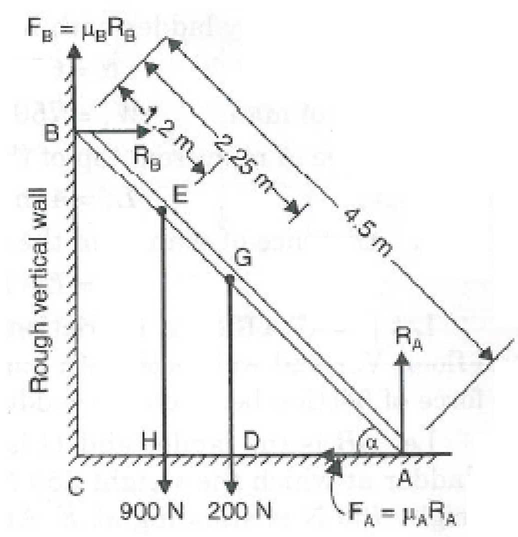
Taking the moments of all forces about A, we get

$$R_B \times BC = 850 \times AD + 750 \times AH$$

$$1600\ \mu \times 5.437 = 850 \times 1.267 + 750 \times 0.8452$$

$$\therefore \mu = 0.199$$

Example 1.48. *A uniform ladder of weight 200 N of length 4.5 m rests on a horizontal ground and leans against a rough vertical wall. The co-efficient of friction between the ladder and floor is 0.4 and between ladder and vertical wall is 0.2. When a weight of 900 N is placed on the ladder at a distance of 1.2 m from the top of the ladder, the ladder is at the point of sliding. Find: i) The angle made by the ladder with horizontal, ii) Reaction at the foot of the ladder, and iii) Reaction at the top of the ladder.*



Solution. Given:
Length of ladder, L = AB = 4.5 *m*
Weight of ladder, W = 200 N
Co-efficient of friction between ladder and floor, $\mu_A = 0.4$
Co-efficient of friction between ladder and wall, $\mu_B = 0.2$
Weight on ladder, W_1 = 900 N
Distance BE = 1.2 *m*
Let α = Angle made by ladder with horizontal
R_A = Normal reaction at A
F_A = Force of friction at A = μR_A
R_B = Normal reaction at B
F_B = Force of friction at B = μR_B
When the ladder AB is on the point of sliding, the foot of the ladder will move towards right and hence a force of friction $F_A = \mu R_A$ will be acting towards left. The top of the ladder will be moving downwards on the vertical wall and hence a force of friction $F_B = \mu R_B$ will be acting upwards.
From triangle ABC, $BC = AB \sin \alpha = 4.5 \sin \alpha$
$AC = AB \cos \alpha = 4.5 \cos \alpha$
$AD = AG \cos \alpha = \frac{AB}{2} \cos cos\, \alpha = 2.25 \cos \alpha$
$AH = AE \cos \alpha = (AB - BE) \cos \alpha = 3.3 \cos \alpha$
Resolving the forces vertically,

$$R_A + F_B = 1100$$

$$R_A + 0.2\, R_B = 1100 \qquad \text{............ } (i)$$

Resolving the force horizontally,

$$R_B = F_A = 0.4\, R_A \qquad \text{............ } (ii)$$

Substituting the value of R_B in equation (*i*), we get

$$R_A + 0.2 \times 0.4\, R_A = 1100$$

$$\therefore R_A = 1018.52\, N$$

Substituting the value of R_A in equation (*ii*), we get

$$R_B = 0.4 \times 1018.52 = 407.41\, N$$

Now taking moments of all forces about A, we get

$$200 \times AD + 900 \times AH = R_B \times BC + FB \times AC$$

$$200 \times 2.25 \cos cos\, \alpha + 900 \times 3.3 \cos cos\, \alpha = 407.41 \times 4.5 \sin sin\, \alpha + 0.2 \times 407.41 \times 4.5 \cos cos\, \alpha$$

$$3053.33 \cos cos\, \alpha\ =\ 1833.345 \sin sin\, \alpha$$

$$\frac{\sin sin\, \alpha}{\cos cos\, \alpha} = \frac{3053.33}{1833.345} = 1.665$$

$\tan tan\, \alpha\ =\ 1.665$

$$\therefore \alpha =\ tan^{-1}(1.665) = 59.01^{o}$$

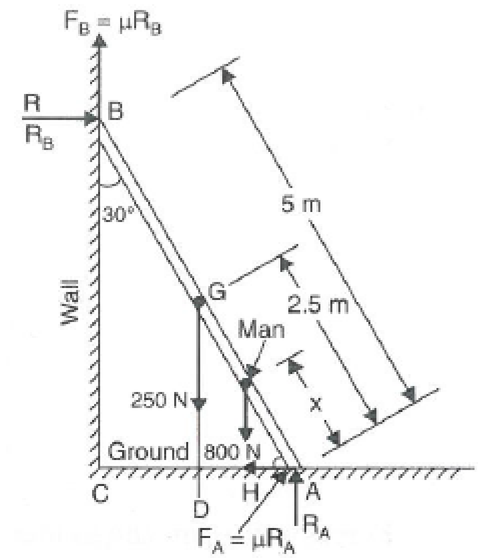


Fig. 1.70

Example 1.49. *A ladder 5 m long and 250 N weight is placed against a vertical wall in a position where its inclination to the vertical is 30º. A man weighing 800 N climbs the ladder. At what position will he induce slipping? The co-efficient of friction for both the contact surfaces of the ladder i.e., with the wall and the floor is 0.2.*

Solution. Given:

Length of ladder, AB = 5 *m*

Angle made by ladder with vertical = 30º

$$\therefore \angle ABC\ =\ 30^{o}$$

And $\angle BAC\ =\ 60^{o}$

Weight of ladder = 250 N

Weight of man = 800 N

Co-efficient of friction between ladder and floor = 0.2

Also co-efficient of friction between ladder and wall = 0.2

Let *x* = the distance climbed by the man up the ladder from A when the ladder is on the point of slipping.

AB is the ladder and G is the middle point of the ladder at which the weight 250 N is acting. The man of weight 800 N is standing at E, at a distance *x* from A, when the ladder is on the point of slipping. This means the ladder at A will start moving towards right and a force of friction $F_A\ =\ \mu R_A$ will be acting towards left at A. The ladder at B will be moving downwards and hence a force of friction $F_B\ =\ \mu\, R_B$ will be acting upwards at B.

R_A = Normal reaction at A

R_B = Normal reaction at B

Resolving the forces vertically, we get

$$R_A\ +\ \mu\, R_B\ =\ 250\ +\ 800$$

$$R_A\ +\ 0.2\, R_B\ =\ 1050 \qquad \ldots\ldots. (i)$$

Resolving the forces horizontally, we get

$$R_B\ =\ \mu\, R_A\ =\ 0.2\, R_A$$

Substituting the value of R_B in equation (*i*), we get

$$R_A\ +\ 0.2\left(0.2\, R_A\right) =\ 1050$$

$$\therefore\ R_A = 1009.6\ N$$

$$\therefore\ R_B = 0.2\times1009.6 = 201.92\ N$$

From triangle AGD,

$$AD = AG\cos 60^\circ = = 2.5\times\frac{1}{2} = 1.25\ m$$

From triangle AEH,

$$AH = x\cos 60^\circ = \frac{x}{2}$$

Taking moments of all forces about A,

$$800\times AH + 250\times AD = R_B\times BC + F_B\times AC \qquad \ldots\ldots\ldots (ii)$$

Where, BC = AB cos 30 = 4.33 m; AC = AB cos 60 = 2.5 m; $F_B = \mu R_B = 40.384$ N

Substituting the above values in equation (*ii*), we get

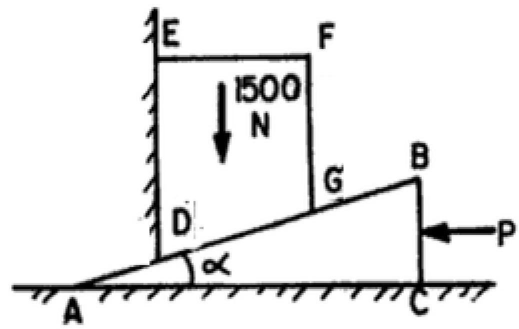


Fig. 1.71

$$800\times\frac{x}{2} + 250\times1.25 = 201.92\times4.33 + 40.384\times2.54$$

$$\therefore x = 1.657\ m$$

Problems on Wedge Friction:

Example 1.50. *A block weighing 1500 N, overlying a 10° wedge on a horizontal floor and leaning against a vertical wall, is to be raised by applying a horizontal force to the wedge. Assuming the coefficient of friction between all the surfaces in contact to be 0.3, determine the minimum horizontal force required to raise the block.*

Solution.

First of all, consider the equilibrium of the block.

Resolving the forces horizontally,

$$R_1 = R_2\sin sin\,10 + \mu R_2\cos cos\,10$$

$$\therefore R_1 = 0.469R_2$$

And now resolving the forces vertically,

$R_2\cos cos\,10 = 1500 + \mu R_1 + \mu R_2\sin sin\,10$ …….. (*i*)

Substituting the value of R_1 in equation (*i*), we get

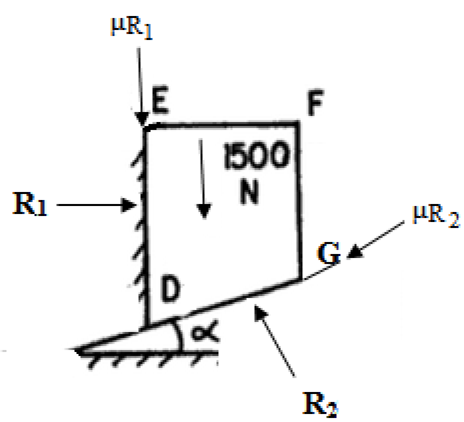


Fig. 1.72

$$R_2 \cos cos\ 10 = 1500 + 0.3 \times 0.469 R_2$$

$\therefore R_2 = 1893.9\ N$

$$\therefore R_1 = 0.469 \times 1893.9 = 888.24\ N$$

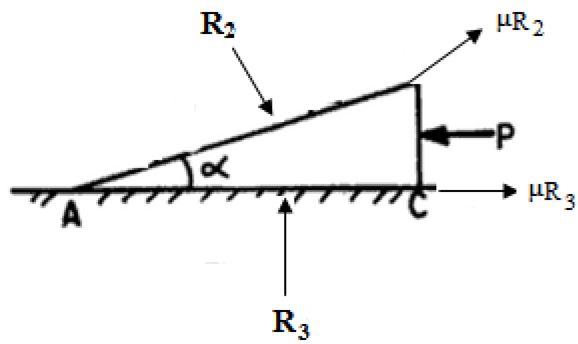


Fig. 1.73

Now consider the equilibrium of the wedge.
Resolving the forces vertically,

$$R_3 + \mu R_2 \sin sin\ 10 = R_2 \cos cos\ 10$$

$$R_3 = 1893.9(\cos cos\ 10 - 0.3 \times \sin sin\ 10\ )$$

$$\therefore R_3 = 1766.46\ N$$

And now resolving the forces horizontally,

$$P = \mu R_2 \cos cos\ 10 + R_2 \sin sin\ 10 + \mu R_3$$

$$= R_2(\mu \cos cos\ 10 + \sin sin\ 10\ ) + \mu R_3$$

$$= 1893.9(0.3 \times \cos cos\ 10 + sin 10\ ) + 0.3 \times 1766.46$$

$$\therefore P = 1418.35\ N$$

Example 1.51. *A block weighing 5000N is to be raised by means of a 12º wedge as shown in Fig. 1.74 Assume μ =0.4 for all the surfaces of contact. What is the horizontal force P that should be applied to raise the block? Weight of the wedge is 150N.*

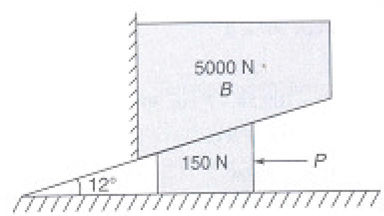


Fig. 1.74

Solution.

First of all, consider the equilibrium of the block.
Resolving the forces horizontally,

$$R_1 = R_2 \sin sin\ 12 + \mu R_2 \cos cos\ 12$$

$$\therefore R_1 = 0.599 R_2$$

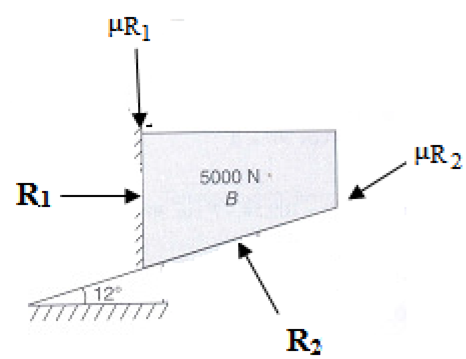


Fig. 1.75

And now resolving the forces vertically,

$R_2 \cos cos\ 12 = 5000 + \mu R_1 + \mu R_2 \sin sin\ 12$ .... (*i*)

Substituting the value of R_1 in equation (*i*), we get

$$R_2 \cos cos\ 12 = 5000 + 0.4 \times 0.599 R_2 + 0.4 R_2 \sin sin\ 12$$

$$\therefore R_2 = 7629.13\ N$$

$$\therefore R_1 = 0.599 \times 7629.13 = 4569.85\ N$$

Now consider the equilibrium of the wedge.

Resolving the forces vertically,

$$R_3 + \mu R_2 \sin sin\, 12 = R_2 \cos cos\, 12$$

$$R_3 = 7629.13(\cos cos\, 12 - 0.4\times \sin sin\, 12\ )$$

$$\therefore R_3 = 6827.94\ N$$

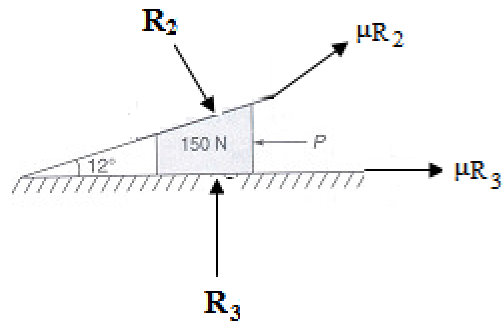


Fig. 1.76

And now resolving the forces horizontally,

$$P = \mu R_2 \cos cos\, 12 + R_2 \sin sin\, 12\ + \mu R_3$$

$$= R_2(\mu \cos cos\, 12 + \sin sin\, 12\ ) + \mu R_3$$

$$= 7629.13(0.4\times \cos cos\, 12 + sin 12\ ) + 0.4\times 6827.94$$

$$\therefore P = 7301.03\ N$$